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# REPRESENTATION OF SOLUTIONS OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAYS AND LINEAR PARTS GIVEN BY NONPERMUTABLE MATRICES*

# ЗОБРАЖЕННЯ РОЗВ'ЯЗКІВ СИСТЕМ ЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З БАГАТЬМА ЗАПІЗНЕННЯМИ ТА ЛІНІЙНИМИ ЧАСТИНАМИ, ВИЗНАЧЕНИМИ НЕКОМУТУЮЧИМИ МАТРИЦЯМИ

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*In recent results on representation of solutions of systems of delayed differential equations the condition that the linear parts are given by pairwise permutable matrices was assumed. In this paper it is shown how this strong condition can be avoided, and representation of solutions of systems of differential equations with nonconstant coefficients and variable delays is derived. The results are applied to a system with two constant delays. Also the nonexistence of blow-up solutions is proved for nonlinear systems.*

*Останні результати про зображення розв'язків систем диференціальних рівнянь із запізненням були отримані за умови комутування матриць, що визначають лінійні частини. У статті показано як можна позбутися цієї сильної умови та отримано зображення розв'язків систем диференціальних рівнянь з несталими коефіцієнтами та запізненнями, що змінюються. Ці результати застосовано до системи зі сталими запізненнями. Також для нелінійних систем доведено відсутність вибухових розв'язків.*

**1. Introduction and preliminaries.** Systems of linear differential equations with one or multiple constant delays were considered in [5, 6] and solutions were represented using matrix polynomials of a time-dependent degree. In the case of multiple delays, pairwise permutability of matrices representing coefficients of linear terms was a necessary condition for deriving the representation. That means that the matrices $A, B_1, \ldots, B_n$ in the equation

$$\dot{x}(t) = Ax(t) + B_1 x(t-\tau_1) + \ldots + B_n x(t-\tau_n) + f(t), \quad t \geq 0,$$

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had to satisfy $AB_i = B_iA$, $B_iB_j = B_jB_i$ for each $i, j = 1, \ldots, n$. For now the most general case with nonconstant coefficients and variable delays was investigated in [9]. We recall this result.

**Theorem 1.1.** *Let* $n \in \mathbb{N}$, $B_i \in C([0,\infty), L(\mathbb{R}^N, \mathbb{R}^N))$, $g_i \in G^0$ *for* $i = 1, \ldots, n$ *and*

$$G^s := \{g \in C([s,\infty), \mathbb{R}) \mid g(t) < t \ \text{ on } \ [s,\infty), g \ \text{ is increasing}\},$$

$f \in C([0,\infty), \mathbb{R}^N)$, $\gamma := \min\{g_1(0), \ldots, g_n(0)\}$, $\varphi \in C([\gamma, 0], \mathbb{R}^N)$. *Then the solution of the equation*

$$\dot{x}(t) = B_1(t)x(g_1(t)) + \ldots + B_n(t)x(g_n(t)) + f(t), \quad t \geq 0, \tag{1.1}$$

*satisfying the initial condition*

$$x(t) = \varphi(t), \quad \gamma \leq t \leq 0, \tag{1.2}$$

*has the form*

$$x(t) = \begin{cases} \varphi(t), & \gamma \leq t < 0, \\ X_{g_1,\ldots,g_n}^{B_1,\ldots,B_n}(t,0)\varphi(0) + \displaystyle\int_0^t X_{g_1,\ldots,g_n}^{B_1,\ldots,B_n}(t,s)\times & \\ \times\left[B_1(s)\psi(g_1(s)) + \ldots + B_n(s)\psi(g_n(s))\right] ds+ & \\ + \displaystyle\int_0^t X_{g_1,\ldots,g_n}^{B_1,\ldots,B_n}(t,s)f(s)ds, & 0 \leq t, \end{cases} \tag{1.3}$$

*where*

$$\psi(t) = \begin{cases} \varphi(t), & t \in [\gamma, 0), \\ \theta, & t \notin [\gamma, 0), \end{cases} \tag{1.4}$$

