UDC 517.9

A POSITIVE SOLUTION OF SINGULAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH NONLINEAR BOUNDARY CONDITION*

ДОДАТНИЙ РОЗВ'ЯЗОК СИНГУЛЯРНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ІМПУЛЬСНИМ ВПЛИВОМ ТА НЕЛІНІЙНОЮ ГРАНИЧНОЮ УМОВОЮ

Weibing Wang

Hunan Univ. Sci. and Technology
Xiangtan, Hunan 411201, China

Jianhua Shen

Hangzhou Normal Univ.
Hangzhou, Zhejiang 310036, China

By using priori estimates and the theory of topological degree, we study the existence of positive solution to a class of second order singular impulsive differential equations with nonlinear boundary conditions. Our result admits that the nonlinear term of differential equation can be singular in its second and third variable.

Із використанням апріорних оцінок та теорії топологічного ступеня вивчено існування розв'язку для класу сингулярних диференціальних рівнянь другого порядку з імпульсним впливом та нелінійними граничними умовами. Отриманий результат має місце також у випадку, коли нелінійний член диференціального рівняння є сингулярним відносно другої та третьої змінних.

1. Introduction. In this paper, we study the singular boundary-value problem with impulsive effect,

$$\begin{aligned} &u''(t) + f(t, u(t), u'(t)) = 0, \quad t \neq t_k, \quad t \in J, \\ &u(t_k^+) = I_k(u(t_k)), \quad 1 \leq k \leq p, \\ &u'(t_k^+) = M_k(u'(t_k)), \quad 1 \leq k \leq p, \\ &u(0) - g(u'(0)) = 0, \quad u'(1) = 0, \end{aligned} \tag{1.1}$$

where $J = [0,1]$, $f : J \times (0,\infty) \times (0,\infty) \to \mathbb{R}$, $I_k, M_k : \mathbb{R} \to \mathbb{R}$ for $1 \leq k \leq p$, $0 = t_0 < t_1 < < t_2 < \ldots < t_p < 1 = t_{p+1}$, $u'(t_k^+) = \lim_{h\to 0^+} h^{-1}[u(t_k + h) - u(t_k)]$, $u'(1) = \lim_{t\to 1^-} u'(t)$, $u'(0) = \lim_{t\to 0^+} u'(t)$. $f(t,u,v)$ may be singular at $u = 0$ or $v = 0$, which means that f tends to infinity when $u \to 0^+$ or $v \to 0^+$, i.e.,

$$\lim_{u\to 0^+} f(\cdot, u, \cdot) = +\infty \quad \text{or} \quad \lim_{v\to 0^+} f(\cdot, \cdot, v) = +\infty.$$

* The work is supported by the NNSF of China (11171085) and Hunan Provincial Natural Science Foundation of China.

Singular differential equations play a very important role in modern applied mathematics due to their deep physical background and broad application, and has been studied extensively, see [1–5]. Recently, boundary-value problems of nonlinear singular impulsive differential equations have received a considerable attention, for example, see [6–12] and the references therein. However to the best knowledge of the authors, there is no paper concerned with the existence of positive solutions to boundary-value problems of impulsive differential equation with nonlinear boundary conditions so far. In the present paper, we investigate the existence of positive solutions of (1.1) by using priori estimates and the theory of topological degree.

Let $J^* = J\backslash\{t_1, t_2, \ldots, t_p\}$, $PC^r(J) = \{u : J \to \mathbb{R} | u^{(j)}(t)$ be continuous at $t \neq t_k$, left continuous at $t = t_k$, and each $u^{(j)}(t_k^+)$ exist for $k = 1, 2, \ldots, p$, where $j = 0, 1, \ldots, r\}$. Note that $PC^0(J) = PC(J)$ and $PC^1(J)$ are Banach spaces with the norms $\|u\|_0 = \sup\{|u(t)| : t \in \in J\}$, $\|u\|_1 = \max\{\|u\|_0, \|u'\|_0\}$, respectively.

By a solution to (1.1) we mean a function $x \in PC^1(J) \cap C^2(J^*)$ that satisfies (1.1).

In the study of problem (1.1), we introduce the following assumptions:

(H_1) $M_k : \mathbb{R} \to \mathbb{R}$ are continuous and strictly increasing functions, $I_k : \mathbb{R} \to \mathbb{R}$ are continuous, nondecreasing functions, $M_k(0) = I_k(0) = 0$, $1 \leq k \leq p$.

(H_2) $g : \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function, $xg(x) > 0$ for $x \neq 0$.

(H_3) $\limsup_{x\to+\infty} \dfrac{I_k(x)}{x} < \infty$, $1 \leq k \leq p$, $\lim_{x\to+\infty} \dfrac{M_1(x)}{x} > 0$, $\lim_{x\to+\infty} \dfrac{M_k(x)}{x} > 1$, $2 \leq k \leq p$.

