UDC 517.9

OSCILLATION CRITERIA FOR CERTAIN SECOND-ORDER SUPERLINEAR DIFFERENTIAL EQUATIONS

КРИТЕРІЇ ОСЦИЛЯЦІЙ ДЛЯ ДЕЯКИХ СУПЕРЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ

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In this paper, a class of second-order superlinear equations is studied. New oscillations criteria are established by using a general class of the parameter functions in the averaging techniques. We extend and improve the oscillation criteria of several authors. One of our results is based on the information of the whole half-line and the other is based on the information on a sequence subintervals of the whole half-line.

Вивчено клас суперлінійних рівнянь другого порядку. Отримано нові критерії осциляцій за допомогою застосування загального класу параметричних функцій у процедурі усереднення. Поширено та покращено критерії осциляцій, які були отримані деякими авторами. Один з отриманих результатів базується на даних на всій півпрямій, інший — на даних на послідовності підінтервалів всієї півпрямої.

1. Introduction. This paper is concerned with the second-order nonlinear differential equation of superlinear type

$$(r(t)|y'|^{p-2}y')' + p(t)|y'|^{p-2}y' + q(t)f(y) = 0, \tag{1.1}$$

where $r \in C(I, R^+)$, $p \in C(I, R)$, $q \in C(I, R)$, $f \in C^1(R, R)$ such that $yf(y) > 0$ and $f'(y) \geq 0$ for $y \neq 0$, $I = [t_0, \infty]$, $R^+ = (0, \infty)$, $R = (-\infty, \infty)$ and $p > 1$ is a real number.

This equation can be considered as a generalization of the second-order equation with damping

$$(r(t)y')' + p(t)y' + q(t)f(y) = 0 \tag{1.2}$$

which have been the subject of intensive studies in the recent years.

By a solution of (1.1), we mean a function $y : [T_y, \infty) \to R$, $T_y \geq t_0$, such that y and $r(t)|y'|^{p-2}y'$ are continuously differentiable and satisfy (1.1) for $t \geq T_y$. A solution is said to be global if it exists on the whole interval. On the other hand a solution, y of (1.1) which exists on some interval $[T_y, \infty)$, $T_y \geq t_0$, is called proper if $\sup\{|y(t)| : t \geq T\} \neq 0$ for all $T \geq T_y$.

The existence of proper (or global) solutions for the nonlinear second-order differential equations was investigated in [4, 5]. In [4] Kiguradze and Chanturia established sufficient conditions for all global solutions to be proper. So we shall suppose that (1.1) has proper solutions and our attention will be restricted to these solutions only.

The oscillation of (1.1) is considered in the usual sense; that is, a solution y of (1.1) is said to be oscillatory if it has arbitrary large zeros on $[T_y, \infty)$; otherwise it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all solutions are oscillatory.

Equation (1.1) is said to be superlinear if

$$0 < \int\limits_{\varepsilon}^{\infty} \frac{du}{f^{\frac{1}{p-1}}(u)}, \quad \int\limits_{-\infty}^{-\varepsilon} \frac{du}{(-f(u))^{\frac{1}{p-1}}} < \infty \tag{1.3}$$

for all $\varepsilon > 0$, and it is said to be sublinear if

$$0 < \int\limits_{0^+}^{\varepsilon} \frac{du}{f^{\frac{1}{p-1}}(u)}, \quad \int\limits_{-\varepsilon}^{0^-} \frac{du}{(-f(u))^{\frac{1}{p-1}}} < \infty \tag{1.4}$$

for all $\varepsilon > 0$.

Here we are interested in the oscillation of solutions of (1.1) when (1.3) is satisfied, including the so-called Emden – Fowler equation

$$y'' + q(t)|y|^{\gamma}\operatorname{sgn} y = 0, \tag{1.5}$$

where γ is a positive real number.

Since many physical problems are modeled by second-order nonlinear differential equations, the oscillatory and nonoscillatory behavior of solutions of such differential equations have been considerably investigated by many authors [1 – 21] and references therein. Probably, the most considered equation is Eq. (1.5), which attracted the attention for the first time around the turn of the century with earlier theories concerning gaseous dynamics in astrophysics. This equation also appears in the study of fluid mechanics, relativistic mechanics and nuclear physics.

Recently, Philos et al. [11, 12], Meng [9], Li and Yan [6], Yu [21], Lu and Meng [7], Manojlovic [8], and Tiryaki [16] have studied the oscillatory behavior of superlinear differential equations. Some of these results involve the Philos type averaging technique. More recently Qing-Hai Hau and Fang Lu [13] obtained some new oscillation criteria for the superlinear equation (1.2) including (1.5), by allowing more general means along the lines given in [2, 3].

