Since each algebra of finite rank can be monomorphically immersed in some complete matrix algebra, this caused, so to speak, an inverse approach to the construction of new algebras. A certain subalgebra stands out from a complete matrix algebra, which is a matrix representation of an algebra of finite rank. It is the implementation of such an approach that makes it possible to endow elements of algebra of finite rank with matrix characteristics, in particular, a canonical representation of algebra elements is constructed through the spectral representation of a matrix, and the algebra itself is endowed with a topological structure through one of the matrix norms. At the same time, an additional condition is often imposed, that it be an algebra over the field of real or complex numbers.

The article constructs a real algebra of finite rank, the elements of which are matrices of the second order with the same sum of rows and columns. We endowed it with a norm and a scalar product, demonstrating that it is a Euclidean space. This algebra is a matrix representation of the algebra of hypercomplex numbers, which we called binary in our research [4].

Key words: *algebra of finite rank, matrix, norm, Euclidean space, hypercomplex system.*

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Kateryna Heseleva, Ph. D.

Kamianets-Podilskyi Ivan Ohiienko National University, Kamianets-Podilskyi

METHODS FOR SOLVING ONE TYPE OF LINEAR INTEGRO-FUNCTIONAL EQUATION

The article considers one of the possible variants of the linear integro-functional equation. A method of transforming such equation into a Fredholm integral equation of the second kind is presented. Approximate solutions of this equation are constructed using collocation and collocation-iterative methods.

Key words: *one type of linear integro-functional equation, Fredholm's equation of the second kind, completely continuous operator, inverse operator, approximate solution, collocation and collocation-iterative methods.*

Introduction. In the study of different problems of a theoretical and applied character of differential, integral and integro-functional equations are totally used [3, 4, 6]. As the construction of exact solutions of such equations is possible only in separate cases. Therefore, methods of construction of approximate solutions of these equations are of great importance. One of these methods is the collocation method and one of its generalizations, is collocation-iterative method [1, 2, 5].

This article considers the possibility of applying these methods to one type of linear integro-functional equation.

Main part. One type of linear integro-functional equation is considered. Equations of somet simpler structure were considered by the author in [3, 4]. We will construct approximate solutions of such equation using collocation and collocation-iterative methods.

Linear Integro-Functional Equation And Its Transformation. In the space $L_2[a,b]$ of real and measurable functions on interval $[a,b]$, which are equivalent in the square, let's consider the integro-functional equation of the next form:

$$y(x)-p(x)y\left(h(x)\right)= f(x)+\int_a^b H_1(x,t)y(t)dt+\int_a^b H_2(x,t)y\left(h(t)\right)dt, \quad x\in\left[a,b\right], \tag{1}$$

$$y\left(x\right)=\psi(x), x\notin\left[a,b\right], \tag{2}$$

where $f\left(x\right), \psi(x)$ are the known functions set on the $[a,b]$ and outside the interval $[a,b]$, appropriately; $y\left(x\right)$ is the sought function from $L_2[a,b]$. Regarding functions $h(x), p(x), H_1(x,t), H_2(x,t)$ we assume that they, respectively, on the interval $[a,b]$ and in the square $\left[a,b\right]^2=\left[a,b\right]\times\left[a,b\right]$ satisfy the conditions:

$$\left|p\left(x\right)\right|\le\overline{p}<\infty, \tag{3}$$

$h\left(x\right)$ – differential on the interval $\left[a;b\right]$ and

$$h'\left(x\right)\ge l>0, x-h\left(x\right)\ge\sigma>0, \tag{4}$$

$$\int_a^b\int_a^b H_i^2(x,t)dxdt=H_i^2<\infty, i=1,2. \tag{5}$$

Let us show that equation (1) under conditions (3)-(5) can be reduced to Fredholm's integral equation of the second kind [3, 5]. To do this, we will rewrite the second integral of the right-hand side of equation (1) taking into account condition (2) as follows:

$$\int_a^b H_2\left(x,t\right)y\left(h(t)\right)dt=\int_a^{h^{-1}(a)} H_2\left(x,t\right)y\left(h(t)\right)dt+\int_{h^{-1}(a)}^b H_2\left(x,t\right)y\left(h(t)\right)dt=$$

$$=\int_a^{h^{-1}(a)} H_2\left(x,t\right)\psi\left(h(t)\right)dt+\int_{h^{-1}(a)}^b H_2\left(x,t\right)y\left(h(t)\right)dt=$$

$$= \varphi(x) + \int_{h^{-1}(a)}^{b} H_2(x,t)\, y(h(t))\, dt,$$

where $\varphi(x) = \int_{a}^{h^{-1}(a)} H_2(x,t)\psi(h(t))\,dt$ – the known function.