(H_4) $f : J \times (0, \infty) \times (0, \infty) \to [0, \infty)$ is continuous. For $(t, u, v) \in J \times (0, \infty) \times (0, \infty)$,

$$f(t, u, v) \leq h(t, u + v) + p(t)w_1(u) + q(t)w_2(v),$$

where the functions p, q are continuous and nonnegative on J, $w_1, w_2 : (0, \infty) \to (0, \infty)$ are continuous and nonincreasing on J, $h : J \times [0, \infty) \to [0, \infty)$ is continuous and nondecreasing in its second argument,

$$\lim_{z\to\infty} \frac{1}{z} \int_0^1 h(t, z)dt = 0.$$

(H_5) There exists $\psi \in C(J, [0, \infty)) : \psi \not\equiv 0$ on J such that

$$f(t, u, v) \geq \psi(t), \quad (t, u, v) \in J \times (0, \infty) \times (0, \infty),$$

$$\int_0^1 (w_1(G(s)) + w_2(H(s)))ds < \infty,$$

where

$$H(t) = \min\left\{ \int_t^1 \psi(s)ds,\ \alpha \right\}, \quad G(t) = \min\left\{ \int_0^t H(s)ds,\ \beta \right\}, \quad t \in J,$$

$$\alpha = \min\left\{ M_p^- \left(\int_{t_p}^1 \psi(s)ds \right), M_{p-1}^- \left(M_p^- \left(\int_{t_p}^1 \psi(s)ds \right) \right), \ldots \right.$$

$$\ldots, M_1^- \left(M_2^- \left(\ldots \left(M_p^- \left(\int\limits_{t_p}^{1} \psi(s)ds \right) \right) \cdots \right) \right) \Bigg\},$$

$$\beta = \min \left\{ I_1 \left(\int\limits_0^{t_1} H(s)ds \right), I_2 \left(I_1 \left(\int\limits_0^{t_1} H(s)ds \right) \right), \ldots \right.$$

$$\left. \ldots, I_p \left(I_{p-1} \left(\ldots \left(I_1 \left(\int\limits_0^{t_1} H(s)\, ds \right) \right) \ldots \right) \right) \right\},$$

where M_k^- is the inverse of M_k.

2. Preliminaries. We establish some results that we shall need in the paper.

Lemma 2.1. *Assume that* (H_1), (H_2) *hold and* $u \in PC^1(J) \cap C^2(J^*)$ *satisfies*

$$\begin{aligned} & -u''(t) \ge 0, \quad t \ne t_k, \quad t \in J, \\ & u(t_k^+) = I_k(u(t_k)), \quad 1 \le k \le p, \\ & u'(t_k^+) = M_k(u'(t_k)), \quad 1 \le k \le p, \\ & u(0) - g(u'(0)) = u'(1) = a > 0. \end{aligned} \tag{2.1}$$

Then $u(t) \ge g(a) + a, u'(t) \ge a$ *for* $t \in J$.

Proof. It is easy to check that $u'(t)$ is nonincreasing in (t_k, t_{k+1}) since $u''(t) \le 0,\ t \in$ $\in (t_k, t_{k+1})$. Suppose that $u(0) \le 0$, then $g(u'(0)) \le u(0) - a < 0$, which implies that $u'(0) < 0$. So $u'(t) < 0$ for $t \in [0, t_1]$ and $u'(t_1^+) = M_1(u'(t_1)) < 0$. In such a way we can show that $u'(t) < 0$ for $t \in J$, a contradiction. Hence, $u(0) > 0$. Similarly, $u'(0) > 0$.

Assume that there exists $s \in (0, 1]$ such that $u(s) < 0$. Let $\tau = \inf\{t \in (0, 1], u(t) < 0\}$. By $u(0) > 0$ and (H_1), we obtain that $u'(\tau) \le 0$.

If $\tau = t_{i_0}$ for some $1 \le i_0 \le p$, then $u(\tau) = 0$, $u'(\tau^+) = M_{i_0}(u'(\tau)) \le 0$. Using the fact that $u'(t)$ is nonincreasing in (t_k, t_{k+1}) and (H_1), one can obtain that $u'(t) \le 0$ for $t \in (\tau, 1]$, a contradiction.

If $\tau \ne t_k$, $1 \le k \le p$, then $u'(\tau) < 0$ and $u'(t) \le 0$ for $t \in (\tau, 1]$, a contradiction.

Hence, $u(t) > 0$ on J. Next, we show that $u'(t) > 0$. If not, there exists $i : 1 \le i \le p$ and sufficiently small $\varepsilon > 0$ such that $u(t)$ is nonincreasing on $(t_i - \varepsilon, t_i)$. Therefore, $u'(t_i) \le 0$, $u'(t_i^+) = M_i(u'(t_i)) \le 0$. One can obtain that $u'(t) \le 0$ for $t \in (t_i, 1]$, a contradiction. The conclusion is true by monotonicity of u.

Lemma 2.1 is proved.

