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The efficiency of numerical realization of mathematical models by the method of supporting cuts in comparison to the widely used traditional method of complete differences is being probed in calculable experiments. The time indexes of the processing of the use of different methods of finding of decisions are being analyzed and the recommendations to the effective use of the method of supporting cuts are formulated.

Key words: *efficiency of numerical realization, time of processing, difference scheme, finite difference method.*

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AN APPROACH TO THE PRECISION PARAMETRIC REDUCTION OF MATHEMATICAL MODELS

The paper considers and analyzes an approach to simplification of mathematical models in order to substantiate possibility of accounting for limited computational resources needed to implement the models. Practical methods are proposed for application and evaluation of possible techniques of models' parametric reduction for considered problem with obtainment of applicability criterion of the given approach.

Key words: *parametric reduction, evaluation of approximations, criterion of applicability.*

Introduction. The more complex the objects and phenomena being studied are, the higher is computational complexity of the models used. Degree of complexity of mathematical descriptions of dynamics of the objects in question goes hand in hand with requirements for computational resources needed to run the models. Basic approach to assurance of accounting for limitation of these resources consists in use of simpler mathematical models instead of complex ones, yet preserving informative value of the results of modeling, i.e., in certain sense preserving adequacy to the object being modeled. Just the meaning of conformity of the simplified model content with the primary one characterizes great variety of approaches to simplification of complex models.

There is a long history of the question of simplification or reduction of mathematical descriptions of objects or phenomena being studied, and it is closely related to the identification problem [1–3], so that various techniques used for identification have been successfully applied to development of simplified descriptions of dynamic objects starting with given complex one. The most important of these techniques are: reduction of the equation system order, separation of motion into «fast» and «slow», replacement of operators, reducing the number of independent variables, linearization of original non-linear equations, neglecting the influence of certain factors.

Setting up the problem of models' precision parametric reduction of. Most commonly the problem of models' simplification is posed as a problem of minimizing a certain measure of deviation of the primary and simplified models' output for given simplification pattern. At the same time and as a rule, it is assumed that parameters of the primary model are known with certainty. It is clear, that in practice such condition does not exist, therefore the issue of mathematical model simplification can be posed as a question of matching model type with precise characteristics of the input data. Intuitively it is obvious that it makes no sense to use complex structure models if their parameters are known with comparatively great inaccuracy. This circumstance preconditions the need for adjustment of such characteristics of model as complexity and accuracy of the input data. Despite the fact, that this statement of the problem of model simplification has long been known while it is derived from approximate calculations technique, definition of practical approaches to its solution is of great importance for study of complex objects and, in particular, for development of algorithms and software for various control and computation systems [4, 5].

It is common to consider modeling accuracy as a basic quality indicator of computer modeling systems. Another indicator, especially for real-time systems, is processing speed required. It is clear, that simplification of models of objects being studied makes it possible to reduce required processing speed, often at the cost of degraded simulation fidelity. Resolution of the contradiction between tendencies of increasing modeling fidelity and decrease in required processing speed can be achieved just on the base of matching mathematical description of given object with accuracy characteristics of input data.

Sufficiently generalized and practical posing of the problem of mathematical model simplification based on the concept of matching model type with accuracy of input data was proposed by A. N. Tikhonov [6, 3], who defined it as a problem of minimizing complexity functional of the class of models, formally comparable in accuracy with observations (input data). The class of formally comparable models is defined by constraints on inequation type for precision measure.

Within this general statement of the problem of mathematical model simplification, series of specific statements can be defined, one of which, namely, precision parametric reduction of models, and approaches to its solution are discussed herein.

Suppose there is some mathematical description (model) M of the simulated phenomena, and the input data (usually numerical parameter values) are given with certain inaccuracy, a measure of which is ε_ρ. Obviously, if $\varepsilon_\rho \to \infty$, then original mathematical description M may be replaced with another description M' (even with simpler one) provided $\varepsilon_\rho \to \infty$, where ε_{yp} is a measure of error of output variables of the primary model, caused by inaccuracy of the input data, i.e. the presence of ε_ρ. Let's assume that $\varepsilon_\rho = 0$, then replacing the original description with another (including simpler) one will originate error $\varepsilon_{ym} \neq 0$. In this case, the question of applicability of the simplified model should be resolved on the basis of analysis of requirements for accuracy of simulation of the phenomenon being studied. In other words, use of simplified model is acceptable if $\varepsilon_{yM} \le \varepsilon_з$, where $\varepsilon_з$ is a given value of modeling error measure, and vice versa. If inequation $\varepsilon_{yM} < \varepsilon_з$ is strong enough, it suggests that primary model can be simplified even more.