Due to condition (4), the continuous function $s = h(t)$ will be increasing and there will be inverse function for it $t = h^{-1}(s),\ dt = \dfrac{ds}{h'\left(h^{-1}(s)\right)}$. The new limits of integration will be the values: $a, h(b)$. Then the last integral will take next form:

$$\int_{h^{-1}(a)}^{b} H_2(x,t)\, y(h(t))\, dt = \int_{a}^{h(b)} \frac{H_2\left(x, h^{-1}(s)\right)}{h'\left(h^{-1}(s)\right)} y(s)\,ds = \int_{a}^{b} \tilde{H}(x,s)\,y(s)\,ds,$$

$$\tilde{H}(x,s) = \begin{cases} \dfrac{H_2\left(x, h^{-1}(s)\right)}{h'\left(h^{-1}(s)\right)}, & s \in \left[a, h(b)\right], \\ 0, s \in (h(b), b], & x \in [a,b]. \end{cases} \tag{6}$$

It should be noted, that the operator $\tilde{H}$ defined by equality

$$\left(\tilde{H}v\right)(x) = \int_{a}^{b} \tilde{H}(x,t)v(t)dt, \forall v(x) \in L_2[a,b], \tag{7}$$

taking into account conditions (4)-(5) will be Fredholm.

Really,

$$\int_a^b\int_a^b \tilde{H}^2(x,s)\,dxds = \int_a^b\int_a^b \frac{H_2^2\left(x,h^{-1}(s)\right)}{\left(h'\left(h^{-1}(s)\right)\right)^2}dxds \le \int_a^b\int_a^b \frac{H_2^2\left(x,h^{-1}(s)\right)}{h^2}dxds =$$

$$= \frac{1}{h^2}\int_a^b\int_a^b H_2^2\left(x, h^{-1}(s)\right)dxds \le \frac{H_2^2}{h^2} < \infty.$$

Taking into account the above considerations, equation (1) with condition (2) will be written as follows

$$y(x) - p(x)y(h(t)) = f(x) + \int_a^b H_1(x,t)\,y(t)\,dt + \varphi(x) + \int_a^b \tilde{H}(x,s)\,y(s)\,ds,$$

or

$$y(x)-p(x)y(h(t))= f_1(x)+\int_a^b T(x,t)y(t)dt, \tag{8}$$

where

$$f_1(x)= f(x)+\varphi(t)= f(t)+\int_a^{h^{-1}(a)} H_2(x,t)\psi(h(t))dt, \tag{9}$$

$$T(x,t)=H_1(x,t)+\tilde{H}(x,t),(x,t)\in[a,b]^2. \tag{10}$$

Let's consider the integral completely continuous the operator H_1, which has the form

$$(H_1v)(x)=\int_a^b H_1(x;t)v(t)dt, \forall v(x)\in L_2[a,b],$$

and the operator S such that

$$(Sv)(x)=\begin{cases} v(x), & x\in\left[a,h^{-1}(a)\right], \\ v(x)-p(x)v(h(x)), & x\in\left[h^{-1}(a),b\right], \end{cases} \tag{11}$$

where $v(x)$ is an arbitrary function of $L_2[a,b]$.

Note that, this operator likes the operator H_1, functionate from $L_2[a,b]$ in $L_2[a,b]$. It is easy to show, that the operator S is linear. Conditions (3), (4) guarantee its unlimited. Really,

$$S = sup\frac{(Sv)(x)}{v(x)} \le 1+\left|\frac{p^2(x)}{h^{'}(x)}\right|^{1/2} \le 1+\frac{\bar{p}}{\sqrt{l}}<\infty,$$

where sup is taken by $v(x)\in L_2[a,b], v(x)\neq 0$.

The same conditions indicate that the operator S is reversible. Its inverse operator has the form

$$(S^{-1}v)(x)=\begin{cases} v(x), x\in\left[a,h^{-1}(a)\right], \\ v(x)+\sum_{i=1}^{S} v(h^i(x)\prod_{k=0}^{i-1} p\left(h^k(x)\right), \end{cases} \tag{12}$$

$$x\in\Delta s, s=\overline{1,m}.$$

Here, as in the future,

$$\Delta s=\left[c_{s-1},c_s\right], c_0=a, c_s=h^{-1}\left(c_{s-1}\right), c_m=b, h^k(x)=h\left(h^{k-1}(x)\right), s=\overline{1,m}.$$

On the other hand, the expression (12) is the solutions of the functional equation

$$y(x)-p(x)y(h(x))=u(x), x\in[a,b], \qquad (12^*)$$
$$y(x)=\psi(x),\ x\notin[a,b],$$

(where $u(x)$ – known, $y(x)$ – searched function) using the method of steps. Condition (3) guarantees the fact that the number of steps m is limited and $m\le\frac{b-a}{\sigma}$.