Lemma 2.2. *Assume that* (H_1), (H_2) *hold and* $\psi \in C(J, [0, \infty))$ *with* $\psi \not\equiv 0$ *on* J. $u \in$

$\in PC^1(J) \cap C^2(J^*)$ *satisfies*

$$-u''(t) \geq \psi(t), \quad t \neq t_k, \quad t \in J,$$
$$u(t_k^+) = I_k(u(t_k)), \quad 1 \leq k \leq p,$$
$$u'(t_k^+) = M_k(u'(t_k)), \quad 1 \leq k \leq p,$$
$$u(0) - g(u'(0)) = u'(1) = a > 0.$$

Then $u'(t) \geq H(t), u(t) \geq G(t)$ *for* $t \in J$.

Proof. Put

$$\bar{h}(t) = \begin{cases} h_p(t), & t_p < t \leq 1, \\ h_k(t), & t_k < t \leq t_{k+1}, \quad 1 \leq k \leq p-1, \\ h_0(t), & t_0 \leq t \leq t_1, \end{cases}$$

$$\bar{g}(t) = \begin{cases} g_0(t), & t_0 \leq t \leq t_1, \\ g_k(t), & t_k < t \leq t_{k+1}, \quad 1 \leq k \leq p-1, \\ g_p(t), & t_p < t \leq 1, \end{cases}$$

where

$$h_p(t) = \int_t^1 \psi(s)ds, \quad h_k(t) = \int_t^{t_{k+1}} \psi(s)ds + M_{k+1}^-(h_{k+1}(t_{k+1})), \quad 0 \leq k \leq p-1,$$

$$g_0(t) = \int_0^t H(s)ds, \quad g_k(t) = \int_{t_k}^t H(s)ds + I_k(g_{k-1}(t_{k-1})), \quad 1 \leq k \leq p.$$

Since $u(t) > 0$, $u'(t) > 0$ on J, $I_k(u(t_k)) > 0$, $M_k(u'(t_k)) > 0$. When $t \in (t_p, 1]$,

$$u'(t) - u'(1) \geq \int_t^1 \psi(s)ds = h_p(t).$$

So $u'(t_p^+) \geq \int_{t_p^+}^1 \psi(s)ds + u'(1) \geq h_p(t_p)$. By $u'(t_p^+) = M_p(u'(t_p))$, we obtain that $u'(t_p) \geq$ $\geq M_p^-(h_p(t_p))$. For $t \in (t_{p-1}, t_p]$,

$$u'(t) \geq \int_t^{t_p} \psi(s)ds + u'(t_p) \geq \int_t^{t_p} \psi(s)ds + M_p^-(h_p(t_p)).$$

By induction, we obtain that $u'(t) \geq \bar{h}(t), t \in J$. It is easy to check that $\bar{h}(t) \geq H(t), t \in J$. So $u'(t) \geq H(t), t \in J$. Similarly, $u(t) \geq G(t)$ for $t \in J$.

Lemma 2.2 is proved.

Lemma 2.3. *Assume that* (H_1) – (H_4) *hold, then there exist constants* $A, B > 0$ *such that for any* $u \in PC^1(J) \cap C^2(J^*)$ *satisfying*

$$\begin{aligned} &0 \le -u''(t) \le h(t, u + u') + p(t)w_1(u) + q(t)w_2(u'(t)), \quad t \ne t_k, \\ &u(t_k^+) = I_k(u(t_k)), \quad 1 \le k \le p, \\ &u'(t_k^+) = M_k(u'(t_k)), \quad 1 \le k \le p, \\ &u(0) - g(u'(0)) = u'(1) = a \in (0, 1], \end{aligned} \tag{2.2}$$

the estimates $u(t) \le A$, $u'(t) \le B$, $t \in J$ *hold true, where* h, p, q, w_1, w_2 *are defined as in* (H_4).

Proof. Note that $u(t) > 0$, $u'(t) > 0$ on J by Lemma 2.1. Set $u'(t_i) = \rho_i$, $0 \le i \le p+1$. Since $u'(t)$ is nonincreasing on (t_i, t_{i+1}),

$$\sup\{u'(t) : t \in (t_i, t_{i+1})\} = u'(t_i) = \rho_i, \quad 0 \le i \le p. \tag{2.3}$$

Put $r = \max\{t_{i+1} - t_i : 0 \le i \le p\}$, $M_0(\rho_0) = \rho_0$, $Q = \sup\{q(t) : t \in J\}$, $P = \sup\{p(t) : t \in \in J\}$, then

$$\rho_0 > \rho_1, \quad M_i(\rho_i) > \rho_{i+1}, \quad 1 \le i \le p, \tag{2.4}$$

$$\sup\{u(t) : t \in [0, t_1]\} = u(t_1) = r\rho_0 + 1 = r_1 > 0, \tag{2.5}$$

$$\sup\{u(t) : t \in (t_i, t_{i+1}]\} = u(t_{i+1}) = I_i(r_i) + r\rho_i =: r_{i+1}, \quad 1 \le i \le p. \tag{2.6}$$

