UDC 519.642

E. V. Semenova, Cand. of Phys. and Mathem. Sciences,
E. A. Volynets, Cand. of Phys. and Mathem. Sciences
Institute of Mathematic of NAS of Ukraine, Kiev

DISCREPANCY PRINCIPLE FOR SOLVING PERIODIC INTEGRAL EQUATIONS OF THE FIRST KIND

Fully discrete projection method with discrepancy principle is considered for solving periodic integral equations of the first kind with unknown smoothness of solution. For proposed approach it is proved the optimality and effectiveness in the sense of computational resource.

Key words: *fully discrete projection method, discrepancy principle, periodic integral equation.*

Introduction. The object of our investigation is periodic integral equation of the first kind with operator of pseudodifferential structure. As it is known pseudodifferential equation of elliptic type are frequently found in various problems of natural sciences that can be described by a boundary value problems such as Laplace, Neumann or Helmholz equations. Such equations are well-known and were investigated, for example, in [1]. The most widely-used approaches for numerical solving periodic integral equations of the first kind are fully discrete collocation and projection methods that applied together with selfregularization principle. In the framework of the paper it will be considered a modification of fully discrete projection method (FDPM). Note that standard variant of FDPM was firstly proposed for solving the integral Symm equation (the particular case of problem under consideration) in [2] and extended on the class of pseudodifferentional equations in [3]. Moreover we introduce some additional discretization in the method to reduce amount of arithmetical operations.

Statement of the problem. In the space $L_2(0,1)$ we consider the following integral equation

$$Au(t) = f(t),\ t \in [0,1]\,, \tag{1}$$

where $f(t)$ is 1-periodic function and operator A has the form

$$A = D + \sum_{p=0}^{q} A_p\,,$$

where $A_p u(t) = \int_0^1 k_p(t-s) a_p(t,s) u(s) ds,\ Du = \int_0^1 k_0(t-s) u(s) ds.$

Let's denote by $C^\infty = C^\infty([0,1]^2)$ the space of infinity smooth 1-biperiodic functions of two variables. Suppose that $a_p \in C^\infty$, $p = 0,..,q$, $a_0(t,t) \neq 0,\ t \in [0,1]$, such that

$$\| a_p \|^2_{\eta_1,\eta_2} := \sum_{k,l=-\infty}^{\infty} |\widehat{a}(k,l)|^2 \, e^{2\eta_2 (|k|^{1/\eta_1} + |l|^{1/\eta_1})} < \infty, \; \eta_1 \geq 1, \eta_2 > 0. \qquad (2)$$

Moreover assume that $k_p(t-s)$ is 1-periodic function with known Fourier coefficients $\widehat{k}_p(n)$ by trigonometric basis. Additionally we suppose that for some $\alpha \in \mathbb{R}$ and $\beta > 0$ the following inequalities

$$\begin{aligned} &c_{00} |n|^{\alpha} \leq |\widehat{k}_0(n)| \leq c_0 |n|^{\alpha}, \quad n \in \mathbb{Z}/0, \\ &|\widehat{k}_0(n) - \widehat{k}_0(n-1)| \leq c |n|^{\alpha-\beta}, \quad n \in \mathbb{Z}, \\ &|\widehat{k}_p(n)| \leq c |n|^{\alpha-\beta}, \quad n \in \mathbb{Z}, \, p = 1,...,q, \end{aligned} \qquad (3)$$

hold true, where $c, c_0, c_{00} > 0$ and $n = \begin{cases} |n|, & n \in \mathbb{Z}/0 \\ 1, & n = 0 \end{cases}$.

