UDC 517.9

WONG'S OSCILLATION THEOREM FOR SECOND-ORDER DELAY DIFFERENTIAL EQUATIONS

ТЕОРЕМА ВОНГА ПРО ОСЦИЛЯЦІЇ ДЛЯ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ З ЗАПІЗНЕННЯМ

A. Özbekler

Atilim Univ.
06836, Incek, Ankara, Turkey
and
Texas A& M Univ. Kingsville
700 University Blvd., Kingsville, TX 78363-8202, USA
e-mail: aozbekler@gmail.com
abdullah.ozbekler@atilim.edu.tr

A. Zafer

Middle East Technical Univ.
06531, Ankara, Turkey
and
Amer. Univ. Middle East
College Engineering and Technology
Block 3, Egaila, Kuwait
e-mail: zafer@metu.edu.tr

Let

$$H(t) := \int\limits_a^t \frac{1}{r(s)z^2(s)} \left(\int\limits_a^s z(k)f(k)dk \right) ds,$$

where z is a positive solution of

$$(r(t)x')' + q(t)x = 0, \quad t \geq a,$$

satisfying

$$\int\limits_a^\infty \frac{1}{r(s)z^2(s)}\, ds < \infty.$$

*It is well known that, see [J. S. W. Wong, J. Math. Anal. and Appl. — 1999. — **231**. — P. 235 – 240], if*

$$\overline{\lim_{t\to\infty}}\, H(t) = -\underline{\lim_{t\to\infty}}\, H(t) = \infty,$$

then every solution of

$$(r(t)x')' + q(t)x = f(t)$$

is oscillatory.

In this paper we extend Wong's result to delay differential equations of the form

$$(r(t)x'(t))' + q(t)x(\tau(t)) = f(t).$$

It is observed that the oscillation behavior may be altered due to presence of the delay. Extensions to Emden–Fowler type delay differential equations are also discussed.

Нехай

$$H(t) := \int_a^t \frac{1}{r(s)z^2(s)} \left(\int_a^s z(k)f(k)dk \right) ds,$$

де z — *додатний розв'язок рівняння*

$$(r(t)x')' + q(t)x = 0, \quad t \geq a,$$

що задовольняє умову

$$\int_a^\infty \frac{1}{r(s)z^2(s)}\, ds < \infty.$$

Відомо (див. [J. S. W. Wong, J. Math. Anal. and Appl. — 1999. — ***231****. — P. 235–240]), що якщо*

$$\overline{\lim_{t\to\infty}}\, H(t) = -\underline{\lim_{t\to\infty}}\, H(t) = \infty,$$

то кожен розв'язок рівняння

$$(r(t)x')' + q(t)x = f(t)$$

є осцилюючим.

У цій статті результат Вонга поширено на диференціальні рівняння з запізненням вигляду

$$(r(t)x'(t))' + q(t)x(\tau(t)) = f(t).$$

Встановлено, що осциляційна поведінка може змінюватись за рахунок запізнення. Також розглянуто узагальнення рівнянь типу Емдена–Фаулера.

1. Introduction. We begin with a well-known theorem by Wong [14].

Theorem 1. *Let* z *be a positive solution of the homogeneous equation*

$$(r(t)x')' + q(t)x = 0, \quad t \geq a,$$

such that

$$\int_a^\infty \frac{1}{r(s)z^2(s)}\, ds < \infty.$$

If

$$\overline{\lim_{t\to\infty}}\, H(t) = -\underline{\lim_{t\to\infty}}\, H(t) = \infty,$$

where

$$H(t) := \int_a^t \frac{1}{r(s)z^2(s)} \left(\int_a^s z(r)f(r)\, dr \right) ds,$$

then every solution of

$$(r(t)x')' + q(t)x = f(t)$$

is oscillatory.

For some extensions of this theorem to impulsive differential equations, dynamic equations on time scales, and nonlinear differential equations, see [8, 9, 15], respectively.

The main purpose of this study is to obtain an oscillation theorem analogous to Theorem 1 for second-order delay differential equations of the form

$$(r(t)x'(t))' + q(t)x(\tau(t)) = f(t), \quad t \geq t_0, \tag{1}$$

where $t_0 \geq 0, r \in C([t_0,\infty),(0,\infty)), q \in C([t_0,\infty),[0,\infty)), f,\tau \in C([t_0,\infty),\mathbb{R}), \tau(t) \leq t$, and $\lim_{t\to\infty}\tau(t) = \infty$.