Our purpose here is to develop oscillation theory for (1.1), without any restriction on the signs of $p(t)$, $q(t)$ and $\rho'(t)$ including the well-known Emden – Fowler and half-linear equations, where ρ' is the differentiation of the test function used in the main section. We obtain some new oscillation criteria extending and improving some earlier results. We believe that our approach is simpler and more general than the recent results for (1.1) with $p(t) \equiv 0$ in [17]. We should note that, in most of the oscillation results for (1.1) and its special cases $(e.g. p(t) \equiv 0)$, the basic condition on f is given by

$$\frac{f'(y)}{|f(y)|^{\frac{p-2}{p-1}}} \geq k > 0 \tag{1.6}$$

for $y \neq 0$ and k is constant.

For instance for $p = 2$, when we take $f(y) = |y|^\gamma \operatorname{sgn} y$, $\gamma > 1$ or when we choose f as

$$f(y) = |y|^\gamma [\lambda + \sin(\ln(1 + |y|))] \operatorname{sgn} y$$

condition (1.6) is not satisfied, hence most of the known methods in the literature for example [1, 14, 15, 17] cannot be applied. To the best of our knowledge there is no oscillation result on (1.1), except for the special case $p = 2$ [13].

We will present theorems in the main results which will be applicable for functions f like the ones given above.

2. Preliminaries. In order to discuss our main results, we introduce the general mean given in [2, 3, 17, 20] and we shall present some properties, which will be used in the proof of our results.

Let $D_1 = \{(t,s) : t_0 \leq s \leq t\}$ denote a subset of R^2 and let $D_2 = \{(t,s) : t_0 \leq s < t\}$. Consider a kernel function $k(t,s)$, which is defined, continuous, and sufficiently smooth on D_1, so that the following conditions are satisfied:

(K_1) $k(t,t) = 0$ and $k(t,s) > 0$ for $(t,s) \in D_2$,

(K_2) $\dfrac{\partial k}{\partial s}(t,s) \leq 0$ and $\lambda(t,s) := \dfrac{-\frac{\partial k}{\partial s}(t,s)}{(k(t,s))^{1/q}}$ for $(t,s) \in D_2$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Let $\rho \in C^1(I)$ and $\rho(t) > 0$ on I. We take the integral operator A_τ^ρ, which is defined in [20] for the first time, in terms of $k(t,s)$ and $\rho(s)$ and as

$$A_\tau^\rho(h;t) = \int\limits_\tau^t k(t,s)h(s)\rho(s)ds, \quad t \geq \tau \geq t_0, \tag{2.1}$$

where $h \in C(I)$. It is easily seen that A_τ^ρ is linear and positive, and in fact satisfies the following:

$$A_\tau^\rho(\alpha_1 h_1 + \alpha_2 h_2; t) = \alpha_1 A_\tau^\rho(h_1;t) + \alpha_2 A_\tau^\rho(h_2;t), \quad A_\tau^\rho(h;t) \geq 0, \tag{2.2}$$

whenever $h \geq 0$

$$A_\tau^\rho(h';t) = -k(t,\tau)h(\tau)\rho(\tau) - A_\tau^\rho\left(\left[-\lambda k^{-\frac{1}{p}} + \frac{\rho'}{\rho}\right] h;t\right) \geq$$

$$\geq -k(t,\tau)h(\tau)\rho(\tau) - A_\tau^\rho\left(\left|-\lambda k^{-\frac{1}{p}} + \frac{\rho'}{\rho}\right| |h|;t\right).$$

Here $h_1, h_2, h \in C(I)$ and α_1, α_2 are real numbers.

Let $D_3 = \{H \in C^1([a,b]) : H(t) \neq 0 \text{ for } t \in I_1 = [a,b] \subset I \text{ and } H(a) = H(b) = 0\}$. We take the integral operator A_a^b in terms of $H \in D_3$ and $\rho(t)$ as

$$A_a^b(h;t) = \int\limits_a^b H^{2(p-1)}(t)h(t)\rho(t)dt, \quad a \leq t \leq b, \tag{2.3}$$

where ρ and h are defined as before. As the operator A_τ^ρ, A_a^b is also linear, positive and also satisfies the following:

$$A_a^b(h',t) = -A_a^b\left(\left[2(p-1)\frac{H'}{H}+\frac{\rho'}{\rho}\right]h;t\right) \geq -A_a^b\left(\left|2(p-1)\frac{H'}{H}+\frac{\rho'}{\rho}\right||h|;t\right). \tag{2.4}$$

Note that the first operator A_τ^ρ is defined on the entire half-line $I = [t_0,\infty)$. The second operator A_a^b is defined on the subinterval $I_1 \subset D$ chosen according to our propose.