Denote by H^{λ}, $-\infty < \lambda < \infty$, Hilbert spaces of 1-periodic functions and by H^{λ_1,λ_2}, $-\infty < \lambda_1, \lambda_2 < \infty$ 1-biperiodic functions with the norms

$$\| u \|_{\lambda} := (|\widehat{u}(0)|^2 + \sum |n|^{2\lambda} |\widehat{u}(n)|^2)^{1/2} < \infty,$$

$$\| a \|_{\lambda_1,\lambda_2} := (\sum |k|^{2\lambda_1} |l|^{2\lambda_2} |\widehat{a}(k,l)|^2)^{1/2} < \infty,$$

respectively. Here $\widehat{u}(n), \widehat{a}(k,l)$ are Fourier coefficients of functions $u(t)$ and $a(t,s)$ by trigonometric basis $\{e_k\}_{k=-\infty}^{+\infty}$, where $e_k(t) = e^{i2\pi kt}$, $t \in [0,1]$.

We suppose that $f \in H^{\mu+1}$ for some unknown $\mu > \alpha + 1/2$. Let instead of $f(t)$ only some its perturbation is given such that for $N = \delta^{-\frac{1}{\lambda-\alpha}}$ we have

$$N^{-1} \left(\sum_{j=1}^{N} | f_{\delta}(jN^{-1}) - f(jN^{-1}) |^2 \right)^{1/2} \leq \delta \| f \|_{\mu-\alpha} .$$

Note that similar class of problems was considered before in [1], [3] and other. In particular, in [3] the optimal order fully discrete projection method for solving (1) with (2)–(3) was proposed. In the paper we consider the same class of problems as in [3] and state the aim to reduce the amount of arithmetical operations with saving optimality of the method.

Auxiliary statements. For further presentation of our results we will use the following notations.

Let's introduce N-dimensional subspace of trigonometric polynomials

$$T_N = \left\{ u_N : u_N(t) = \sum_{k \in \mathbb{Z}_N} c_k e_k(t) \right\},$$

where $\mathbb{Z}_N = \left\{ k : -\frac{N}{2} < k \le \frac{N}{2},\ k = 0, \pm 1, \pm 2, \ldots \right\}$.

Denote by P_N and P_Ω orthoganal projectors

$$P_N u(t) = \sum_{k \in \mathbb{Z}_N} \hat{u}(k) e_k(t) \in T_{N,}$$

$$P_{\Omega_N} u(t) = \sum_{l,k \in \Omega_N} \hat{a}(k,l) e_k(t) e_l(s) \in T_N \times T_{N,}$$

where Ω_N is some domain on the coordinate plane restricted by square $(-N/2, N/2] \times (-N/2, N/2]$. Also denote by $Q_{N,}$ and $Q_{N,N}$ interpolation projectors, such that $Q_N u(t) \in T_N$, $Q_{N,N} u(t) \in T_N \times T_N$ and on the uniform grid it holds true $(Q_N u)(jN^{-1}) = u(jN^{-1}), j = 1, 2, \ldots, N,$

$$(Q_{N,N} a)(jN^{-1}, iN^{-1}) = a(jN^{-1}, iN^{-1}), \quad j, i = 1, 2, \ldots, N\,.$$

Modified fully discrete projection method. As Ω_N we take the following domain on the coordinate plane

$$D_M^{\eta_1} = \left\{ (k,l) : |k|^{1/\eta_1} + |l|^{1/\eta_1} < \left(\frac{M}{2} \right)^{\frac{1}{\eta_1}},\ k, l = 0, \pm 1, \pm 2, \ldots \right\}.$$

Assume that the discrete information about kernels $a_p(t,s)$ and right hand side f are given in the knots of uniform grids $\left(\frac{j_1}{N}, \frac{j_2}{N} \right)$ and $\left(\frac{j}{N} \right)$ respectively, where $j, j_1, j_2 = 1, \ldots, N$.

The right-hand side of equation (1) we approximate as $f_{\hat{N}} := Q_{\hat{N}} f_\delta$. Thus, instead of the kernels $a_p(t,s)$ we take the following finite dimensional elements

$$a_{p,M} = P_{D_M^{\eta_1}} Q_{M,M} a_p, \tag{4}$$

where M is discretization parameter such that $\hat{N} > M$.

