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## CONDITIONS OF EQUILIBRIUM FOR EUROPEAN OPTION

The article deals with the Black-Scholes model where parameters depend on the time and the environmental state, conditions under which the fair price of an option before and after averaging coincide are considered. Furthermore, the main mathematical characteristics for the fair price of the European call option under the finite discrete-time homogenous Markov chain process are given.

**Key words**: *option, fair price, Black-Scholes model, Markov chain, stationary distribution, switching probability.*

**1. Introduction.** The transactions related to securities trading play important role on the stock markets. However, such kind of operations on the stock market require from traders to use the appropriate financial instruments which are characterized by a certain level of risk. Therefore, investing money at risky and risk-free assets stock market members faced with the problem of «portfolio decision making». The problem of «portfolio decision making» confronting individuals can roughly be described as the problem of the best allocation (investment) of funds (with due attention to possible risks) among property, gold, securities (bonds, stock, options, futures) etc. Options which belong to derivative financial instruments occupy significant place among securities [11]. Development and analysis of option pricing theory have been started in the first half of the XX-th century. In the framework of option theory was a thesis of Bachelier [1] where he derived a formula of option price, based on an assumption that stock prices generated by Brownian motion. Later on this model was modified by Samuelson [10] that gave impetus to genuine revolution in pricing methods after publications of Black and Scholes [2] and Merton [9]. Nowadays those theoretical works are a source of inspiration for multiple studies of more complex options and other types of derivatives.

In this paper we consider a problem of European price option calculation under uncertainty (unpredictable economic or political circumstances). The underlying shares prices on a given model described by geometric Brownian motion and interest rate, drift and volatility which depend on regime of

 

Markov process, are is widely used in financial analysis. Related works in this type have been studied and documented in the recent years. In particular, Di Masi, Kabanov and Runggaldier [4] consider the problem of hedging an European call option for a diffusion model where drift and volatility are function of a two state Markov jump process. Guo [8] also considers the same model and gives a closed-form formula for the European call option. Fuh Cheng-Der,Wang Ren-Her and Cheng Jui-Chi [7] proposed an approximation formula for option price when the number of states for the underlying Markov process is bigger than two. In this article we provide formulae of the main mathematical characteristics of the fair price of the European call option under finite discrete-time homogenous Markov chain process.

**2. Main Results.** Different types of options exist on the real financial market, but in this paper we will focus only on a European call option, which is based on the stock of value described by the random sequence $S=\left(S(t)\right)_{0\le t\le T}$.

A European option can be shown for exercise only at some fixed time $T$ (maturity time) due to a strike price $K$. If the final asset price exceeds the strike price, i.e., if $S(T)>K$, then this situation is suitable for the buyer of the option, since under the terms of the contract he has the right to buy shares at a price $K$ and then immediately sell them for the market at a price $S(T)$. In this case, the buyer has the payoff of $S(T)-K$.

On the contrary, if $S(T)<K$, then the buyer would be better off purchasing the assets at a market price $S(T)$, thus the buyer will simply refuse the contract incurring the payoff of $0$.

Obviously, in order to hold such a financial instrument, the buyer has to pay some premium $C(S,T)$ which is called a fair price. Therefore, the seller and the buyer are constantly facing the problem of a fair price determination. A large number of outstanding scholars from Merton [9] to Black and Scholes [2] have been working on this problem lately. Although the formulae that have been developed are useful, a lot of them are not adequate on account of a more complicated situation on the stock market nowadays. For instance, how to calculate a fair price of a call option under conditions of uncertainty of the development of market prices, interest rates, the unforeseeable nature of the actions and decisions of market operators, and so on.

**2.1. The fair price of a European call option.** Let us consider the Black-Scholes continuous time model on a given probability space $(\Omega,F,\mathrm{P})$, with filtration $\{F_t,t\ge 0\}$ and two assets. As we have mentioned before, external determinants have an impact on the option price.

Let us assume that they are described by a set of mutually exclusive events $A_1,...,A_n$ and hereafter are identified as the environmental state. For example, the first state of the environment $A_1 = $ {the financial market falls into stagnation}, $A_2 = $ {the financial market falls into recession}, $A_3 = $ {the financial market gains growth}, etc. At time $t \geq 0$ one of the $n$ environmental states could occur with a corresponding probability

$$P\left(A_i\right) = p_i, p_i > 0 \quad \forall i, \ \sum_{i=1}^{n} p_i = 1.$$

Nevertheless, it is unknown which state will definitely occur at time $t > 0$. Therefore, a question arises as how to calculate a fair price of an option in this case. For simplicity we consider only a one-step model. A multi-step model is much more complicated and it requires further investigation.