*with the $N$-dimensional vector $\theta$ of zeros, and*

$$X_{g_1,\ldots,g_m}^{B_1,\ldots,B_m}(t,s) := \begin{cases} \Theta, & t < s, \\ Y(t,s), & s \leq t < g_m^{-1}(s), \\ Y(t,s) + \displaystyle\int_{g_m^{-1}(s)}^t Y(t,q_1)B_m(q_1)Y(g_m(q_1),s)dq_1 + \ldots & \\ \ldots + \displaystyle\int_{g_m^{-k}(s)}^t Y(t,q_1)B_m(q_1)\times & \\ \times \displaystyle\int_{g_m^{-(k-1)}(s)}^{g_m(q_1)} Y(g_m(q_1),q_2)B_m(q_2)\times\ldots & \\ \ldots\times \displaystyle\int_{g_m^{-1}(s)}^{g_m(q_{k-1})} Y(g_m(q_{k-1}),q_k)\times & \\ \times B_m(q_k)Y(g_m(q_k),s)dq_k\ldots dq_1, & \\ & g_m^{-k}(s) \leq t < g_m^{-(k+1)}(s), k \in \mathbb{N}, \end{cases} \tag{1.5}$$

*where* $Y(t,s) = X^{B_1,\ldots,B_{m-1}}_{g_1,\ldots,g_{m-1}}(t,s)$, *for* $m = 2,\ldots,n$, *the* $N \times N$ *zero matrix* $\Theta$, *and*

$$X^B_g(t,s) := \begin{cases} \Theta, & t < s, \\ \mathbb{I}, & s \le t < g^{-1}(s), \\ \mathbb{I} + \int_{g^{-1}(s)}^t B(q_1)dq_1 + \ldots + \int_{g^{-k}(s)}^t B(q_1) \times & \\ \times \int_{g^{-(k-1)}(s)}^{g(q_1)} B(q_2)\ldots \int_{g^{-1}(s)}^{g(q_{k-1})} B(q_k)dq_k\ldots dq_1, & \\ & g^{-k}(s) \le t < g^{-(k+1)}(s), \quad k \in \mathbb{N}. \end{cases}$$

Note that the pairwise permutability of $B_1(t),\ldots,B_n(t)$ was not needed. However, a result from [9] on the equation involving a nondelayed term on the right-hand side was stated only for the case of pairwise permutable constant coefficients. In the present paper we show how to tackle this problem without permutability condition and even for nonconstant coefficients. So this is the most general case for the delay functions from the class $G^0$.

To illustrate our results, we consider a system of delayed differential equations with constant coefficients and two constant delays in Section 3, and derive a formula for its solution. A known result for permutable matrices [10] (see also [3]) is obtained as a particular case.

In the final section we apply results of Section 2 to prove a criterion on a nonexistence of blow-up solutions for differential equations with variable coefficients and nonconstant delays. By such a solution we understand a function $x\colon [a,b) \to \mathbb{R}^N$ with $a \in \mathbb{R}$, $a < b \le \infty$ such that $\lim_{t\to T^-} \|x(t)\| = \infty$ for some $a < T < b$ and a vector norm $\|\cdot\|$. We note that the main result of this section (Theorem 4.1) is a generalization of a weaker result from [7].

Further applications of results of this paper can be achieved, e.g., in stability theory [6–9] or Fredholm boundary-value problems [2].

**2. Representation of solutions of general time dependent systems.** In this section we shall investigate systems of differential equations with multiple variable delays. For the simplicity we start to consider the equation with constant coefficients,

$$\dot{x}(t) = Ax(t) + B_1x(g_1(t)) + \ldots + B_nx(g_n(t)) + f(t), \quad t \ge 0. \tag{2.1}$$

**Theorem 2.1.** *Let* $n \in \mathbb{N}$, $A$, $B_i$, $i = 1,\ldots,n$, *be* $N \times N$ *matrices,* $g_i \in G^0$ *for* $i = 1,\ldots,n$, $\gamma := \min\{g_1(0),\ldots,g_n(0)\}$, $f \in C([0,\infty),\mathbb{R}^N)$ *and* $\varphi \in C([\gamma,0],\mathbb{R}^N)$ *be given functions. Then the solution of the initial value problem (2.1), (1.2) has the form*

$$x(t) = \begin{cases} \varphi(t), & \gamma \le t < 0, \\ \widetilde{X}(t,0)\varphi(0) + \int_0^t \widetilde{X}(t,s)[B_1\psi(g_1(s)) + \ldots & \\ \ldots + B_n\psi(g_n(s))]ds + \int_0^t \widetilde{X}(t,s)f(s)ds, & 0 \le t, \end{cases}$$