In the general case, applicability of mathematical model M or M' is determined by condition

$$\varepsilon_{yM} + \varepsilon_{yp} \le \varepsilon_з . \qquad (1)$$

Here ε_{yM} is either a measure of error for description of the phenomenon being studied with model M, or a measure of error conditioned by replacement of given model M with another one, i.e. M' (including a simpler one).

Since this paper addresses the issue of mathematical model simplification, hereinafter ε_{yM} will stand just for measure of error due to replacement of the initially given model with a simpler one.

As a characteristic property of consistency with error in the input data of the mathematical model M', that replaces the model M, we shall accept

$$\alpha = \varepsilon_{yM} / \varepsilon_{yp} \qquad (2)$$

which we will call concordance coefficient. Herewith applicability condition (1) of the model M' takes the form

$$(\alpha+1)\varepsilon_{yp} \le \varepsilon_3 . \qquad (3)$$

Definition 1. Mathematical model M' is called α - concordant with error of the input data ε_p of the model M , if $\varepsilon_{yM} = \alpha\varepsilon_{yp}$ and $\alpha > 0$.

The problem of concordance of the model M with accuracy of the input data means obtaining model M' with a certain concordance coefficient. Whereby α =0.01 can be accepted as a lower limit, since further reduction of concordance coefficient, meaning the use of models, which differ little from the primary one, would not lead to significant gain in the accuracy of modeling. On the other hand, if $\alpha > 10$, the overall modeling error will be almost entirely determined by ε_{yM} , in which case it is practically impossible to obtain reasonable values of ε_{yM} while modeling complex objects. Thus, it is practical to constrain maximum concordance coefficient of mathematical models with the interval (0.01 $\div$ 10).

While addressing specific problems, maximum value of concordance coefficient is determined by the applicability condition (3) as follows

$$\alpha_{\max} = \varepsilon_3 / \varepsilon_{yp} - 1. \qquad (4)$$

Now let N_M be computational complexity measure of mathematical model M (for instance, for dynamic objects described by ordinary differential equations in normal form, number of operations reduced to the same type and required to compute the right side can be accepted as N_M), and μ_α be a set of mathematical models, concordance coefficient of which with input data error does not exceed α .

If $\alpha < \alpha_{\max}$, then all the models from μ_α are equivalent in terms of applicability, while applicability condition for any model is fulfilled (3). At the same time, models from μ_α may not be equivalent in terms of computational complexity. Just under this condition it is possible to simplify given model M by replacing it with model $M' \in \mu_\alpha$, for which $N_{M'} < N_M$.

Definition 2. Mathematical model M'_0 is called optimally simplified, if

$$N_{M'_0} = \min_{M' \in \mu_\alpha} N_{M'} \qquad (5)$$

Obviously, the measure of computational complexity $N_{M'}$ is a decreasing function of concordance coefficient. Consequently, concordance coefficient of optimally simplified model M'_0 exceeds α .

Development of α -concordant, as well as optimally simplified models pursues dual aim. On the one hand, it is a substantiation of certain assumptions allowing us to simplify the model, on the other hand, it is reduction of requirements to processing speed of computer modeling systems. Yet the real-world solution of the problem in hand faces considerable difficulties, conditioned mainly by complexity of the problem of estimation of values ε_{yM}, ε_{yp}, as well as by difficulties in formal characterization of the set μ_{α}.

Parametric approach to reduction of models. The realizable way to overcome the difficulties is to use parametric approach to development of simplified models. Its essence consists in replacement of certain parameter values of the original model M with such other values, which reduce the measure of computational complexity. Usually it is replacement of given values of some parameters of the initial model with zero values (both for additive and multiplicative parameters) and/or with single values (for multiplicative parameters). Such parametric simplification is helpful if mathematical models of the processes or phenomena being studied contain sufficiently great number of parameters, values of which are determined from experimental data. These models may include models in the form of linear or nonlinear algebraic equation systems of higher dimension, differential equation systems (both ordinary and partial) with coefficients which are composite functions of input and output variables, integral equations with kernels, form of which is determined by actual characteristics of the objects being studied etc.