For each environmental state $A_i$ we can compute the value of non-risky and risky assets according to the next formulae:

$$B_{A_i}\left(t\right) = \exp\left\{\int_0^t r_{A_i}\left(s\right)ds\right\}, \quad t \geq 0, \tag{1}$$

$$S_{A_i}(t) = S_{A_i}(0)\exp\left\{\int_0^t\left(\mu_{A_i}\left(s\right) - \frac{1}{2}\sigma_{A_i}^2\left(s\right)\right)ds + \int_0^t \sigma_{A_i}(s)dW_{A_i}(s)\right\}, \tag{2}$$

where $r_{A_i}\left(t\right)$ — risk-free interest rate, $\mu_{A_i}\left(t\right)$ — measure of average rate of growth of an asset price (also known as a drift), $\sigma_{A_i}(t)$ — volatility, $W_{A_i}(t)$ — standard Wiener process, $S_{A_i}(0)$ — initial value of risky assets. Taking into account the absence of arbitrage opportunities we require the fulfillment of the following conditions: $\int_0^T \left(\frac{\mu_{A_i}\left(s\right) - r_{A_i}\left(s\right)}{\sigma_{A_i}\left(s\right)}\right)^2 ds < \infty$, $\int_0^T \sigma_{A_i}^2\left(s\right)ds < \infty$, $\int_0^T \left|\mu_{A_i}\left(s\right)\right| ds < \infty$, $\forall i = 1,2,...,n$. According to [3] and [6] the fair price of the European call option $C_{A_i}(S_{A_i}, t)$ satisfies the following boundary value problem:

$$\begin{aligned} &\frac{\partial C_{A_i}\left(S_{A_i}, t\right)}{\partial t} + \frac{1}{2}\sigma_{A_i}^2\left(t\right)S_{A_i}^2 \frac{\partial^2 C_{A_i}(S_{A_i}, t)}{\partial S_{A_i}^2} + \\ &+ r_{A_i}(t)S_{A_i} \frac{\partial C_{A_i}(S_{A_i}, t)}{\partial S_{A_i}} - r_{A_i}\left(t\right)C_{A_i}(S_{A_i}, t) = 0, \end{aligned} \tag{3}$$

$$C_{A_i}(S_{A_i},T)=\max(S_{A_i}-K,0),\ \ (S_{A_i},t)\in R^{+}\times[0,T]. \qquad (4)$$

The following theorem is true [12]:

**Theorem 1.** We suppose that for $\forall i=1,2,...,n$ the $r_{A_i}(t)$ and $\sigma_{A_i}(t)$ are continuous on the interval $[0,T]$, $\sigma_{A_i}(t)>0$ for all $t\in[0,T]$. Then a solution of differential equation (3) with boundary conditions (4) has the following representation:

$$C_{A_i}(S_{A_i},t)=S_{A_i}\Phi(d_1^{A_i})-K\exp\left\{-\int_t^T r_{A_i}(s)ds\right\}\Phi(d_2^{A_i}),\ i=1,...,n$$

where

$$d_1^{A_i}=\frac{\ln\frac{S_{A_i}}{K}+\int_t^T (r_{A_i}(s)+\frac{1}{2}\sigma_{A_i}^2(s))ds}{\sqrt{\int_t^T \sigma_{A_i}^2(s)ds}},\ \ d_2^{A_i}=\frac{\ln\frac{S_{A_i}}{K}+\int_t^T (r_{A_i}(s)-\frac{1}{2}\sigma_{A_i}^2(s))ds}{\sqrt{\int_t^T \sigma_{A_i}^2(s)ds}}$$

and

$$\Phi(d_j^{A_i})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_j^{A_i}}\exp\{-\frac{1}{2}s^2\}ds,\ \ j=1,2$$

cumulative distribution function for a standardized normal random variable.

According to the previous theorem, at time we can compute the value of $C_{A_i}(S_{A_i},t)$ for any given state $A_i$ having probability $P(A_i)=p_i$, since the values of $r_{A_i}$, $\mu_{A_i}$, $\sigma_{A_i}$ are known.