It turns out that the delay term may cause solutions to oscillate. For instance, we will see that the equation

$$(t^{5/2}x'(t))' + \frac{1}{2}\,t^{1/2}x(\lambda t) = t^{\alpha}\sin(\ln t), \quad \alpha \in \left(-\frac{1}{2}, 0\right),$$

is oscillatory when $\lambda \in (0,1)$ but nonoscillatory if $\lambda = 1$.

Similar to [14] the main assumption is the existence of a nonoscillatory solution of

$$(r(t)x')' + q(t)x = 0, \quad t \geq a. \tag{2}$$

It turns out that a nonpricipal solution sufficies for our purpose. Recall that a nontrivial solution u of Eq. (2) is called principal if for every solution v of Eq. (2), linearly independent of u, one has

$$\lim_{t\to\infty}\frac{u(t)}{v(t)} = 0.$$

Such a solution v is called a nonprincipal solution of Eq. (2). It is known that if Eq. (2) is nonoscillatory, then the principal and nonprincipal solutions exist, and that a principal solution u is unique up to a multiplicative constant. For other characterizations of these solutions, see [5, 6]. Note that the function z employed in Theorem 1 is indeed a nonprincipal solution of Eq. (2). The principal and nonprincipal solutions play important role for the investigation of oscillation and asymptotic behavior of solutions of some related equations [1–14].

As usual it is tacitly assumed that Eq. (1) has a solution $x(t)$ defined on an interval $[t_0,\infty)$ and nontrivial on $[T,\infty)$ for any $T \geq t_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, otherwise it is said to be *nonoscillatory*. Eq. (1) is called oscillatory (nonoscillatory) if all solutions are oscillatory (nonoscillatory).

2. Main result. Let v denote a nonprincipal solution of Eq. (2) which is positive for $t \geq a$. Note that

$$\int\limits_a^{\infty}\frac{ds}{r(s)v^2(s)} < \infty. \tag{3}$$

The main result is the following theorem.

Theorem 2. *Let $v(t)$ be a positive solution of (2) satisfying (3), i.e., a nonprincipal solution. Define*

$$\mathcal{H}_0(t) := \int\limits_a^t \frac{1}{r(s)v^2(s)}\int\limits_a^s q(k)v(k)\int\limits_{\tau(k)}^k \frac{1}{r(\eta)}\,d\eta\,dk\,ds$$

and

$$\mathcal{H}(t) := \int\limits_a^t \frac{1}{r(s)v^2(s)} \int\limits_a^s \left\{ f(k) + q(k) \int\limits_{\tau(k)}^k \frac{1}{r(\eta)} \int\limits_a^\eta f(\lambda)\, d\lambda\, d\eta \right\} v(k)\, dk\, ds.$$

If

$$\lim_{t\to\infty} \mathcal{H}_0(t) < \infty, \quad \overline{\lim_{t\to\infty}}\, \mathcal{H}(t) = -\underline{\lim_{t\to\infty}}\, \mathcal{H}(t) = \infty, \tag{4}$$

then Eq. (1) is oscillatory.

Proof. Suppose that there is a nonoscillatory solution $x(t)$ of Eq. (1). We may assume that $x(\tau(t)) > 0$ on $[a, \infty)$ for some $a \geq t_0$ sufficiently large.

Put

$$w(t) = \frac{x(t)}{v(t)}, \quad t \geq a.$$

It follows from (1) that w satisfies

$$(r(t)w'(t)v(t))' + r(t)w'(t)v'(t) = f(t) + q(t) \int\limits_{\tau(t)}^t x'(s)\, ds. \tag{5}$$

Integrating (1) from a to t, we see that

$$x'(t) \leq \frac{\alpha}{r(t)} + \frac{1}{r(t)} \int\limits_a^t f(s)\, ds, \quad t \geq a,$$

where $\alpha = r(a)x'(a)$. Using this estimate in (5) leads to

$$(r(t)w'(t)v(t))' + r(t)w'(t)v'(t) \leq f(t) + \alpha q(t) \int\limits_{\tau(t)}^t \frac{ds}{r(s)} + q(t) \int\limits_{\tau(t)}^t \frac{1}{r(s)} \int\limits_a^s f(k)\, dk\, ds. \tag{6}$$