Theoretical substantiation of possibility of parametric simplification of mathematical models for development of α-concordant and optimized simplified models is based on the results of the precision theory in relation to inherent error analysis.

Let $M(y,V,\rho)$ be mathematical model; y,V,ρ — vectors of output, input variables and parameters, respectively, that belong to normed spaces with metrics ρ_y,ρ_V,ρ_p, that are determined by relevant norms; N_p is computational complexity measure [7].

Let ρ_0 be an unknown vector of precise parameter values of the model and ρ_3 be a vector of given parameter values. It is known that

$$\rho_p(\rho_3,\rho_0)\le\varepsilon_p. \quad (6)$$

Condition (6) determines set Ω_p, and from $\rho\in\Omega_p$ follows $\rho_p(\rho_3,\rho)\le\varepsilon_p$. Thus we defined the set μ of possible models

$$\mu=\left\{M(y,V,\rho):\rho\in\Omega\rho,\rho_p(\rho_3,\rho)\le\varepsilon_p\right\},$$

and model $M(y,V,\rho)$ appears to be primary mathematical description.

Suppose further that $\rho_y(\rho_3,\rho)$ is measure of error in the space of output variables, conditioned by inaccuracy of assignment of the vector ρ_3. $\rho_y(\rho_3,\rho)$ is measure of error, conditioned by replacement of given model $M(y,V,\rho_3)$ with another one, particularly with a simpler model $M(y,V,\rho_r)$; $\rho_y(\rho_3,\rho)$ — measure of error that characterizes distinction between the exact model $M(y,V,\rho_0)$ and the model $M(y,V,\rho_r)$.

Following estimates can be determined for the above errors

$$\rho_y(\rho_3,\rho) / \rho \in \Omega\rho, V \in \Omega\rho \le \varepsilon_{yp}, \tag{7}$$

$$\rho_y(\rho_3,\rho_r) / V \in \Omega v \le \varepsilon_{yM}, \tag{8}$$

$$\rho_y(\rho_0,\rho_r) / V \in \Omega v \le \varepsilon_{yr}. \tag{9}$$

(Ωv is the range of possible values of input variables), and the triangle inequality defines their relationship

$$\varepsilon_{yr} \le \varepsilon_{yM} + \varepsilon_{yp} = (\alpha+1)\varepsilon_{yp}. \tag{10}$$

$\varepsilon_{yr} \le \varepsilon_{yM}$ If $\rho_r \in \Omega\rho$, then $\varepsilon_{yM} \le \varepsilon_{yr}$ and, consequently, the set μ of possible models is a set of α - concordant models for α =1, i.e. $\mu = \mu_1$. However, application of parametric approach to the search for simplified models in the set μ has limited possibilities. Indeed, usually the set $\Omega\rho$ does not contain points that belong to coordinate axes, and therefore with this model simplification one can only count on reduction in the number of significant digits in representation of given parameters. Note that such simplification finds use in the art of approximate calculations and can be used also in organization of computational processes in modeling systems on the step of computing tools selection.

Generally speaking, if $\rho_r \in \Omega\rho$, then $\varepsilon_{yM} \le \varepsilon_{yr}$, and therefore concordance coefficient of appropriate model will be greater than 1. Nevertheless, there is a possibility in principle to determine the set of α-concordant models for $\alpha \le 1$ and $\rho_r \overline{\in} \Omega\rho$. This possibility is conditioned by the fact that error in the output variables space, caused by inaccuracy of vector ρ_3 assignment, is estimated in the metric ρ_y.