Then the operators $A_{p,M}$ can be approximated by

$$A_{p,M} u(t) = \int_0^1 k_p(t-s) a_{p,M}(t,s) u(s) ds,$$

where function $a_{p,M}$ has the form (4).

Additional discretization of operators $A_{p,M}$ consists in replacing $A_{p,M}$ on the operator $P_l A_{p,M} P_l$. Thus we approximate A in terms of

$$A_M = D + P_l \sum_{p=0}^{q} A_{p,M} P_l, \tag{5}$$

where $l = \widehat{N}^{\tau}$ for some $0 < \tau < 1$ and $\widehat{N}, M$ are discretization parameters that should be chosen in appropriate way. Note that approximation (5) is distinguished from respective approximation from [3] by using additional projectors P_l and $P_{D_M^{m}}$. Such approach help to reduce the amount of arithmetical operations.

We propose the following modification of the fully discrete projection method (FDPM) for numerical solving (1):

$$A_M u_{\widehat{N}} := D u_{\tilde{N}} + P_l \sum_{p=0}^{q} A_{p,M} P_l u_{\widehat{N}} = Q_{\widehat{N}} f_{\delta}\ , \tag{6}$$

where $A_{p,M}$ has the view (5) and $u_{\widehat{N}} \in T_{\widehat{N}}$ is taken as approximate solution of (1).

Note that by virtue of (3) it holds true

$$A_{p,M} \in L(H^{\lambda}, H^{\lambda-\alpha+\beta}),\ p = 0,...q.$$

Following [1] for fast solving (6) we propose to use GMRES. The procedure is following: for $n = 1, 2, ...$ we construct the sequence $u_{n,\widehat{N}}$ that satisfies the condition

$$\| S_{\widehat{N}} u_{n,\widehat{N}} - f_{\widehat{N}} \|_{\alpha} = \min_{u \in K_n(S_{\widehat{N}}, f_{\widehat{N}})} \| S_{\widehat{N}} u - f_{\widehat{N}} \|_{\alpha}, \tag{7}$$

where $S_{\widehat{N}} = D + P_l \sum_{p=0}^{q} A_{p,M} P_l$ and $K_n(S_{\widehat{N}}, f_{\widehat{N}})$ is well-known Krylov space. As the stopping rule we consider the discrepancy principle

$$\| S_{\widehat{N}} u_{n,N} - f_{\widehat{N}} \|_{\alpha} \le c\delta \| f_{\widehat{N}} \|_{\alpha}\ ,$$

where $u_{n,\widehat{N}}$ is n-th iteration of GMRES that we consider as approximate solution to $u_{\widehat{N}}$.

Theorem 1. Let n be the first number that fulfils (7). Then the accuracy of GMRES is

$$\| u_{n,\widehat{N}} - u_{\widehat{N}} \|_{\lambda} \le c \left(\frac{\widehat{N}}{2} \right)^{\lambda - 2\alpha} \delta\ ,$$

where c is some positive constant. Moreover the amount of arithmetical operations for solving (6) by (7) is $O(\widehat{N}\log\widehat{N})$.

Main results. In view of self regularization on the pair of spaces H^{λ} and $H^{\lambda-\alpha}$ the problem under consideration doesn't need any additional regularization but it is necessary to chose appropriate discretization parameters. For this aim we propose to use discrepancy principle that is describe below.