3. Main results. In this section we establish some oscillation criteria for (1.1) and its special cases in the superlinear case. Additional assumptions on the function f will be imposed.

Suppose that

$$\int\limits_{\varepsilon}^{\infty}\left(\frac{f'(u)}{f^2(u)}\right)^{\frac{1}{p}}du < \infty, \quad \int\limits_{-\varepsilon}^{-\infty}\left(\frac{f'(u)}{f^2(u)}\right)^{\frac{1}{p}}du < \infty \tag{3.1}$$

for all $\varepsilon > 0$, and

$$\min\left\{\inf_{x>0}\frac{f'(x)}{f^{\frac{p-2}{p-1}}(x)}G(x),\ \inf_{x<0}\frac{f'(x)}{(-f(x))^{\frac{p-2}{p-1}}}G(x)\right\} > 0, \tag{3.2}$$

where

$$G(x) = \begin{cases} \displaystyle\int_x^{\infty}\left(\frac{f'(u)}{f^2(u)}^{\frac{1}{p}}\right)du, & x > 0, \\ \displaystyle\int_x^{-\infty}\left(\frac{f'(u)}{f^2(u)}\right)^{\frac{1}{p}}du, & x < 0. \end{cases}$$

We define the following functions that will be used in the proof of our results. Suppose that there exists a function $\phi \in C^1(I,R^+)$ such that

$$\xi(t) := r(t)\phi'(t) - p(t)\phi(t),$$

$$\eta(t) := \frac{1}{r(t)\phi(t)},$$

and

$$\nu(t,T) := \eta^{\frac{1}{p-1}}(t)\left(\int\limits_T^t \eta(s)^{\frac{1}{p-1}}ds\right)^{-1}, \quad T \geq t_0.$$

Let us state the main results.

Theorem 3.1. *Let conditions (1.3), (3.1), (3.2) and $p > 1$ and $p \neq 2$ hold. Assume that $k(t,s)$ satisfies conditions (K_1) and (K_2) and A_τ^ρ is defined by (2.1). If there exist $k(t,s)$, $\phi \in C(I,R^+)$, and $\rho \in C^1(I,R^+)$ for any constant $C > 0$, such that*

$$\xi(t) \geq 0, \quad \xi'(t) < 0$$

for $t \geq t_0$,

$$\int\limits_T^\infty (\eta(s))^{\frac{1}{p-1}}\, ds = \infty, \tag{3.3}$$

$$\liminf_{t\to\infty} \int\limits_T^t \phi(s)q(s)ds > -\infty, \tag{3.4}$$

and

$$\limsup_{t\to\infty} \frac{1}{k(t,t_0)}\, A_{t_0}^{\rho}\left[q\phi - \left(\frac{p-1}{p}\right)^p \frac{1}{p-1}\left(\frac{1}{C}\right)^{(p-1)}\left(\left|\lambda k^{-\frac{1}{p}} + \frac{\rho'}{\rho}\right| + \xi\eta\right)^p \nu^{-(p-1)}; t\right] = \infty, \tag{3.5}$$

then any solution $y(t)$ *of Eq. (1.1) such that* $y'(t)$ *is bounded is oscillatory.*

Proof. Suppose (1.1) possesses a nonoscillatory solution $y(t)$ on $[T,\infty)$, $T \geq t_0$. Without loss of generality, we assume that $y(t) > 0$ for every $t \geq T$. We observe that the substitution $x = -y$ transforms (1.1) into the equation

$$(r(t)|x'|^{p-2}x')' + p(t)|x'|^{p-2}x' + q(t)f^*(x) = 0,$$

where $f^*(x) = -f(-y)$, $y \in R$. Since the function $f^*(x)$ is subject to the same conditions as on $f(y)$, we can restrict our discussion to the case where the solution $y(t)$ is positive on $[T,\infty)$. Further assume that $|y\prime(t)| \leq L$ for some $L > 0$. Let $w(t)$ be defined by

$$w(t) = \phi(t)\frac{r(t)|y\prime(t)|^{p-2}y\prime(t)}{f(y(t))}, \quad t \geq T, \tag{3.6}$$

we have

$$w'(t) = -q(t)\phi(t) + \xi(t)\frac{|y'(t)|^{p-2}y'(t)}{f(y(t))} - \frac{f'(y(t))|w(t)|^q}{(f(y(t)))^{\frac{p-2}{p-1}}(\phi(t)q(t))^{\frac{1}{p-1}}}. \tag{3.7}$$