Indeterminancy of values of the primary model parameter vector (presence of $\Omega\rho$) leads to indeterminancy of values of the output variables, quantitative measure of which is error

$$\Delta y = y(\rho,V) - y(\rho_3,V) / V \in \Omega_V. \tag{11}$$

The range $\Omega_{\Delta y}$ of possible values of error Δy has complex configuration in the space of output variables, that depends on the type of Ω_ρ, properties of mathematical description $M(y,V,\rho)$ and is defined by the inherent error equation. Detailed characteristic of $\Omega_{\Delta y}$ with techniques used to analyze inherent error that were developed in precision theory, can be obtained only for the simplest problems. More often, however, the range to which the error can be attributed, is characterized by a sphere S (in the given metric ρy), radius of which (in this particular case ε_{yp}) is determined considering the upper estimate (7) for the norm corresponding to metric ρy. Due to coarseness of these estimates, $S \supset \Omega\Delta y$ is the case. The sphere S in the space of output variables is matched with certain region $\Omega'\rho$ in the space of parameters, where $\Omega'\rho \supset \Omega\rho$ and $\rho \in \Omega'\rho$ follows $\rho y(\rho_3,\rho) \le \varepsilon_{yp}$. Consequently, if $\rho_r \in \Omega'\rho$, then $\varepsilon_{yM} \le \varepsilon_{yp}$ and set μ' of possible models that corresponds to the set $\Omega'\rho$ is a set of consistent models for $\alpha = 1$.

Due to inclusion of $\Omega'\rho$, application of parametric approach to simplification of the original description in the model class μ' has a great potential as compared to the class μ. It can be attributed to the fact that for complex problems with a great number of parameters defined with less-than-high accuracy, the range $\Omega'\rho$ contains points that belong to the coordinate axes of the parameter space. Consequently, vector with some zero components can be selected as a vector ρ_r, and thus decrease in value of computational complexity measure N_{pr} can be achieved.

In the general case, $\Omega'\rho$ may not meet with the coordinate axes, however even under these conditions, as it is not hard to show, there is a set $\Omega'\rho(\alpha) = \left\{\rho : \rho_y(\rho_3,\rho) \le \alpha\varepsilon_{yp}, \alpha > 1\right\}$ that includes points of the coordinate axes and defines a set of models μ'_α, consistent with the input data error with consistency coefficient $\alpha > 1$. Indeed, assume $\Omega'\rho(\beta) = \left\{\rho : \rho_p(\rho_3,\rho) \le \rho\varepsilon_p\right\}$. It is clear that $\Omega_p(1) = \Omega_p$, and there is such a value $\beta_0 = 1$, that for $\beta > \beta_0$ the sphere $\Omega_p(\beta)$ meets at least one of the coordinate axes of the parameter space. Set $\Omega_p(\beta)$ defines a range of possible values of the output variable errors $\Omega_{\Delta y}(\beta)$, which is estimated in the metric ρy by the sphere S_α with radius $\alpha(\beta)\varepsilon_{yp}$, i.e.

$$\rho_y(\rho_3,\rho) / \rho \in \Omega'\rho(\beta) \le \alpha(\beta)\varepsilon_{yp}. \quad (12)$$

Here, $\alpha(\beta)$ is non-negative non-decreasing function, and $\alpha(1)$ = 1, we assume that function $\alpha(\beta)$ is increasing and confined one. The first assumption means that error of output variables increases if error of the input data increases (for practical tasks we can consider this condition as satisfied). The second condition means correctness of all the models with parameters from $\Omega_p(\beta)$, which is a limitation for applicability of the parametric approach to model simplification.

Now let $\Omega'\rho(\alpha) = \left\{\rho : \rho_y(\rho_3, \rho) \le \alpha(\beta)\varepsilon_{yp}\right\}$ be prototype in the space of parameters of the sphere S_α. While $\Omega'(\alpha) \supset \Omega(\beta)$, the set $\Omega'\rho(\alpha)$ also intersects with at least one of the coordinate axes, as was to be demonstrated. Thus, the following theorem is proved.

Theorem 1. For any $\Omega\rho = \left\{\rho : \rho_p(\rho_3, \rho) \le \varepsilon_p, \varepsilon_p > 0\right\}$ there is such $\alpha_0 > 0$, that for $\alpha > \alpha_0$ in the set μ_α. of α -concordant with the error of input data well-posed models $M(y, V, \rho)$, defined by a set of parameters $\Omega\rho(\alpha) = \left\{\rho : \rho_y(\rho_3, \rho) \le \alpha\varepsilon_{yp}\right\}$, there is a non-empty subset $\overline{\mu}_\alpha$ of models $M(y, V, \overline{\rho})$ for which $N_{\overline{p}} < N_p \,/\, \rho \in \Omega\rho$.