Integrating the inequality $-u''(t) \le h(t, u + u') + p(t)w_1(u) + q(t)w_2(u'(t))$ yields

$$\begin{aligned} \rho_0 + \sum_{i=1}^{p} M_i(\rho_i) - \sum_{i=1}^{p} \rho_i &\le \int_0^1 \left[h(t, u(t) + u'(t)) + p(t)w_1(u(t)) + q(t)w_2(u'(t))\right] dt \le \\ &\le \sum_{i=0}^{p} \int_{t_i}^{t_{i+1}} h(t, r_{i+1} + M_i(\rho_i))dt + P\sum_{i=1}^{p} \int_{t_i}^{t_{i+1}} w_1(G(t))dt + \\ &+ Q\sum_{i=1}^{p} \int_{t_i}^{t_{i+1}} w_2(H(t))dt. \end{aligned}$$

By the condition (H_4), we obtain that

$$L := P\sum_{i=1}^{p} \int_{t_i}^{t_{i+1}} w_1(G(t))dt + Q\sum_{i=1}^{p} \int_{t_i}^{t_{i+1}} w_2(H(t))dt < \infty,$$

$$1 \le \frac{\sum_{i=1}^{p} \rho_i}{\sum_{i=0}^{p} M_i(\rho_i)} + \frac{1}{\sum_{i=0}^{p} M_i(\rho_i)} \left(\sum_{i=0}^{p} \int\limits_{t_i}^{t_{i+1}} h(t, r_{i+1} + M_i(\rho_i))dt + L \right). \tag{2.7}$$

Suppose $\{u_m\}_{m\in N}$ satisfy (2.2). Put $u'_m(t_i) = \rho_{i,m}, 0 \le i \le p{+}1$. Assume that $k \in \{0, 1, \ldots, p+ +1\}$ such that

$$\lim_{m\to\infty} \rho_{k,m} = \infty. \tag{2.8}$$

We derive a contradiction in the following way. Let k be the largest number satisfying (2.8), i.e., if $i > k$, $\rho_{i,m}$ are bounded and

$$\lim_{m\to\infty} \rho_{i,m} = \infty, \quad 1 \le i \le k. \tag{2.9}$$

In fact, (2.9) is obvious if $k = 1$. If $k \ge 2$, from $M_i(\rho_i) > \rho_{i+1}$, we obtain that

$$\lim_{m\to\infty} M_{i-1}(\rho_{i-1,m}) = +\infty,$$

which implies by (H_3) that $\lim_{m\to\infty} \rho_{i-1,m} = \infty$. By induction, we know that (2.9) holds.

Put

$$\begin{aligned} r_{1,m} &= r\rho_0 + 1, \\ r_{i+1,m} &= I_i(r_{i,m}) + rM_i(\rho_{i,m}), \quad 1 \le i \le p, \end{aligned} \tag{2.10}$$

and $M_0(\rho_{0,m}) = \rho_{0,m}$, then

$$1 \le \frac{\sum_{i=1}^{p} \rho_{i,m}}{\sum_{i=0}^{p} M_i(\rho_{i,m})} + \frac{1}{\sum_{i=0}^{p} M_i(\rho_{i,m})} \left(\sum_{i=0}^{p} \int\limits_{t_i}^{t_{i+1}} h(t, r_{i+1,m} + M_i(\rho_{i,m}))dt + L \right). \tag{2.11}$$

Consider the first term in the right-hand side of (2.7). By (H_3) there exist $\delta > 0$ and $m_0 \in N$ such that if $m \ge m_0$,

$$\begin{aligned} &M_1(\rho_{1,m}) \ge \rho_{1,m}\delta, \\ &M_i(\rho_{i,m}) \ge (1+\delta)\rho_{i,m}, \quad \text{if} \quad k \ge 2, \quad 2 \le i \le k. \end{aligned} \tag{2.12}$$

Hence,

$$S_m = \frac{\sum_{i=1}^{p} \rho_{i,m}}{\sum_{i=0}^{p} M_i(\rho_{i,m})} < \frac{\sum_{i=1}^{k} \rho_{i,m}}{(1+\delta)\sum\limits_{i=1}^{k} \rho_{i,m}} + \frac{\sum_{i=k+1}^{p} \rho_{i,m}}{\rho_{0,m}},$$

$$\lim_{m\to\infty} S_m \le \frac{1}{1+\delta} < 1.$$

Next, we consider the second term in the right-hand side of (2.7). Set

$$z_{i,m} = r_{i+1,m} + M_i(\rho_{i,m}), \quad 0 \le i \le p. \tag{2.13}$$

From (2.8), (2.9) and (2.12), we have

$$\lim_{m\to\infty} r_{i,m} = \infty, \quad 1 \le i \le k+1,$$
$$\lim_{m\to\infty} z_{i,m} = \infty, \quad 1 \le i \le k. \tag{2.14}$$

By (H_4), we have

$$\lim_{m\to\infty} \frac{1}{z_{i,m}} \int\limits_{t_i}^{t_{i+1}} h(t, z_{i,m})\, dt = 0. \tag{2.15}$$

When $1 \le i \le p$,

$$\frac{z_{i,m}}{\sum_{i=0}^{p} M_i(\rho_{i,m})} < \frac{I_i(r_{i,m})}{\sum_{i=0}^{p} M_i(\rho_{i,m})} + 1 + r < \frac{I_i(r_{i,m})}{r_{i,m}} \frac{r_{i,m}}{\sum_{i=0}^{p} M_i(\rho_{i,m})} + 1 + r <$$
$$< \frac{I_i(r_{i,m})}{r_{i,m}} \left(\frac{I_{i-1}(r_{i-1,m})}{r_{i-1,m}} \frac{r_{i-1,m}}{\sum_{i=0}^{p} M_i(\rho_{i,m})} + r \right) + 1 + r.$$