Let us consider the boundedness of the following term:

$$\int\limits_{t_1}^t \xi(s)\frac{|y'(s)|^{p-2}y'(s)}{f(y(s))}\, ds, \quad t \geq t_1 \geq T. \tag{3.8}$$

Applying the integral mean value theorem and using the boundedness of $y'(t)$, there exists $t^* \in [t_1,t]$ for every $t \geq t_1$ such that

$$\int\limits_{t_1}^t \xi(s)\frac{|y'(s)|^{p-2}y'(s)}{f(y(s))}\, ds \leq \xi(t_1)L^{p-2}\int\limits_{y(t_1)}^{y(t)} \frac{du}{f(u)} \leq \xi(t_1)L^{p-2}\int\limits_{y(t_1)}^{\infty} \frac{du}{f(u)} < K_1, \tag{3.9}$$

where $K_1 > 0$ is a constant. We consider the following three cases for the behavior of $y'(t)$.

Case 1. Suppose that $y'(t) > 0$ for $t \geq t_1 \geq T$ then $w(t) > 0$ for $t \geq t_1$. Integrating both sides of (3.7) from t_1 to t, we have

$$w(t) = w(t_1) - \int\limits_{t_1}^{t} q(s)\phi(s)ds + \int\limits_{t_1}^{t} \xi(s)\frac{|y'(s)|^{p-2}y'(s)}{f(y(s))}\,ds - \int\limits_{t_1}^{t} \frac{f'(y(s))|w(s)|^q ds}{[[f(y(s))]^{p-2}(\phi(s)r(s))]^{\frac{1}{p-1}}}. \quad (3.10)$$

Hence, for $t \geq t_1$, we obtain

$$w(t) \leq N - \int\limits_{t_1}^{t} q(s)\phi(s)ds - \int\limits_{t_1}^{t} \frac{f'(y(s))|w(s)|^q ds}{[(f^{p-2}(y(s)))(\phi(s)r(s))]^{\frac{1}{p-1}}}, \quad (3.11)$$

where $N = K_1 + w(t_0)$. Using (3.4) we see that

$$\int\limits_{t_1}^{\infty} \frac{f'y(s))|w(s)|^q ds}{[(f^{p-2}(y(s)))(\phi(s)r(s))]^{\frac{1}{p-1}}} < \infty. \quad (3.12)$$

There exists a constant $M > 0$ such that

$$\int\limits_{t_1}^{\infty} \frac{f'(y(s))|w(s)|^q ds}{[(f^{p-2}(y(s)))(\phi(s)r(s))]^{\frac{1}{p-1}}} \leq M, \quad t \geq t_1. \quad (3.13)$$

By using Hölder's inequality and (3.13), we get

$$\left|\int\limits_{t_1}^{t} y'(s)\left(\frac{f'(y(s))}{f^2(y(s))}\right)^{\frac{1}{p}} ds\right| = \left|\int\limits_{t_1}^{t} (\phi(s)r(s))^{\frac{1}{p}}(\phi(s)r(s))^{-\frac{1}{p}}y'(s)\left(\frac{f'(y(s))}{f^2(y(s))}\right)^{\frac{1}{p}} ds\right| \leq$$

$$\leq \left|\left(\int\limits_{t_1}^{t} \frac{r(s)\phi(s)|y'(s)|^p f'(y(s))ds}{f^2(y(s))}\right)^{\frac{1}{p}} \left(\int\limits_{t_1}^{t} \frac{ds}{(r(s)\phi(s))^{\frac{1}{p-1}}}\right)^{\frac{1}{q}}\right| \leq$$

$$\leq M^{\frac{1}{p}}\left(\int\limits_{t_1}^{t} \eta^{\frac{1}{p-1}}(s)\right)^{\frac{1}{q}}. \quad (3.14)$$

Applying the condition (3.2), we see that

$$\left(\frac{f'(y(t))}{f^{\frac{p-2}{p-1}}(y(t))}\right)^{\frac{1}{q}} \int\limits_{y(t)}^{\infty} \left(\frac{f'(u)}{f^2(u)}\right)^{\frac{1}{p}} du \geq N_1, \quad t \geq t_1, \quad (3.15)$$

where N_1 is a positive constant. Let

$$N_2 = \int\limits_{y(t_1)}^{\infty} \left(\frac{f'(u)}{f^2(u)}\right)^{\frac{1}{p-1}} du > 0.$$