On the other hand, we can calculate the averages of $r_{A_i}$, $\mu_{A_i}$, $\sigma_{A_i}$ over all states of the environment and after that find out a fair price $C(S,t)$. That is, if $r(t)=\sum_{i=1}^{n} r_{A_i}p_i$, $\mu(t)=\sum_{i=1}^{n}\mu_{A_i}p_i$, $\sigma(t)=\sum_{i=1}^{n}\sigma_{A_i}p_i$ then we can find out a share price $S(t)$ according to formula (2) which allows to calculate the price of a European call option $C(S,t)$.

The goal of this paper is to find the conditions under which the fair price of a European call option will satisfy the equality:

$$C(S,t)=p_1C_{A_1}(S_{A_1},t)+p_2C_{A_2}(S_{A_2},t)+...+p_nC_{A_n}(S_{A_n},t) \qquad (5)$$

In other words, under which conditions the fair price of an option before and after averaging will coincide. Equality (5) could be written as:

$$S\Phi(d_1)-K\exp\left\{-\int_t^T r(s)ds\right\}\Phi(d_2)=$$
$$=\sum_{i=1}^{n} p_i\left(S_{A_i}\Phi(d_1^{A_i})-K\exp\left\{-\int_t^T r_{A_i}(s)ds\right\}\Phi(d_2^{A_i})\right) \tag{6}$$

Obviously, (6) is true if

$$S\Phi(d_1)=\sum_{i=1}^{n} p_i S_{A_i}\Phi(d_1^{A_i}), \tag{7}$$

$$\exp\left\{-\int_t^T r(s)ds\right\}\Phi(d_2)=\sum_{i=1}^{n} p_i\exp\left\{-\int_t^T r_{A_i}(s)ds\right\}\Phi(d_2^{A_i}) \tag{8}$$

Taking into consideration that $S=S(0)\exp\left\{\sum_{i=1}^{n} p_i X_i\right\}$, where

$$X_i=\int_0^t\left(\mu_{A_i}(s)-\frac{1}{2}\sigma_{A_i}^2(s)\right)ds+\int_0^t\sigma_{A_i}(s)dW_{A_i}(s),$$

then (7) could be rewritten as:

$$\exp\left\{\sum_{i=1}^{n} p_i X_i\right\}\int_{-\infty}^{d_1}\exp\left(-\frac{s^2}{2}\right)ds=\sum_{i=1}^{n}\left(p_i\exp\{X_i\}\int_{-\infty}^{d_1^{A_i}}\exp\left(-\frac{s^2}{2}\right)ds\right) \tag{9}$$

After simplifying equation (8), the following equality is deduced:

$$\exp\left\{-\sum_{i=1}^{n}\left(p_i\int_t^T r_{A_i}(s)ds\right)\right\}\int_{-\infty}^{d_2}\exp\left(-\frac{s^2}{2}\right)ds=$$
$$=\sum_{i=1}^{n}\left(p_i\exp\left\{-\int_t^T r_{A_i}(s)ds\right\}\int_{-\infty}^{d_2^{A_i}}\exp\left(-\frac{s^2}{2}\right)ds\right). \tag{10}$$

Thus, the fair price of a European call option before and after averaging will coincide if and only if conditions (9) and (10) are satisfied.

**2.2. Main Mathematical Characteristics of the European Call Option.** In this section we are going to determine the main mathematical characteristics such as mathematical expectation, variance and expected risk for the function $C=\left\{C_{A_1}(S_{A_1},t),C_{A_2}(S_{A_2},t),\ldots,C_{A_n}(S_{A_n},t)\right\}$ under finite discrete-time homogenous Markov chain process. Let us label $A=(A_1,A_2,\ldots)$ as some finite or countable set whose elements are called states of Markov chain. Consider a random function $x(t)$ which may take values on $A$, defined for $t = 0,1,2,...,$ and homogeneous Markov chain that has the following property:

$$\Pi(t) = \Pi(t)^t,$$

where $\Pi(t)$ — matrix of transition probabilities. Obviously, that for a homogeneous Markov process with discrete time is enough to set up an initial matrix $\Pi(1)$, since to homogeneity restriction the probability does not depend on the time $t$, but rather remains constant over time.