For any $1 \le i \le p$,

$$\frac{z_{i,m}}{\sum_{i=0}^{p} M_i(\rho_{i,m})} < \frac{I_i(r_{i,m})}{r_{i,m}} \left(\frac{I_{i-1}(r_{i-1,m})}{r_{i-1,m}} \left(\frac{I_{i-2}(r_{i-2,m})}{r_{i-2,m}} \times \right.\right.$$
$$\left.\left. \times \left(\ldots \frac{I_1(r\rho_{0,m}+1)}{r\rho_{0,m}+1} \frac{r\rho_{0,m}+1}{\rho_{0,m}} + r \ldots \right) + r \right) + r \right) + 1 + r.$$

Hence,

$$\frac{z_{i,m}}{\sum_{i=0}^{p} M_i(\rho_{i,m})} < A_i(A_{i-1}(A_{i-2}(\ldots(A_1 r + r)\ldots) + r) + r) + 1 + r,$$

where $A_i = \lim_{x\to\infty} I_i(x)/x < \infty$, $1 \le i \le p$. Combining (2.15) we obtain that

$$\lim_{m\to\infty} \frac{1}{\sum_{i=0}^{p} M_i(\rho_{i,m})} \int\limits_{t_i}^{t_{i+1}} h(t, r_{i+1,m} + M_i(\rho_{i,m}))dt = 0.$$

Hence,

$$\lim_{m\to\infty} \frac{1}{\sum_{i=0}^{p} M_i(\rho_{i,m})} \left(\sum_{i=0}^{p} \int\limits_{t_i}^{t_{i+1}} h(t, r_{i+1,m} + M_i(\rho_{i,m}))dt + L \right) = 0,$$

$$1 \le \lim_{m\to\infty} \frac{\sum_{i=1}^{p} \rho_{i,m}}{\sum_{i=0}^{p} M_i(\rho_{i,m})} + \lim_{m\to\infty} \frac{\sum_{i=0}^{p} \int_{t_i}^{t_{i+1}} h(t, r_{i+1,m} + M_i(\rho_{i,m}))dt + L}{\sum_{i=0}^{p} M_i(\rho_{i,m})} \le \frac{1}{1+\delta} < 1,$$

a contradiction. Hence, there exists $B > 0$ such that $u'(t) \le B \ \forall t \in [0,1]$. Next, we show that there exists $A > 0$ such that $u(t) \le A \ \forall t \in [0,1]$. Clearly, $u(0) \le 1 + g(B)$. Using the mean value theorem, we obtain that $u(t_1) - u(0) \le Bt_1$. Hence,

$$u(t_1) \le Bt_1 + 1 + g(B) := \lambda_1.$$

When $t \in (t_1, t_2]$, we have

$$u(t) - u(t_1^+) \le B(t - t_1),$$

$$u(t_2) \le J_1(u(t_1)) + B(t_2 - t_1) \le J_1(\lambda_1) + B(t_2 - t_1) := \lambda_2.$$

By induction, there is a constant $A > 0$ such that $u(t) \le A \ t \in [0,1]$.

Lemma 2.3 is proved.

Lemma 2.4. *Assume that* $v \in PC(J)$, $d_k, e_k \in \mathbb{R}$, *then*

$$\begin{aligned} & -u''(t) = v(t), \quad t \ne t_k, \quad t \in J, \\ & u(t_k^+) = u(t_k) + d_k, \quad 1 \le k \le p, \\ & u'(t_k^+) = u'(t_k) + e_k, \quad 1 \le k \le p, \\ & u(0) - g(u'(0)) = u'(1) = a \end{aligned} \tag{2.16}$$

has a unique solution,

$$\widetilde{u}(t) = a + g\left(a + \int_0^1 v(s)ds - \sum_{k=1}^{p} e_k\right) + at + \int_0^1 G(t,s)v(s)ds + \sum_{0<t_k<t} [d_k + (t - t_k)e_k] - t\sum_{k=1}^{p} e_k,$$

where

$$G(t,s) = \begin{cases} t, & 0 \le t \le s \le 1, \\ s, & 0 \le s \le t \le 1. \end{cases}$$

Proof. Assume that u_1, u_2 are solutions of (2.16) and $U = u_1 - u_2$. It ia easy to check that $U \in C^2(J)$ and $U'' \equiv 0$ on J. Hence, $U(t) = bt + c$, where b, c are constants. From $U'(1) = u_1'(1) - u_2'(1) = 0$, we obtain that $b = 0$. Thus $u_1'(0) = u_2'(0) = 0$ and $c = U(0) = u_1(0) - u_2(0) = g(u_1'(0)) - g(u_2'(0)) = 0$. That is, $u_1 = u_2$ on J.