Then, applying (3.15), we have

$$\frac{f'(y(t))}{f^{\frac{p-2}{p-1}}(y(t))} \geq N_1^q \left[N_2 - \int\limits_{y(t_1)}^{y(t)} \left(\frac{f'(u)}{f^2(u)}\right)^{\frac{1}{p}} du\right]^{-q} = N_1^q \left[N_2 - \int\limits_{t_1}^{t} y'(s) \left(\frac{f'(y(s))}{f^2(y(s))}\right)^{\frac{1}{p}} ds\right]^{-q} \geq$$

$$\geq N_1^q \left[N_2 + \left|\int\limits_{t_1}^{t} y'(s) \left(\frac{f'(y(s))}{f^2(y(s))}\right)^{\frac{1}{p}} ds\right|\right]^{-q}.$$

A use of (3.14) in the above inequality leads to

$$\frac{f'(y(t))}{f^{\frac{p-2}{p-1}}(y(t))} \geq \frac{N_1^q}{\left[N_2 + M^{\frac{1}{p}} \left(\int_{t_1}^{t} \eta^{\frac{1}{p-1}}(s)ds\right)^{\frac{1}{q}}\right]^q}.$$

Hence, there exists a constant $C > 0$ and $t_2 > t_1$ such that

$$\frac{f'(y(t))}{f^{\frac{p-2}{p-1}}(y(t))} \geq \frac{C}{\left[\int_{t_1}^{t} \eta^{\frac{1}{p-1}}(s)ds\right]} \tag{3.16}$$

for all $t \geq t_1$, where C is a positive constant which depends on N_1, N_2, M and p.

Now from (3.16), Eq. (3.7) gives

$$w'(t) \leq -q(t)\phi(t) + \xi(t)\eta(t)|w(t)| - C\nu(t, t_1)|w(t)|^q. \tag{3.17}$$

Applying the operator A_τ^ρ to (3.17) and using (2.2), we obtain

$$A_\tau^\rho(q\phi; t) \leq k(t,\tau)w(\tau)\rho(\tau) + A_\tau^\rho\left(\left(\left|\lambda k^{-\frac{1}{p}} + \frac{\rho'}{\rho}\right| + \xi\eta\right)|w| - Cv(\cdot, t_1)|w|^q; t\right). \tag{3.18}$$

By using the inequality

$$Du - Eu^q \leq \left(\frac{p-1}{p}\right)^p \frac{1}{p-1} D^p E^{-(p-1)}, \quad D \geq 0, \quad E \geq 0, \quad u \geq 0, \tag{3.19}$$

which can be easily obtained by using the extremum of one variable function, we get

$$A^{\rho}_{\tau}(q\phi,t) \le k(t,\tau)w(\tau)\rho(\tau) + A^{\rho}_{\tau}\Bigg(\left(\frac{p-1}{p}\right)^p \frac{1}{p-1}\left(\left|\lambda k^{-1/p}+\frac{\rho'}{\rho}\right|+\xi\eta\right)^p \times$$

$$\times\ (C\nu(\cdot,t_1))^{-(p-1)}\,;t\Bigg). \tag{3.20}$$

If we set $\tau = t_0$ and divide (3.20) through by $k(t,t_0)$, then we have

$$\frac{1}{k(t,t_0)}\,A^{\rho}_{t_0}\left(q\phi - \left(\frac{p-1}{p}\right)^p \frac{C^{-(p-1)}}{p-1}\left(\left|\lambda k^{-1/p}+\frac{\rho'}{\rho}\right|+\xi\eta\right)^p \nu(\cdot,t_1)^{-(p-1)};t\right) \le \rho(t_0)w(t_0). \tag{3.21}$$

Taking $\limsup$ in (3.21) as $t \to \infty$, condition (3.5) gives the desired contradiction in (3.21). Thus, the existence of a nonoscillatory solution $y(t)$ is ruled out, so Eq. (1.1) is oscillatory.

Case 2. Suppose that $y'(t)$ is oscillatory. Then, there exists a sequence $\{t_m\}_{m=1,2,\ldots}$ such that $\lim_{m\to\infty} t_m = \infty$ and $y'(t_m) = 0$, $m = 1,2,\ldots$. Choose m such that $t_m \ge t_0$. Without loss of generality we assume that $y'(t) > 0$ for $t \in (t_m, t_{m+1})$. Further, in view of (3.4),

$$\int\limits_{t_m}^{t_{m+1}} \frac{f'(y(s))|w(s)|^q}{[f^{p-2}(y(s))(r(s)\phi(s))]^{1/p-1}}ds \le N - \int\limits_{t_m}^{t_{m+1}} q(s)\phi(s)ds. \tag{3.22}$$

There is an infinite number of m's such that $y'(t) > 0$ for $t \in (t_m, t_{m+1})$. Summing all these inequalities (3.22) and using (3.4), we have

$$\int\limits_{t_{m_1}}^{\infty} \frac{f'(y(s))|w(s)|^q}{[f^{p-2}(y(s))(r(s)\phi(s))]^{1/p-1}}ds < \infty.$$

The rest of the proof is as in Case 1.