It isn't difficult to make sure that, the operator S^{-1}, as well as the operator S, linear and bounded. So, considering the above, we can hold equation (1) as an operator equation

$$(Sy)(x)=f(x)+(Ty)(x), \qquad (13)$$

where $f(x)$ is given, $y(x)$ is the desired function in $L_2[a,b]$.

Let $(Sy)(x)=u(x)$, then $y(x)=(S^{-1}u)(x)$ and we pass from equation (13) to equation

$$u(x)=f(x)+(\tilde{T}u)(x). \qquad (14)$$

The operator $\tilde{T}=TS^{-1}=H_1S^{-1}+\tilde{H}S^{-1}$ Fredholm as a superposition of Fredholm's and linear bounded operators [2]. In other words, by applying the above-mentioned substitution, we transform the integro-functional equation (1) into Fredholm's integral equation of the second kind

$$u(x)=f(x)+\int_a^b\tilde{T}(x;t)u(t)dt \qquad (14^*)$$

with the completely continuous integral operator T,

$$\tilde{T}(x,t)=\begin{cases}T(x,t)+\sum_{i=1}^{m-s}T\left(x,\left(h^{-1}\right)^k(t)\right), & t\in\Delta s,\\ T(x,t), t\in(c_{m-1};b), s=\overline{1,m-1}, & x\in(a;b),\end{cases}$$

where $\left(h^{-1}\right)^k(t)=h^{-1}\left(\left(h^{-1}\right)^{k-1}(t)\right)$.

The following theorem is true.

Theorem 1. Equation (1) with condition (2), when conditions (3)-(5) are fulfilled, is equivalent to the integral equation(14*) with the completely continuous operator $\tilde{T}$.

Collocation Method of Solving Linear Integro-Functional Equation. The idea of this method in relation to equation (1) is that we look for an approximate solution $y_m(x)$. It has the next form

$$y_m(x) = \sum_{j=1}^{m} a_j \varphi_j(x),$$

and we determine from the functional equation

$$y_m(x) - p(x) y_m(h(x)) =$$

$$= f(x) + \int_a^b H_1(x,t) y_m(t) dt + \int_a^b H_2(x,t) y_m(h(t)) dt, x \in [a,b], \quad (15)$$

$$y_m(x) = \psi(x),\ x \notin [a,b],$$

where $\{\varphi_j(x)\}$ is a system of linearly independent $[a,b]$ functions, $j = \overline{1,m}$, and the unknown parameters $a_j = a_j(n)$ are found from the condition

$$\gamma_m(x_i) = 0, i = \overline{0,n}, \quad (16)$$

$$\gamma_m(x) = y_m(x) - p(x) y_m(h(x)) -$$

$$- f(x) - \int_a^b H_1(x;t) y_m(t) dt - \int_a^b H_2(x;t) y_m(h(t)) dt. \quad (17)$$

To find the parameters, a_j we obtain the system of linear algebraic equations

$$\sum_{j=1}^{n} \beta_{ij} a_j = b_i, i = \overline{1,n}, \quad (18)$$

in which β_{ij} is calculated according to the formula

$$\beta_{ij} = \varphi_j(x_i) - \tilde{T}_j(x_i), b_i = f(x_i), i = \overline{1,n}\,, \quad (19)$$

$$\tilde{T}_j(x) = \int_a^b \tilde{T}(x,t) \varphi_j(t) dt.$$

the system of equations (18) in vector form is $\Lambda a_k = b_k$.

Theorem 2. If the matrix Λ is nondegenerate, then there will be only one solution of the system of equations (18) and the approximate solution $y_m(x)$ is unequivocally found.

Remark. Often, collocation nodes take uniformly spaced $[a,b]$ system of points $x_i = a + i\dfrac{b-a}{n}$.

Collocation-Iterative Method of Solving Linear Integro-Functional Equation. Let's apply the collocation-iterative method to equation (1). We determined the approximate solution $y_k(x)$ according to the formulas

$$y_k(x) - p(x) y_k(h(x)) = \\ = f(x) + \int_a^b H_1(x,t) z_k(t) dt + \int_a^b H_2(x,t) z_k(h(t)) dt, x \in [a,b], \tag{20}$$

$$y_k(x) = \psi(x), x \notin [a,b], k = 1,2,...$$

$$z_k(x) = y_{k-1}(x) + \omega_k(x), \tag{21}$$

$$\omega_k(x) = \sum_{j=1}^{n} a_j^k \eta_j(x),$$

$$\eta_j(x) = (S^{-1}\varphi_j)(x).$$

the unknown parameters $a_j^k = a_j^k(n)$ will find from the condition $\gamma_k(x_i) = 0$ where $x_i \in [a,b]$, $i = \overline{1,n}$ – nodes of collocation and

$$\gamma_k(x) = f(x) + \int_a^b H_1(x,t) z_k(t) dt + \int_a^b H_2(x,t) z_k(h(t)) dt - \\ - y_k(x) + p(x) y_k(h(x)), \ x \in [a,b]. \tag{22}$$

$$\mathcal{E}_k(x) = f(x) + \int_a^b H_1(x,t) y_{k-1}(t) dt + \int_a^b H_2(x,t) y_{k-1}(h(t)) dt - \\ - y_{k-1}(x) + p(x) y_{k-1}(h(x)), x \in [a,b]$$

and substituting the function $z_k(x)$ defined by formula (20) into expression (22) to find the parameters a_j^k we obtain the system of linear algebraic equations.