Next, we show that $\widetilde{u}$ is a solution of (2.16). From the definition of $\widetilde{u}$, we have

$$\widetilde{u}'(t) = a + \int_t^1 v(s)ds + \sum_{0<t_k<t} e_k - \sum_{k=1}^{p} e_k, \quad \widetilde{u}''(t) = -v(t), \quad t \ne t_k,$$

$$\widetilde{u}(t_k^+) - \widetilde{u}(t_k) = \sum_{i\le k} d_i - \sum_{i<k} d_i = d_k,$$

$$\widetilde{u}'(t_k^+) - \widetilde{u}'(t_k) = \sum_{i\le k} e_i - \sum_{i<k} e_i = e_k,$$

$$\widetilde{u}'(0) = a + \int_0^1 v(s)ds - \sum_{k=1}^p e_k, \quad \widetilde{u}'(1) = a + \int_1^1 v(s)ds + \sum_{0<t_k<1} e_k - \sum_{k=1}^p e_k = a,$$

$$\widetilde{u}(0) = a + g\left(a + \int_0^1 v(s)ds - \sum_{k=1}^p e_k\right) + \int_0^1 G(0,s)v(s)ds = a + g(\widetilde{u}'(0)).$$

That is, $\widetilde{u}$ is a unique solution of (2.16).

Lemma 2.4 is proved.

Lemma 2.5. *Assume that* $(H_1)-(H_3)$ *hold,* $v \in C(J,[0,\infty))$, $\tilde{f}(t,x,y) \in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ *and* $0 \le \tilde{f}(t,x,y) \le v(t)$ *for any* $(t,x,y) \in J\times\mathbb{R}\times\mathbb{R}$. *Then the equation*

$$\begin{aligned} &-u''(t) = \tilde{f}(t,u(t),u'(t)), \quad t \ne t_k, \quad t \in J, \\ &u(t_k^+) = I_k(u(t_k)), \quad 1 \le k \le p, \\ &u'(t_k^+) = M_k(u'(t_k)), \quad 1 \le k \le p, \\ &u(0) - g(u'(0)) = u'(1) = a \in (0,1] \end{aligned} \tag{2.17}$$

has a positive solution $u \in PC^1(J)\cap C^2(J^*)$.

Proof. Define an operator $T_\lambda : PC^1(J) \to PC^1(J)$ by

$$(T_\lambda u)(t) = a + g\left(a + \lambda\int_0^1 \tilde{f}(s,u(s),u'(s))ds - \sum_{k=1}^p (M_k(u'(t_k)) - u'(t_k))\right) + at +$$

$$+ \lambda\int_0^1 G(t,s)\tilde{f}ds + \sum_{0<t_k<t}[I_k(u(t_k)) - u(t_k)) + (t-t_k)(M_k(u'(t_k)) - u'(t_k))] -$$

$$- t\sum_{k=1}^p (M_k(u'(t_k)) - u'(t_k)), \quad \lambda \in [0,1].$$

Using Lemma 2.4, one can check that $u \in PC^1(J)\cap C^2(J^*)$ is a solution of (2.16) if and only if $u \in PC^1(J)$ is fixed point of T_1. Moreover, T_0 has a unique fixed point $u_0 \in PC^1(J)$.

It is not difficult to prove that $T_\lambda(\lambda \in [0,1])$ is completely continuous. For any $\lambda \in [0,1]$, we show that there exists a constant $M > 0$ independent of λ such that u_λ, with $T_\lambda u_\lambda = u_\lambda$, satisfies $\|u_\lambda\|_{PC^1} < M$. If $\lambda \in (0,1]$,

$$0 \le -u_\lambda''(t) = \lambda\tilde{f}(t,u_\lambda(t),u_\lambda'(t)) \le v(t), \quad t \ne t_k, \quad t \in J,$$

$$u_\lambda(t_k^+) = I_k(u_\lambda(t_k)), \quad 1 \le k \le p,$$

$$u'_\lambda(t_k^+) = M_k(u'_\lambda(t_k)), \quad 1 \le k \le p,$$

$$u_\lambda(0) - g(u'_\lambda(0)) = u'_\lambda(1) = a \in (0,1].$$

By Lemma 2.3, we have that $0 < u_\lambda(t) \le A$, $0 < u'_\lambda(t) \le B$ ($h(t,z) \equiv v(t), p = q \equiv 0$ in Lemma 2.3). Put $M > 1 + A + B + \|u_0\|_{PC^1}$, $\|u_\lambda\|_{PC^1} < M$ for any $\lambda \in [0,1]$. Set $\Omega = \{u \in \in PC^1(J), \|u\|_{PC^1} < M\}$ and $\Psi_\lambda = I - T_\lambda$, here I is the identity operator, then

$$\deg(\Psi_1, \Omega, 0) = \deg(\Psi_0, \Omega, 0) = \deg(I - T_0, \Omega, 0) \ne 0,$$

here, deg is the Lery–Schauder degree. Hence, $\Psi_1(u) = u - T_1 u = 0$ has a solution $u \in \bar{\Omega}$. Since the fixed point of T_1 is the solution of (2.17) and due to Lemma 2.1, (2.17) has a positive solution.