Case 3. Suppose that $y'(t) < 0$ for $t \ge t_1 \ge t_0$. Meanwhile if the inequality (3.12) holds, then we can have a similar contradiction as in Case 1. If the integration in (3.12) is divergent, we obtain the following inequality from (3.4) and (3.10):

$$N_3 + \int\limits_{t_1}^{t} \frac{f'(y(s))|w(s)|^q}{f^{\frac{p-2}{p-1}}(y(s))}\,\eta(s)^{\frac{1}{p-1}}\,ds \le -w(t), \quad t \ge t_1, \tag{3.23}$$

where N_3 is a constant. By choosing $t_2 \ge t_1$, we can get

$$N_4 = N_3 + \int\limits_{t_1}^{t_2} \eta^{\frac{1}{p-1}}(s)\frac{f'(y(s))|w(s)|^q}{f^{\frac{p-2}{p-1}}(y(s))}ds > 1. \tag{3.24}$$

From (3.23) and (3.10), we obtain

$$w(t) < 0, \quad t \geq t_2.$$

Then using (3.23), we find

$$\frac{\dfrac{\eta^{\frac{1}{p-1}}(t)|w(t)|^q f'(y(t))}{f^{\frac{p-2}{p-1}}(y(t))}}{N_3 + \int_{t_1}^t \frac{\eta^{\frac{1}{p-1}}(s)|w(s)|^q f'(y(s))}{f^{\frac{p-2}{p-1}}(y(s))} ds} \geq -\frac{y'(t)f'(y(t))}{f(y(t))}, \quad t \geq t_2.$$

Integrating the above inequality, we get

$$\ln \left[N_3 + \int_{t_1}^{t} \frac{\eta^{\frac{1}{p-1}}(s)|w(s)|^q f'(y(s))}{f^{\frac{p-2}{p-1}}(y(s))} ds \right] \geq \ln \frac{f(y(t_2))}{f(y(t))}.$$

Hence

$$N_3 + \int_{t_1}^{t} \frac{\eta^{\frac{1}{p-1}}(s)|w(s)|^q f'(y(s))}{f^{\frac{p-2}{p-1}}(y(s))} ds \geq \frac{f(y(t_2))}{f(y(t))}, \quad t \geq t_2. \tag{3.25}$$

Applying (3.23) and (3.24), we have

$$y(t) \leq y(t_2) - f^{\frac{1}{p-1}}(y(t_2)) \int_{t_2}^{t} \eta^{\frac{1}{p-1}}(s) ds.$$

In this inequality, the right-hand side tends to $-\infty$ as $t \to \infty$ by (3.3). However, the left-hand side is positive, which is a contradiction.

Theorem 3.1 is proved.

For $p = 2$, we do not require the boundedness condition for $y'(t)$ and we have the following result.

Theorem 3.2. *Let conditions (1.3), (3.1) and (3.2) hold with* $p = 2$. *Assume that* $k(t,s)$ *satisfies conditions* (K_1) *and* (K_2) *and* A_τ^ρ *is defined by (2.1). If there exist* $k(t,s)$, $\phi \in C(I, R^+)$ *and* $\rho \in C^1(I, R^+)$ *for any constant* $C > 0$, *such that*

$$\xi(t) \geq 0, \quad \xi'(t) < 0$$

for $t \geq t_0$,

$$\int_{T}^{\infty} \eta(s) ds = \infty, \tag{3.26}$$

$$\liminf_{t \to \infty} \int_{T}^{t} \phi(s) q(s) ds > -\infty, \tag{3.27}$$

and

$$\limsup_{t\to\infty} \frac{1}{k(t,t_0)} A_{t_0}^{\rho} \left[q\phi - \frac{1}{4C} \left(\left| \lambda k^{-\frac{1}{2}} + \frac{\rho'}{\rho} \right| + \xi\eta \right)^2 \nu^{-1}; t \right] = \infty, \tag{3.28}$$

then any solution $y(t)$ of Eq. (1.2) is oscillatory.

Proof. Let $p = 2$. As in [13], by using conditions (1.3) and (3.6) we have

$$\int_{t_1}^{t} \xi(s) \frac{y'(s)}{f(y(s))}\, ds \le \xi(t_1) \int_{y(t_1)}^{\infty} \frac{du}{f(u)} = K_2, \tag{3.29}$$

where $K_2 > 0$ is a constant. Replacing K_1 with K_2 in the proof of Theorem 3.1, the rest of the proof stays the same.