*for* $\psi$ *given by (1.4), and*

$$\widetilde{X}(t,s) = e^{At}X^{\widetilde{B}_1,\ldots,\widetilde{B}_n}_{g_1,\ldots,g_n}(t,s)e^{-As}$$

*for* $X^{B_1,\ldots,B_n}_{g_1,\ldots,g_n}(t,s)$ *defined by (1.5), and* $\widetilde{B}_i(t) = e^{-At}B_ie^{Ag_i(t)}$ *for* $i = 1,\ldots,n$.

***Proof.*** Let $x(t) = e^{At}y(t)$. Then $y(t)$ satisfies

$$\begin{aligned} \dot{y}(t) &= \widetilde{B}_1(t)y(g_1(t)) + \ldots + \widetilde{B}_n(t)y(g_n(t)) + \tilde{f}(t), \quad t \geq 0, \\ y(t) &= \widetilde{\varphi}(t), \quad \gamma \leq t \leq 0, \end{aligned} \tag{2.2}$$

for $\tilde{f}(t) = e^{-At}f(t)$, $\widetilde{\varphi}(t) = e^{-At}\varphi(t)$. Theorem 1.1 implies

$$y(t) = \begin{cases} \widetilde{\varphi}(t), & \gamma \leq t < 0, \\ X_{g_1,\ldots,g_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n}(t,0)\widetilde{\varphi}(0) + \int_0^t X_{g_1,\ldots,g_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n}(t,s)\times & \\ \times\left[\widetilde{B}_1(s)\widetilde{\psi}(g_1(s)) + \ldots + \widetilde{B}_n(s)\widetilde{\psi}(g_n(s))\right]ds + & \\ + \int_0^t X_{g_1,\ldots,g_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n}(t,s)\tilde{f}(s)ds, & 0 \leq t, \end{cases} \tag{2.3}$$

where

$$\widetilde{\psi}(t) = \begin{cases} \widetilde{\varphi}(t), & t \in [\gamma, 0), \\ \theta, & t \notin [\gamma, 0). \end{cases} \tag{2.4}$$

Note that $\widetilde{\varphi}(0) = \varphi(0)$, and $\widetilde{\psi}(g_i(t)) = e^{-Ag_i(t)}\psi(g_i(t))$ for $i = 1, \ldots, n$. The statement is obtained when one returns back to $x(t)$.

Now, we turn to variable coefficients. So we shall consider the equation

$$\dot{x}(t) = A(t)x(t) + B_1(t)x(g_1(t)) + \ldots + B_n(t)x(g_n(t)) + f(t), \quad t \geq 0. \tag{2.5}$$

**Theorem 2.2.** *Let $n \in \mathbb{N}$, $A, B_i \in C([0,\infty), L(\mathbb{R}^N, \mathbb{R}^N))$, $g_i \in G^0$ for $i = 1, \ldots, n$, $\gamma := \min\{g_1(0), \ldots, g_n(0)\}$, $f \in C([0,\infty), \mathbb{R}^N)$ and $\varphi \in C([\gamma, 0], \mathbb{R}^N)$ be given functions. Then the solution of the initial value problem (2.5), (1.2) has the form*

$$x(t) = \begin{cases} \varphi(t), & \gamma \leq t < 0, \\ \widetilde{X}(t,0)\varphi(0) + \int_0^t \widetilde{X}(t,s)[B_1(s)\psi(g_1(s)) + \ldots & \\ \ldots + B_n(s)\psi(g_n(s))]ds + \int_0^t \widetilde{X}(t,s)f(s)ds, & 0 \leq t, \end{cases} \tag{2.6}$$