This theorem is a theoretical substantiation for possibility of application of parametric approach to development of simplified mathematical models based on concordance of the model type with accuracy of the input data. The fact (proven by the theorem) of existence of α -concordant models with a lower measure of computational complexity than the primary model is the main point of the principle of precision parametric reduction of mathematical models that can be stated as follows.

For any model with inaccurately defined parameters, provided error of the output variables is an increasing and confined function of the parameter error, there is a simpler α -concordant model that differs from the primary model by values of some parameters that reduce computational complexity measure.

Of course, applicability of parametrically simplified models resulting from precision parametric reduction is defined by condition (3), hence obtaining of simpler mathematical descriptions than original ones with moderate value of concordance coefficient is of practical interest. That's why it is desirable to be able to evaluate concordance coefficient of parametrically simplified models a priori, with fairly simple means.

Evaluation of inherent error within the linear precision theory for a broad class of problems (mathematical descriptions of which represent models that can be simplified) leads to a linear dependence of ε_{yp} from ε_p, i.e.

$$\varepsilon_{yp} = C_M \varepsilon_p \,, \tag{13}$$

where C_M — constant determined by character of the problem (for well-posed problems it is norm of inverse operator of inherent error linear equations).

In this case, it is easy to get a dependable upper bound for value α_0 of concordance coefficient of the models, which can be obtained as a result of the precision parametric reduction.

Theorem 2. For the minimum value α_0 of concordance coefficient of the parametrically simplified model $M(y, V, \overline{\rho})$ the following estimate is valid:

$$\alpha_0 \le \frac{\min\limits_{i=\overline{1,m}} \rho_p(\rho_3, \wp i)}{\varepsilon\rho} = \alpha_b \,, \tag{14}$$

where $\wp i, i = \overline{1,m}$ — coordinate axes in the parameter space.

Proof. Condition (13) implies that function $\alpha(\beta)$ has the form of $\alpha(\beta) = \beta$. Value β_0 can be defined from condition that the sphere $\Omega\rho(\beta)$ with radius $\beta_0\varepsilon_p$ touches the nearest of the coordinate axes of the parameter space $\wp i, i = \overline{1,m}$, i.e.

$$\beta_0 \le \frac{\min\limits_{i=\overline{1,m}} \rho_p(\rho_3, \wp i)}{\varepsilon\rho}$$

From inclusion $\Omega'(\alpha(\beta_0)) \supset \Omega(\beta_0)$ follows $\alpha_0 \le \beta_0$. QED.

As a matter of fact, estimate (14) is criterion of applicability of the precision parametric reduction of mathematical models. Thus, higher values of α_b (on the order of tens) demonstrate impracticality of parametric model simplification and, on the contrary, small values of α_b (on the order of unities) evidence feasibility of application of the precision parametric reduction for development of acceptable simplified models.

Parametric model simplification under discussion covers also the case with linearization of complex functional relationships. Herewith the use of linear terms means «zeroing» of coefficients for all other terms of the expansion.

Furthermore, it is possible to use the principle of precision parametric reduction to assess insignificance of separate fragments of the model by introducing fictitious multiplicative parameters (nominally equal to unity) for the fragments under consideration with subsequent evaluation of insignificance of these parameters, i.e. with assessment of possibility of replacement of their nominal values with zero.

Conclusions. Based on the concept of mathematical model simplification, the principle and elements of the theory of the precision parametric reduction of mathematical descriptions of dynamic objects are exposed as one of the ways of accounting for computer means' speed requirements.

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Розглядається і аналізується підхід до спрощення математичних моделей з метою обґрунтування можливості обліку обмеженості ресурсів комп'ютерних засобів, що реалізують моделі. Запропоновано конструктивні способи використання і оцінки можливих прийомів параметричної редукції моделей в розглянутій задачі з отриманням критерію застосовності даного підходу.

Ключові слова: *параметрична редукція, оцінка наближень, критерій застосовності.*

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