Let $D_N = \{1, 2, 2^2, ..., 2^{N_1} = \delta^{\frac{1}{\lambda-\alpha}}\}$ be a set of possible discretization parameter. We take the discretization parameters $\widehat{N}$ and M by the rule:

$$\widehat{N} = \min\{\widehat{N} \in D_N : \| A_N u_{\widehat{N}} - Q_{\widehat{N}} f_\delta \| \le b_2\delta\}, \tag{8}$$

$$M = O(\log\widehat{N}).$$

Theorem 2. Let M and $\widehat{N}$ be chosen according (8). Then the error bound of FDPM for (1) is

$$\| u - u_{\widehat{N}} \|_\lambda \le c(\delta^{\frac{\mu-\lambda}{\mu-\alpha}}),$$

where c is some positive constant that doesn't depend on $\widehat{N}, \delta$. Moreover to find numerical solution by FDPM (6) with (7) and (8) it is necessary to execute $O(N\log^2 N)$ arithmetical operations.

Remark. Taking into account Theorems 1 and 2 the general accuracy of the method (6) in combination with (7) and (8) is the following

$$\| u - u_{n,\widehat{N}} \|_\lambda \le c\delta^{\frac{\mu-\lambda}{\mu-\alpha}}.$$

Besides the total amount of arithmetical operations for solving (1) by (6) in combination with (7) and (8) is $O(N\log^2 N)$.

Conclusion. In the paper for solving periodic integral equations of the first kind a problem of reduction amount of arithmetical operations is considered. For this we propose some modification of fully discrete projection method in combination with discrepancy principle for choosing appropriate discretization parameter and GMRES for fast solving of system of linear equations. It was proved that such approach guarantees the best possible accuracy of recovering the solution in the metric of Sobolev spaces with minimal computational costs in comparison with the methods investigated earlier.

References:

1. Saranen J., Vainikko G. Periodic Integral and Pseudodifferential Equations with Numerical Approximation. Berlin: Springer, 2002. 452 p.

2. Pereverzev S. V., Prossdorf S. On the characterization of self-regularization properties of a fully discrete projection method for Symm's integral equation. *J. Integral Equations Appl*. 2000. V. 12, N 2. P. 113–130.
3. Solodky S. G., Semenova E. V. A class of periodic integral equation with numerical solving by a fully discrete projection method. UMV. 2014. V. 11, N 3, P. 400–416.

Розглянуто повністю дискретний проекційний метод у комбінації з принципом рівноваги для розв'язування періодичних інтегральних рівнянь у апостеріорному випадку. Доведена оптимальність та економічність такого підходу.

Ключові слова: *повністю дискретний проекційний метод, принцип нев'язки, періодичні інтегральні рівняння.*

Date received 06.03.2017

УДК 519.8

Н. В. Семенова, д-р фіз.-мат. наук,
Т. Т. Лебєдєва, канд. екон. наук,
Т. І. Сергієнко, канд. фіз.-мат. наук

Інститут кібернетики імені В. М. Глушкова НАН України, м. Київ

ОПТИМАЛЬНІСТЬ ТА КОРЕКТНІСТЬ У ВЕКТОРНИХ ЗАДАЧАХ ДИСКРЕТНОЇ ОПТИМІЗАЦІЇ

Сформульовано умови оптимальності розв'язків векторної задачі дискретної оптимізації на допустимій множині, що описується псевдоопуклими функціями обмежень, отримано достатні умови оптимальності різних видів розв'язків задачі та п'яти типів її стійкості. Встановлено топологічні властивості підмножин простору вхідних даних задачі, на яких зберігається оптимальність її розв'язків.

Ключові слова: *дискретна оптимізація, векторна задача, стійкість, коректність.*

Вступ. Встановлення необхідних і достатніх умов оптимальності та стійкості розв'язків векторних дискретних задач це актуальна проблема, оскільки їх знання дає основу для розробки способів перевірки оптимальності та якості того чи іншого обраного розв'язку, та побудови ефективних методів знаходження множин оптимальних розв'язків, які мають деякі наперед задані властивості інваріантності при можливих збуреннях вхідних даних задачі [1].

У доповіді сформульовано умови оптимальності розв'язків векторної задачі дискретної оптимізації на допустимій множині, яка описується псевдоопуклими функціями обмежень [2], отримано дос-