*for $\psi$ given by (1.4), and $\widetilde{X}(t,s) = \Phi(t)X_{g_1,\ldots,g_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n}(t,s)\Phi^{-1}(s)$ for $X_{g_1,\ldots,g_n}^{B_1,\ldots,B_n}(t,s)$ defined by (1.5), $\widetilde{B}_i(t) = \Phi^{-1}(t)B_i(t)\Phi(g_i(t))$ for $i = 1, \ldots, n$ and $\Phi(t)$ is a fundamental matrix satisfying*

$$\begin{aligned} \dot{\Phi}(t) &= \overline{A}(t)\Phi(t), \quad t \geq \gamma, \\ \Phi(0) &= \mathbb{I} \end{aligned} \tag{2.7}$$

*with*

$$\overline{A}(t) = \begin{cases} A(t), & t \geq 0, \\ A(0), & t < 0, \end{cases}$$

*where $\dot{\Phi}$ at 0 is considered one-sidedly.*

***Proof.*** Let $x(t) = \Phi(t)y(t)$. Then $y(t)$ solves (2.2) with $\tilde{f}(t) = \Phi^{-1}(t)f(t), \widetilde{\varphi}(t) = \Phi^{-1}(t)\varphi(t)$ and $\widetilde{B}_i(t)$ as in the statement of the theorem. So, $y(t)$ has the form (2.3). When one returns to $x(t)$ the statement immediately follows.

***Remark 2.1.*** Note that from (1.5) it follows that $X_{g_1,\ldots,g_n}^{B_1,\ldots,B_n}(t,s)$ does not depend on the values of $B_i(q)$ on $[0, g_i^{-1}(s))$ for each $i = 1,\ldots,n$. Consequently, we can take any extension of $A(t)$ onto $[\gamma,\infty)$ instead of $\overline{A}(t)$ and the same solution $x(t)$ given by (2.6) results.

Next, we consider a particular case of matrix functions $A, B_1,\ldots,B_n$.

***Corollary 2.1.*** *Let $n \in \mathbb{N}$, $B_i \in C([0,\infty), L(\mathbb{R}^N,\mathbb{R}^N))$ for $i = 1,\ldots,n$, $Q \in C^1(\mathbb{R}, L(\mathbb{R}^N, \mathbb{R}^N))$ be a nonsingular $T$-periodic matrix with $Q(0) = \mathbb{I}$, $R$ be a constant $N\times N$ matrix, $g_i \in G^0$ for $i = 1,\ldots,n$, $\gamma := \min\{g_1(0),\ldots,g_n(0)\}$, $f \in C([0,\infty),\mathbb{R}^N)$ and $\varphi \in C([\gamma,0],\mathbb{R}^N)$ be given functions. Then the solution of the initial value problem consisting of the equation*

$$\dot{x}(t) = \left(\dot{Q}(t) + Q(t)R\right)Q^{-1}(t)x(t) + B_1(t)x(g_1(t)) + \ldots + B_n(t)x(g_n(t)) + f(t), \quad t \geq 0,$$

*and initial condition (1.2) has the form (2.6) for $\psi$ given by (1.4), $\widetilde{X}(t,s) = \Phi(t)X_{g_1,\ldots,g_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n}(t,s)\times$ $\times\Phi^{-1}(s)$ for $X_{g_1,\ldots,g_n}^{B_1,\ldots,B_n}(t,s)$ defined by (1.5),*

$$\widetilde{B}_i(t) = \Phi^{-1}(t)B_i(t)\Phi(g_i(t)), \quad i = 1,\ldots,n,$$

*and*

$$\Phi(t) = \begin{cases} Q(t)e^{Rt}, & t \geq 0, \\ e^{\left(\dot{Q}(0)+R\right)t}, & t < 0, \end{cases}$$

***Proof.*** Noting that $\Phi(t)$ is a $C^1$ function satisfying (2.7) with

$$\overline{A}(t) = \begin{cases} \left(\dot{Q}(t) + Q(t)R\right)Q^{-1}(t), & t \geq 0, \\ \dot{Q}(0) + R, & t < 0, \end{cases}$$

the statement follows from Theorem 2.2.