For a given stochastic matrix

$$\Pi = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix},$$

always exist such numbers $q_1,...,q_n$ that $\sum_{k=1}^{n} q_k p_{ki} = q_i$ and $\sum_{i=1}^{n} q_i = 1$, for all $i = 1,...,n$ where $p_{ij}$ is the conditional probability to jump from a given state $A_i$ to state $A_j$. If $x(t)$ homogeneous Markov process with $n$ states of environment having one-step transition matrix $\Pi$ and initial distribution $P\{x(0) = A_k\} = q_k$, then $P\{x(t) = k\} = q_k$ is true for all $t$. Distribution $\{q_1,...,q_n\}$ called stationary distribution for a given Markov chain.

Suppose that the matrix of transition probabilities has a following representation:

$$p_{12} = 1, \; p_{kk-1} = p_{kk+1} = \frac{1}{2}, \; 1 < k < n, \; p_{nn-1} = 1.$$

Having this matrix we can find stationary distribution of $\{q_i\}_{i=1}^{n}$:

$$2q_1 = q_2 = ... = q_{n-1} = 2q_n = \frac{1}{n}. \qquad (11)$$

From (11) it is easy to see that the stationary distribution is described by $n$-dimensional vector $q = \left(\frac{1}{2n}, \frac{1}{n}, ..., \frac{1}{n}, \frac{1}{2n}\right)$. Using all those results and assumptions we can start to find mathematical expectation, variance and expected risk for European call option

$$C = \left\{C_{A_1}(S_{A_1}, t), C_{A_2}(S_{A_2}, t), ..., C_{A_n}(S_{A_n}, t)\right\}$$

applying ergodic theorem for discrete time [5]. Firstly, we have to check necessary condition of limit existence $\sum q_x C(x) < \infty$.

$$\sum_x q_x C(x) = \sum_{i=1}^{n} q_i C_{A_i} = \frac{1}{2n} C_{A_1} + \frac{1}{n} C_{A_2} + ... + \frac{1}{n} C_{A_{n-1}} + \frac{1}{2n} C_{A_n} =$$

$$= \frac{1}{2n}(C_{A_1} + C_{A_n}) + \frac{1}{n}(C_{A_2} + ... + C_{A_{n-1}}) < \infty.$$

As we have a finite number of terms, therefore the sum is also a finite number. Taking this into account, the expected value is calculated according to the formula (12):

$$E(C) = \frac{1}{2n}(C_{A_1} + C_{A_n}) + \frac{1}{n}(C_{A_2} + ... + C_{A_{n-1}}). \quad (12)$$

In order to get second characteristic namely variance of the European call option we will apply classical definition of dispersion

$$Var(C) = E(C^2) - (E(C))^2.$$

Using ergodic theorem we can find $E(C^2)$:

$$E(C^2) = \sum_{i=1}^{n} q_i C_{A_i}^2 = \frac{1}{2n}(C_{A_1}^2 + C_{A_n}^2) + \frac{1}{n}(C_{A_2}^2 + ... + C_{A_{n-1}}^2).$$

Since the $E(C^2)$ and $(E(C))^2$ are known, the formula for the variance can be written as:

$$Var(C) = \frac{1}{2n}(C_{A_1}^2 + C_{A_n}^2) + \frac{1}{n}\sum_{i=2}^{n-1} C_{A_i}^2$$

$$-\left(\frac{1}{2n}(C_{A_1} + C_{A_n}) + \frac{1}{n}\sum_{i=2}^{n-1} C_{A_i}\right)^2. \quad (13)$$

It is easy to see, that the formula of expected risk can be obtained directly from (12) and (13) as

$$\delta = \frac{\sqrt{Var(C)}}{E(C)}. \quad (14)$$

Thus, in this chapter we calculated the main mathematical characteristics such as mathematical expectation, variance and expected risk for European price option according to (12), (13), and (14) respectively.

**3. Summary.** In this paper formulae of risky and non-risky assets price under uncertainty were considered. This article explores conditions of equilibrium for European call option under which the fair price before and after averaging coincide. Furthermore, the basic mathematical characteristics of the fair price under homogeneous Markov process were calculated.

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У статті розглянуто модель Блека-Шоулса та Мертона з параметрами, що залежать від часу та стану зовнішнього середовища, встановлено умови за яких ціна Європейського опціону до і після усереднення у даній моделі співпадає. Також виведено формули основних математичних характеристик для оцінки справедливої ціни Європейського опціону на скінченному однорідному ланцюгу Маркова з дискретним часом.

**Ключові слова:** *опціон, справедлива ціна, модель Блека-Шоулса, ланцюг Маркова, стаціонарний розподіл, перехідні ймовірності.*

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