$$\sum_{j=1}^{n} \beta_{ij} a_j^k = b_i^k, i = \overline{1,n}, \tag{23}$$

in which

$$\beta_{ij} = \varphi_j(x_i) - H_j(x_i), b_i^k = \mathcal{E}_k(x_i),$$

$$H_j(x) = \int_a^b H_1(x,t) \eta_j(t) dt + \int_a^b H_2(x,t) \eta_j(h(t)) dt, j = \overline{1,n},$$

$$\eta_j(x) = (S^{-1}\varphi_j)(x),$$

$$b_i^k = \mathcal{E}_k(x_i).$$

The system of equations (23) can be written in the form $\Lambda a_k = b_k$, where b_k, a_k are written in the vector form and Λ – the matrix mentioned above, composed of elements β_{ij}.

Note that in the role of approximation to the desired solution, you can take both the function $y_k\left(x\right)$ and the function $z_k\left(x\right)$. Attention should be paid to the fact that based on the analysis of formulas (20)-(22) at $\omega_k\left(x\right)=0, k=1,2,3,\ldots,$ approximation $y_k\left(x\right)$ is by the method of successive approximations.

Algorithms (15)-(17) and (20)-(22) for constructing approximate solutions of equation (1) are equivalent to the appropriate algorithms of collocation and collocation-iterative methods for Fredholm's integral equation of the second kind.

Conclusions. The article considers the problem of constructing approximate solutions of one type of linear integro-functional equation. It is shown that, under certain conditions, this equation can be reduced to Fredholm's integral equation of the second kind through some transformations. Collocational and collocational-iterative methods of finding approximate solutions were applied to the mentioned integro-functional equation.

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МЕТОДИ РОЗВ'ЯЗУВАННЯ ОДНОГО ТИПУ ЛІНІЙНОГО ІНТЕГРО-ФУНКЦІОНАЛЬНОГО РІВНЯННЯ

У статті розглядається один тип лінійного інтегро-функціонального рівняння. Наведено спосіб перетворення такого рів-

няння до інтегрального рівняння Фредгольма другого роду. Наближені розв′язки цього рівняння побудовані за допомогою колокаційного та колокаційно-ітеративного методів.

Ключові слова: *один тип лінійного інтегро-функціонального рівняння, рівняння Фредгольма другого роду, цілком неперервний оператор, обернений оператор, наближений розв′язок, колокаційний та колокаційно-ітеративний методи.*

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М. В. Гритчук, аспірант,
І. І. Клевчук, д-р фіз.-мат. наук

Чернівецький національний університет
імені Юрія Федьковича, м. Чернівці

ПОБУДОВА ОБЛАСТЕЙ СТІЙКОСТІ ЛІНІЙНИХ АВТОНОМНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ІЗ ЗАПІЗНЕННЯМ

Метою цієї статті є дослідження стійкості розв’язків лінійних автономних диференціальних рівнянь із запізненням аргументу. Дослідження стійкості можна звести до проблеми розміщення коренів характеристичного рівняння. Для лінійного диференціального рівняння із кількома запізненнями одержано необхідні і достатні умови, при яких всі корені характеристичного рівняння мають від’ємні дійсні частини (отже, нульовий розв’язок відповідного диференціального рівняння є асимптотично стійким). Для скалярного диференціального рівняння із запізненням одержано область стійкості на площині параметрів. Досліджено умови обмеженості і побудовано області стійкості лінійного автономного диференціального рівняння із кількома запізненнями. Для побудови області стійкості використано принцип аргументу, метод *D*-розбиттів і числові методи. У цій статті ми досліджуємо стійкість розв’язків лінійних автономних диференціальних рівнянь із кількома запізненнями. Одержано необхідні і достатні умови, при яких всі корені характеристичного рівняння мають від’ємні дійсні частини. Одержано обмеження на коефіцієнти рівняння за допомогою принципу аргументу і побудовано область стійкості лінійного автономного диференціального рівняння із двома запізненнями. Використано принцип аргументу, метод *D*-розбиттів і числові методи для побудови області стійкості лінійного автономного диференціального рівняння із двома запізненнями. В методі *D*-розбиттів ми шукаємо значення параметрів, для яких характеристичне рівняння має хоча б один нуль на