A close look at the proof of Theorem 3.1 reveals that condition (3.5) may be replaced by the conditions

$$\limsup_{t\to\infty} \frac{1}{k(t,t_0)} A_{t_0}^{\rho}(q\phi) = \infty \tag{3.30}$$

and

$$\limsup_{t\to\infty} \frac{1}{k(t,t_0)} A_{t_0}^{\rho} \left(\left(\left| \lambda k^{-\frac{1}{p}} + \frac{\rho'}{\rho} \right| + \xi\eta \right)^p \nu^{-(p-1)}; t \right) < \infty. \tag{3.31}$$

This leads to the following result.

Corollary 3.1. *Let the conditions of Theorem 3.1 be satisfied except that condition (3.5) is replaced by (3.30) and (3.31). Then any solution $y(t)$ of Eq. (1.1) such that $y'(t)$ is bounded is oscillatory.*

By making the same replacement in Theorem 3.2, we get similar results.

Note that the above oscillation criteria as well as most of the known results in the literature require information of (1.1) on the entire half-line I. Now, motivated by [2, 3, 13], we present the following oscillation criteria for equation (1.1) which depend on the behavior of the coefficients only on a sequence of subintervals I_1 of I.

Theorem 3.3. *Let conditions (1.3), (3.1), (3.2) and $p > 1$ and $p \neq 2$ hold. Assume that for any $T \ge t_0$, there exist a, b satisfying $T \le a < b$ such that D_3 and A_a^b be defined as before. If there exist $H \in D_3$, $\phi \in C(I, R^+)$ and $\rho \in C^1(I, R^+)$ for any constant C such that (3.1), (3.3) and (3.4) hold and*

$$A_a^b(q\phi, t) > \left(\frac{p-1}{p}\right)^p \frac{1}{p-1} \left(\frac{1}{C}\right)^{p-1} A_a^b \left(\left(\left| 2(p-1)\frac{H'}{H} + \frac{\rho'}{\rho} \right| + \xi\eta \right)^p \nu^{-(p-1)}; t \right), \tag{3.32}$$

then any solution $y(t)$ of Eq. (1.1) such that $y'(t)$ is bounded is oscillatory.

Proof. Again let $y(t)$ be nonoscillatory solution of (1.1), say $y(t) > 0$ for $t \ge t_0$. Further assume that $|y\prime(t)| \le L$ for some $L > 0$. We shall discuss three cases for the behavior of $y'(t)$, namely $y'(t) > 0$, $y'(t)$ is oscillatory and $y'(t) < 0$.

Case 1. $y'(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. We proceed as in the proof of Theorem 3.1, by the assumption, we can choose a, $b \geq t_1$ and $a < b$, i.e., for a given $T \geq t_1$, there exist a, b satisfying $T \leq a < b$. Applying the operator A_a^b to (3.17) using (2.4) and the fact that $H(a) = H(b) = 0$, we obtain

$$A_a^b(q\phi;t) \leq A_a^b\left(\left(\left|2(p-1)\frac{H'}{H} + \frac{\rho'}{\rho}\right| + \xi\eta\right)|w| - C\nu|w|^q;t\right). \tag{3.33}$$

By using the inequality (3.19),

$$A_a^b(q\phi;t) \leq A_a^b\left(\frac{p-1}{p}\right)^p \frac{1}{p-1}\left(\left(\left|2(p-1)\frac{H'}{H} + \frac{\rho'}{\rho}\right) + \xi\eta\right)^p (C\nu)^{-(p-1)};t\right)$$

which contradicts the assumption (3.32).

Theorem 3.3 is proved.

Proof of the other cases is similar to the cases given in the proof of the Theorem 3.1. Thus, the existence of the nonoscillatory solution $y(t)$ is ruled out, so Eq. (1.1) is oscillatory.

For the sake of completeness, we should note that the following result is given as Theorem 2.1 in [13]:

Theorem 3.4. *Let conditions (1.3), (3.1) and (3.2) hold with $p = 2$. Assume that for any $T \geq t_0$, there exist a, b satisfying $T \leq a < b$ such that D_3 and A_a^b be defined as before. If there exist $H \in D_3$, $\phi \in C(I,\ R^+)$ and $\rho \in C^1(I,R^+)$ for any constant C such that (3.2), (3.26) and (3.27) hold and*

$$A_a^b(q\phi,t) > \frac{1}{4C}A_a^b\left(\left(\left|2\frac{H'}{H} + \frac{\rho'}{\rho}\right| + \xi\eta\right)^2 \nu^{-1};t\right), \tag{3.34}$$

then any solution $y(t)$ of Eq. (1.2) is oscillatory.