Multiplying both sides of (6) by $v(t)$ gives

$$(r(t)v^2(t)w'(t))' \leq f(t)v(t) + \alpha q(t)v(t) \int\limits_{\tau(t)}^t \frac{ds}{r(s)} + q(t)v(t) \int\limits_{\tau(t)}^t \frac{1}{r(s)} \int\limits_a^s f(k)\, dk\, ds. \tag{7}$$

It is not difficult to see from (7) that

$$w(t) \leq \beta + \gamma \int\limits_a^t \frac{ds}{r(s)v^2(s)} + \alpha \mathcal{H}_0(t) + \mathcal{H}(t),$$

where $\beta = w(a)$ and $\gamma = r(a)v^2(a)w'(a)$ are constants. Thus, in view of (4), from

$$\varliminf_{t\to\infty} w(t) \le \beta + \gamma \int\limits_a^\infty \frac{ds}{r(s)v^2(s)} + \alpha \lim_{t\to\infty} \mathcal{H}_0(t) + \varliminf_{t\to\infty} \mathcal{H}(t)$$

we have

$$\varliminf_{t\to\infty} w(t) = -\infty. \tag{8}$$

Obviously, (8) contradicts the positivity of the nonprincipal solution $v(t)$.

In case $x(t)$ is eventually negative, a similar argument reveals that $\varlimsup_{t\to\infty} w(t) = \infty$, which again results in the same contradiction.

Theorem 2 is proved.

Remark 1. If $\tau(t) \equiv 0$, then Theorem 2 reduces to Theorem 1.

Example 1. Consider the delay differential equation

$$(t^{5/2}x'(t))' + \frac{1}{2}\, t^{1/2}x(\lambda t) = t^\alpha \sin(\ln t), \quad t \ge 1, \tag{9}$$

where $\alpha \in \left(-\frac{1}{2}, 0\right)$ and $\lambda \in (0,1)$.

Let $v(t) = t^{-1/2}$ and $a = 1$, then we calculate that

$$\mathcal{H}_0(t) = \frac{1}{2} \int\limits_1^t s^{-3/2} \int\limits_1^s \int\limits_{\lambda\sigma}^\sigma k^{-5/2}\, dk\, d\sigma\, ds = \frac{2}{3}\, (1 - \lambda^{-3/2})(t^{-1/2} - t^{-1} - 1).$$

By tedious calculations, we also see that

$$\mathcal{H}(t) = \{k_1 \sin(\ln t) + k_2 \cos(\ln t)\}\, t^{\alpha+1/2} + \{k_3 \sin(\ln t) + k_4 \cos(\ln t)\}\, t^\alpha + (k_5 \ln t + k_6)t^{-1/2} + k_7,$$

where $k_1, \ldots, k_7$ are real suitable constants. It is clear that

$$\lim_{t\to\infty} \mathcal{H}_0(t) = -\frac{2}{3}\, (1 - \lambda^{-3/2}) < \infty, \quad \varlimsup_{t\to\infty} \mathcal{H}(t) = \infty, \quad \varliminf_{t\to\infty} \mathcal{H}(t) = -\infty.$$

Thus, condition (4) is also satisfied. By Theorem 2, we may conclude that equation (9) is oscillatory.

Note that if the delay is absent, then (9) takes the form

$$(t^{5/2}x')' + \frac{1}{2}\, t^{1/2}x = t^\alpha \sin(\ln t), \quad t \ge 1,$$

and this equation has the nonoscillatory solution $x(t) = t^{\alpha-1/2}\sin(\ln t)$.

3. Nonlinear delay equations. In this section we extend our result to Emden – Fowler type delay differential equations of the form

$$(r(t)x'(t))' + q(t)|x(\tau(t))|^{\beta-1}x(\tau(t)) = f(t), \quad \beta > 1, \tag{10}$$

and

$$(r(t)y'(t))' - p(t)|y(\sigma(t))|^{\gamma-1}y(\sigma(t)) = g(t), \quad 0 < \gamma < 1, \tag{11}$$

where r, q, f, and τ are as defined previously, $p \in C([t_0, \infty), [0, \infty))$, σ, $g \in C([t_0, \infty), \mathbb{R})$, $\sigma(t) \le t$, and $\lim_{t\to\infty} \sigma(t) = \infty$.