Lemma 2.5 is proved.

3. Main result. Our main result is the following theorem.

Theorem 3.1. *Assume that (H_1)–(H_5) are satisfied, then (1.1) has at least one positive solution.*

Proof. Let $m \in N$,

$$F(t,x,y) = f(t, \delta_1(m,x), \delta_2(m,y))$$

for $x, y \in \mathbb{R}$, where

$$\delta_1(m,x) = \begin{cases} \dfrac{1}{m}, & x \le \dfrac{1}{m}, \\ x, & \dfrac{1}{m} \le x \le 1 + A, \\ 1 + A, & x > 1 + A, \end{cases} \qquad \delta_2(m,y) = \begin{cases} \dfrac{1}{m}, & y \le \dfrac{1}{m}, \\ y, & \dfrac{1}{m} \le y \le 1 + B, \\ 1 + B, & y > 1 + B, \end{cases}$$

here, A, B are defined as in Lemma 2.3.

Consider the equation

$$\begin{aligned} & u''(t) + F(t, u(t), u'(t)) = 0, \quad t \ne t_k, \quad t \in J, \\ & u(t_k^+) = I_k(u(t_k)), \quad 1 \le k \le p, \\ & u'(t_k^+) = M_k(u'(t_k)), \quad 1 \le k \le p, \\ & u(0) - g(u'(0)) = u'(1) = \frac{1}{m}. \end{aligned} \tag{3.1}$$

Note that

$$\psi(t) \le F(t,x,y) \le V(t) < +\infty,$$

where

$$V(t) = \sup\left\{ f(t,x,y) : x \in \left[\frac{1}{m}, 1 + A\right], y \in \left[\frac{1}{m}, 1 + B\right] \right\}.$$

By Lemma 2.5, (3.1) has a positive solution $u_m(t)$ for any $m \in \mathbb{N}$. Using Lemmas 2.1 and 2.3, we obtain that for any $t \in J$,

$$\frac{1}{m} \le u_m(t) < 1 + A, \quad \frac{1}{m} \le u'_m(t) < 1 + B.$$

Hence, u_m satisfies

$$\begin{aligned} &u''_m(t) + f(t, u_m(t), u'_m(t)) = 0, \quad t \ne t_k, \quad t \in J, \\ &u_m(t_k^+) = I_k(u_m(t_k)), \quad 1 \le k \le p, \\ &u'_m(t_k^+) = M_k(u'_m(t_k)), \quad 1 \le k \le p, \\ &u_m(0) - g(u'_m(0)) = u'_m(1) = \frac{1}{m}. \end{aligned} \tag{3.2}$$

By (H_5),

$$-u''_m(t) = f(t, u_m(t), u'_m(t)) \ge \psi(t), \quad t \in J.$$

By Lemma 2.3,

$$u_m(t) \ge G(t), \quad u'_m(t) \ge H(t), \quad t \in J.$$

If $s_1, s_2 \in (t_i, t_{i+1}]$, $s_1 < s_2$,

$$|u_m(s_2) - u_m(s_1)| \le \left| \int_{s_1}^{s_2} u'_m(t)dt \right| \le B(s_2 - s_1),$$

$$\begin{aligned} |u'_m(s_2) - u'_m(s_1)| = \left| \int_{s_1}^{s_2} u''_m(t)dt \right| = \left| \int_{s_1}^{s_2} f(t, u_m(t), u'_m(t))dt \right| \le \\ \le \int_{s_1}^{s_2} h(t, u_m(t) + u'_m(t))dt + \int_{s_1}^{s_2} p(t)w_1(u_m(t))dt + \\ + \int_{s_1}^{s_2} q(t)w_2(u'_m(t))dt \le \\ \le \int_{s_1}^{s_2} h(t, A + B + 2)dt + P \int_{s_1}^{s_2} w_1(u_m(t))dt + Q \int_{s_1}^{s_2} w_2(u'_m(t))dt \le \\ \le \int_{s_1}^{s_2} h(t, A + B + 2)dt + P \int_{s_1}^{s_2} w_1(G(t))dt + Q \int_{s_1}^{s_2} w_2(H(t))dt \le \end{aligned}$$

$$\leq (s_2 - s_1)\max\{h(t, A+B+2) : t \in [0,1]\} + (P+Q)\int\limits_{s_1}^{s_2}(w_1(G(t)) + w_2(H(t)))dt,$$

which by the condition (H_5) implies that $\{u'_m\}$ is equicontinuous on (t_i, t_{i+1}). Hence, there exist a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ which converges to $u \in PC^1(J)$. Since

$$u_m(t) = \frac{1}{m} + g\left(\frac{1}{m} + \int\limits_0^1 f(s, u_m(s), u'_m(s))ds - \sum_{k=1}^{p}(M_k(u'_m(t_k)) - u'_m(t_k))\right) + \frac{t}{m} +$$

$$+ \int\limits_0^1 G(t,s) f(s, u_m(s), u'_m(s))ds - t\sum_{k=1}^{p}(M_k(u'_m(t_k)) - u'_m(t_k)) +$$