Remarks. 3.1. In the above results, the conditions that the integral $\int \frac{dt}{r(t)}$ is either convergent or divergent, and the dampeing coefficient $p(t)$ is a "small" function are not necessary. Therefore, the restraint for $r(t)$ and $p(t)$ is relaxed. Also there is no sign conditions on $p(t)$ and $q(t)$.

3.2. When $p = 2$, it is not necessary to assume that $y'(t)$ is bounded. For $p > 1$ and $p \neq 2$, we require a boundedness condition for the derivative of the solution. However the conditions on $f(y)$ are relaxed. Removing the boundedness condition still remains as an open problem and will be interesting.

3.3. Since the conditions on $f(y)$ are relaxed, Theorem 3.1 improves some results in [17] related to the special case of Eq. (1.1) when $p(t) \equiv 0$.

3.4. Note that Theorem 3 given in [16], which also uses the averaging technique, contain different sufficient conditions than Theorem 3.1 with $p = 2$ and $p(t) \equiv 0$.

4. An example. Let us consider the following second-order superlinear equation with damping

$$(t^{\lambda-1}|y'|^{p-2}y')' - t^{\lambda-2}|y'|^{p-2}y' + Kt^{\lambda}|y|^{\gamma}\operatorname{sgn} y = 0, \quad t \geq 1. \tag{4.1}$$

Let the constants γ, λ and K satisfy the following conditions depending on the value of p:

For $1 < p < 2$, let

(1) $p = \dfrac{2k}{2k-1}$, where k is a positive integer,

(2) $0 < \lambda < p-1 < \gamma$,

(3) $K = \left(\dfrac{p-1}{p}\right)^p \dfrac{1}{p-1}\left(\dfrac{1}{C}\right)^{p-1}(2p+\lambda+1)^p \dfrac{\Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{2p-1}{2}\right)}$;

for $p = 2$, let

(1) $0 < \lambda \le 1 < \gamma$,

(2) $K = \dfrac{(5+\lambda)^2}{C}$;

for $p > 2$, let

(1) p be an integer,

(2) $0 < \lambda \le 1$ and $p-1 < \gamma$,

(3) $K = \left(\dfrac{p-1}{p}\right)^p \dfrac{1}{p-1}\left(\dfrac{1}{C}\right)^{p-1}(2p+\lambda+1)^p \dfrac{\Gamma(p)}{\Gamma\left(\frac{p}{2}\right)}$.

Let $H(t) = \sin t$, $\phi(t) = t$ and $\rho(t) = t^{-(\lambda+1)}$. For any $T \ge 1$, choose n sufficiently large so that $n\pi = 2k\pi > T$ and set $a = 2k\pi$, $b = (2k+1)\pi$. It is easy to verify that

$$
\begin{aligned}
A_a^b(\phi q, t) &= \int\limits_a^b H^{2(p-1)}(t)\phi(t)q(t)\rho(t)dt = \int\limits_{2k\pi}^{(2k+1)\pi} K\sin^{2(p-1)}tdt = \\
&= K\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2p-1}{2}\right)}{\Gamma(p)}\left(\frac{p-1}{p}\right)^p \frac{1}{p-1}\left(\frac{1}{C}\right)^{p-1} \times \\
&\times A_a^b\left(\left(\left|2(p-1)\frac{H'}{H} + \frac{\rho'}{\rho}\right| + \xi\eta\right)^p (\nu)^{-(p-1)}; t\right) \le \\
&\le \left(\frac{p-1}{p}\right)^p \frac{1}{p-1}\left(\frac{1}{C}\right)^{p-1}(2p+\lambda+1)^p \int\limits_{2k\pi}^{(2k+1)\pi} \sin^{(p-2)} tdt = \\
&= \left(\frac{p-1}{p}\right)^p \frac{1}{p-1}\left(\frac{1}{C}\right)^{p-1}(2p+\lambda+1)^p \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}.
\end{aligned}
\tag{4.2}
$$

Now, we have three cases depending on the value of p:

When $p > 1$ and $p \ne 2$ choosing suitable K value in (4.2), the conditions in Theorem 3.3 are satisfied, hence any solution $y(t)$ of Eq. (4.1) such that $y'(t)$ is bounded, is oscillatory.

When $p = 2$, choosing suitable K value in (4.2), the conditions in Theorem 3.4 are satisfied, hence any solution $y(t)$ of Eq. (4.1) is oscillatory. This case is already given in [13].

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Received 30.10.13