Following [9] we consider slightly more general equations

$$(r(t)x'(t))' + q(t)F(x(\tau(t))) = f(t) \tag{12}$$

and

$$(r(t)y'(t))' - p(t)G(y(\sigma(t))) = g(t), \tag{13}$$

where F and G are continuous functions satisfying the following conditions:

(C_1) $uF(u) > 0$ and $uG(u) > 0$ for $u \ne 0$;

(C_2) (a) $\lim_{|u|\to\infty} u^{-1}F(u) > 1$, $\lim_{|u|\to 0} u^{-1}F(u) < 1$,

(b) $\lim_{|u|\to\infty} u^{-1}G(u) < 1$, $\lim_{|u|\to 0} u^{-1}G(u) > 1$.

Using (C_1) and (C_2), it is easy to see that there exist positive constants ρ_0, α_0, δ_0, μ_0 such that

$$\begin{gathered} \max_{u\ge 0}[u - F(u)] = \rho_0, \quad \min_{u\le 0}[u - F(u)] = -\alpha_0, \\ \max_{u\le 0}[u - G(u)] = \delta_0, \quad \min_{u\ge 0}[u - G(u)] = -\mu_0. \end{gathered} \tag{14}$$

The proofs of the theorems below are similar to the counterparts in [9] and the arguments developed in the previous section. We omit the proofs.

Theorem 3. *Let $v(t)$ be a positive solution of (2) satisfying (3) and $\mathcal{H}_0$ be as in Theorem 2. Define*

$$\mathcal{H}_1(t) := \int_a^t \frac{1}{r(s)v^2(s)} \int_a^s \{f(k) + [\mathcal{R}(k) + \rho_0]q(k)\} v(k)\, dk\, ds$$

and

$$\mathcal{H}_2(t) := \int_a^t \frac{1}{r(s)v^2(s)} \int_a^s \{f(k) + [\mathcal{R}(k) - \alpha_0]q(k)\} v(k)\, dk\, ds,$$

where the constants ρ_0, α_0 are defined as in (14) and

$$\mathcal{R}(t) := \int_{\tau(t)}^t \frac{1}{r(s)} \int_a^s f(k)\, dk\, ds.$$

If

$$\lim_{t\to\infty} \mathcal{H}_0(t) < \infty, \quad \overline{\lim_{t\to\infty}}\, \mathcal{H}_j(t) = -\underline{\lim_{t\to\infty}}\, \mathcal{H}_j(t) = \infty, \quad j = 1, 2,$$

then Eq. (12) is oscillatory.

When $F(x) = |x|^{\beta-1}x, \beta > 1$, (12) reduces to the Emden – Fowler superlinear delay equation (10). Then, we have the following result.

Theorem 4. *Let $v(t)$ be a positive solution of (2) satisfying (3). If*

$$\lim_{t\to\infty} \mathcal{H}_0(t) < \infty, \quad \overline{\lim_{t\to\infty}}\, \mathcal{H}_+(t) = - \underline{\lim_{t\to\infty}}\, \mathcal{H}_-(t) = \infty,$$

where

$$\mathcal{H}_\pm(t) := \int_a^t \frac{1}{r(s)v^2(s)} \int_a^s \Big\{ f(k) + \Big[\mathcal{R}(k) \pm (\beta-1)\beta^{\beta/(1-\beta)}\Big] q(k) \Big\} v(k)\, dk\, ds,$$

then (10) is oscillatory.

Remark. If we take $\tau(t) \equiv t$ in (10), then Theorem 4 reduces to Corollary 2.3 in [9].

Theorem 5. *Let*

$$\widetilde{\mathcal{H}}_0(t) := \int_a^t \frac{1}{r(s)\tilde{v}^2(s)} \int_a^s p(k)\tilde{v}(k) \int_{\sigma(k)}^k \frac{1}{r(\eta)}\, d\eta\, dk\, ds$$

and that

$$\widetilde{\mathcal{H}}_2(t) := \int_a^t \frac{1}{r(s)\tilde{v}^2(s)} \int_a^s \Big\{ g(k) - [\widetilde{\mathcal{R}}(k) - \mu_0]p(k) \Big\} \tilde{v}(k)\, dk\, ds,$$

$$\widetilde{\mathcal{H}}_3(t) := \int_a^t \frac{1}{r(s)\tilde{v}^2(s)} \int_a^s \Big\{ g(k) - [\widetilde{\mathcal{R}}(k) + \delta_0]p(k) \Big\} \tilde{v}(k)\, dk\, ds,$$

where $\tilde{v}(t)$ is a nonprincipal solution of

$$(r(t)y')' - p(t)y = 0,$$

the constants μ_0, δ_0 are as in (14), and

$$\widetilde{\mathcal{R}}(t) := \int_{\sigma(t)}^t \frac{1}{r(s)} \int_a^s g(k)\, dk\, ds.$$