$$+ \sum_{0<t_k<t}[I_k(u_m(t_k)) - u_m(t_k)) + (t - t_k)(M_k(u'_m(t_k)) - u'_m(t_k))],$$

taking $m_j \to \infty$, we have

$$u(t) = g\left(\int\limits_0^1 f(s, u(s), u'(s))ds - \sum_{k=1}^{p}(M_k(u'(t_k)) - u'(t_k))\right) +$$

$$+ \int\limits_0^1 G(t,s) f(s, u(s), u'(s))ds - t\sum_{k=1}^{p}(M_k(u'(t_k)) - u'(t_k)) +$$

$$+ \sum_{0<t_k<t}[I_k(u(t_k)) - u(t_k)) + (t - t_k)(M_k(u'(t_k)) - u'(t_k))],$$

which follows that

$$u''(t) + f(t, u(t), u'(t)) = 0, \quad t \neq t_k,$$

$$u(t_k^+) = I_k(u(t_k)), \quad 1 \leq k \leq p,$$

$$u'(t_k^+) = M_k(u'(t_k)), \quad 1 \leq k \leq p,$$

$$u(0) - g(u'(0)) = u'(1) = 0.$$

Hence, u is one positive solution of (1.1).

Theorem 3.1 is proved.

Example 3.1. Consider the equation

$$u''(t) + t^2 + (u'(t))^{-\frac{1}{2}} + (u(t))^{-\frac{1}{2}} = 0, \quad t \neq t_k, \quad t \in J,$$

$$u(t_k^+) = 2u(t_k), \quad u'(t_k^+) = 2u'(t_k), \quad k = 1, 2, \tag{3.3}$$

$$u(0) - (u'(0))^3 = u'(1) = 0,$$

where $0 < t_1 < t_2 < 1$.

In fact, $f(t,x,y) = t^2 + y^{-\frac{1}{2}} + x^{-\frac{1}{2}}$. Put $\psi(t) = t^2$, then

$$H(t) = \min\left\{\frac{1}{3}(1-t^3), \frac{1}{12}(1-t_2^3)\right\} \geq \frac{1}{12}(1-t_2^3)(1-t),$$

$$G(t) \geq \frac{1}{24}(1-t_2^3)t_1 t.$$

Setting $h(t,x) = 1$, $w_1(x) = w_2(x) = x^{-\frac{1}{2}}$, then for $(t,x,y) \in [0,1] \times (0,\infty)^2$,

$$\psi(t) \leq f(t,x,y) \leq h(t,x+y) + w_1(x) + w_2(y),$$

$$\int\limits_0^1 (w_1(G(s)) + w_2(H(s)))ds \leq \frac{8\sqrt{6}}{\sqrt{t_1(1-t_2^3)}} < \infty.$$

By Theorem 3.1, (3.3) has at least one positive solution.

1. *O'Regan D.* Theory of singular boundary value problems. — Singapore: World Sci., 1994.
2. *O'Regan D.* Nonlinear superlinear singular and nonsingular second order boundary value problems // J. Different. Equat. — 1998. — **143**. — P. 60–95.
3. *Lan, Webb J.* Positive solutions of semilinear differential equations with singularities // J. Different. Equat. — 1998. — **148**. — P. 407–421.
4. *Agarwal R., O'Regan D.* Singular boundary value problems with integrable singularities // Miskolc Math. Notes. — 2004. — **5**. — P. 119–127.
5. *Agarwal R., O'Regan D., Staněk S.* Positive solutions of singular problems with sign changing Carathéodory nonlinearities depending on x' // J. Math. Anal. and Appl. — 2003. — **279**. — P. 597–616.
6. *Xian X., O'Regan D., Agarwal R. P.* Multiplicity results via topological degree for impulsive boundary value problems under non-well-ordered upper and lower solution conditions // Boundary Value Problems. — 2008. — **2008**. — Article ID 197205. — 21 p.
7. *Rachůnková I.* Singular Dirichlet second-order BVPs with impulses // J. Different. Equat. — 2003. — **193**. — P. 435–459.
8. *Li J., Shen J.* Existence of positive solution to a singular second-order initial value problem with impulses // J. Comput. and Appl. Math. — 2007. — **206**. — P. 699–712.
9. *Li J., Nieto J. J.* Existence of positive solutions for multipoint boundary value problem on the half-line with impulses // Boundary Value Problems. — 2009. — **2009**. — Article ID 834158. — 12 p.
10. *Lee Y. H., Liu X.* Study of singular boundary value problems for second order impulsive differential equation // J. Math. Anal. and Appl. — 2007. — **331**. — P. 159–176.
11. *Lin X., Jiang D.* Multiple positive solutions of Dirichlet boundary value problems for second order impulsive differential equation // J. Math. Anal. and Appl. — 2006. — **321**. — P. 501–514.
12. *Lee E. L., Lee Y. H.* Multiple positive solutions of two point boundary value problems for second order impulsive differential equation // Appl. Math. and Comput. — 2004. — **158**. — P. 745–759.

Received 08.07.10,
after revision — 13.03.12