If

$$\lim_{t\to\infty} \widetilde{\mathcal{H}}_0(t) < \infty \tag{15}$$

and

$$\overline{\lim_{t\to\infty}}\, \widetilde{\mathcal{H}}_j(t) = - \underline{\lim_{t\to\infty}}\, \widetilde{\mathcal{H}}_j(t) = \infty, \quad j = 2, 3,$$

then Eq. (13) is oscillatory.

If $G(y) = |y|^{\gamma-1}y$, $0 < \gamma < 1$, then (13) reduces to the Emden – Fowler sublinear delay equation (11). The following theorem is immediate.

Theorem 6. *Let (15) hold. If*

$$\overline{\lim_{t\to\infty}}\, \widetilde{\mathcal{H}}_+(t) = -\underline{\lim_{t\to\infty}}\, \widetilde{\mathcal{H}}_-(t) = \infty,$$

where

$$\widetilde{\mathcal{H}}_\pm(t) := \int_a^t \frac{1}{r(s)\tilde{v}^2(s)} \int_a^s \left\{ g(k) - \left[\widetilde{\mathcal{R}}(k) \pm (1-\gamma)\gamma^{\gamma/(1-\gamma)} \right] q(k) \right\} \tilde{v}(k)\, dk\, ds,$$

then (11) is oscillatory.

Remark 3. If we take $\sigma(t) \equiv t$ in (11), then Theorem 6 reduces to Corollary 2.4 in [9].

References

1. *Agarwal R. P., Grace S. R., O'Regan D.* Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations. — Dordrecht: Kluwer Acad. Publ., 2002.
2. *Chen S.* Asymptotic integrations of nonoscillatory second order differential equations // Trans. Amer. Math. Soc. — 1991. — **327**. — P. 853 – 865.
3. *Došlý O.* Principal solutions and transformations of linear Hamiltonian systems (English summary) // Arch. Math. (Brno). — 1992. — **28**. — P. 113 – 120.
4. *Došlý O., Řehák P.* Half-linear differential equations. — Heidelberg: Elsevier Ltd., 2005.
5. *Hartman P.* Ordinary differential equations. — Philadelphia: SIAM, 2002.
6. *Kelley W., Peterson A.* The theory of differential equations classical and qualitative. — New Jersey: Pearson Education Inc., 2004.
7. *Morse M., Leighton W.* Singular quadratic functionals // Trans. Amer. Math. Soc. — 1936. — **40**. — P. 252 – 286.
8. *Özbekler A., Zafer A.* Principal and nonprincipal solutions of impulsive differential equations with applications // Appl. Math. and Comput. — 2010. — **216**. — P. 1158 – 1168.
9. *Özbekler A., Wong J. S. W., Zafer A.* Forced oscillation of second-order nonlinear differential equations with positive and negative coefficients // Appl. Math. Lett. — 2011. — **24**. — P. 1225 – 1230.
10. *Trench W. F.* Linear perturbation of a nonoscillatory second order differential equation // Proc. Amer. Math. Soc. — 1986. — **97**. — P. 423 – 428.
11. *Wong J. S. W.* On the generalized Emden – Fowler equation // SIAM Rev. — 1975. — **17**. — P. 339 – 360.
12. *Wong J. S. W.* Second order nonlinear forced oscillations // SIAM J. Math. Anal. — 1988. — **19**. — P. 667 – 675.
13. *Wong J. S. W.* An oscillation theorem for second order sublinear differential equations // Proc. Amer. Math. Soc. — 1990. — **110**. — P. 633 – 637.
14. *Wong J. S. W.* Oscillation criteria for forced second-order linear differential equation // J. Math. Anal. and Appl. — 1999. — **231**. — P. 235 – 240.
15. *Zafer A.* On oscillation and nonoscillation of second order dynamic equations // Appl. Math. Lett. — 2009. — **22**. — P. 136 – 141.

Received 13.10.15