**3. Derivation of a formula for solutions of time independent systems.** Here, we apply our results to find a solution of a system with linear terms represented by nonpermutable constant matrices and constant delays.

Let us consider the initial value problem consisting of the equation

$$\dot{x}(t) = B_1x(t-\tau_1) + B_2x(t-\tau_2) + f(t), \quad t \geq 0, \tag{3.1}$$

and initial condition (1.2) where $\tau_1,\tau_2 > 0$, $B_1$, $B_2$ are constant $N\times N$ matrices and $f \in C([0,\infty),\mathbb{R}^N)$, $\varphi \in C([\gamma,0],\mathbb{R}^N)$ are given functions with $\gamma = -\max\{\tau_1,\tau_2\}$. The assumptions of Theorem 2.1 are satisfied with $A = \Theta$ and $g_i(t) = t-\tau_i$ for $i = 1,2$. Hence the theorem gives a solution of (3.1), (1.2). In this case, $\widetilde{X}(t,s) = X_{g_1,g_2}^{B_1,B_2}(t,s)$ and the formula for the solution can be simplified. Note that $X_{g_1}^{B_1}(\cdot,s)$ is a matrix solution of

$$\dot{X}(t) = B_1X(t-\tau_1), \quad t \geq s,$$

$$X(s) = \mathbb{I}.$$

Thus we can use [5] to write

$$X_{g_1}^{B_1}(t,s) = \sum_{i=0}^{\left\lfloor \frac{t-s}{\tau_1} \right\rfloor} B_1^i \frac{(t-i\tau_1-s)^i}{i!} \tag{3.2}$$

with the floor function $\lfloor\cdot\rfloor$. Here we used the property of empty sum, i.e.,

$$\sum_{i=a}^{b} z(i) = \sum_{i\in\varnothing} z(i) = 0$$

for any function $z$ provided that $a > b$. We shall compute $X_{g_1,g_2}^{B_1,B_2}(t,s)$ for $s + k\tau_2 \leq t < s+(k+1)\tau_2$, $k \in \mathbb{N}$. In the following we use the step function

$$\sigma(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

Let $1 \leq K \leq k$ be an arbitrary fixed integer to extend the sums to infinity. Then

$$\begin{aligned} X_K(t,s) := \int\limits_{s+K\tau_2}^{t} X_{g_1}^{B_1}(t,q_1)B_2 \int\limits_{s+(K-1)\tau_2}^{q_1-\tau_2} X_{g_1}^{B_1}(q_1-\tau_2,q_2)B_2 \ldots \\ \ldots \int\limits_{s+\tau_2}^{q_{K-1}-\tau_2} X_{g_1}^{B_1}(q_{K-1}-\tau_2,q_K)B_2 X_{g_1}^{B_1}(q_K-\tau_2,s)dq_K \ldots dq_1 = \\ = \int\limits_{s+K\tau_2}^{t} \sum_{i_0=0}^{\infty} B_1^{i_0} \frac{(t-i_0\tau_1-q_1)^{i_0}}{i_0!} \sigma(t-i_0\tau_1-q_1)B_2 \ldots \\ \ldots \int\limits_{s+2\tau_2}^{q_{K-2}-\tau_2} \sum_{i_{K-2}=0}^{\infty} B_1^{i_{K-2}} \frac{(q_{K-2}-\tau_2-i_{K-2}\tau_1-q_{K-1})^{i_{K-2}}}{i_{K-2}!} \times \\ \times \sigma(q_{K-2}-\tau_2-i_{K-2}\tau_1-q_{K-1})B_2 \times \\ \times \int\limits_{s+\tau_2}^{q_{K-1}-\tau_2} \sum_{i_{K-1}=0}^{\infty} B_1^{i_{K-1}} \frac{(q_{K-1}-\tau_2-i_{K-1}\tau_1-q_K)^{i_{K-1}}}{i_{K-1}!} \times \\ \times \sigma(q_{K-1}-\tau_2-i_{K-1}\tau_1-q_K)B_2 \sum_{i_K=0}^{\infty} B_1^{i_K} \frac{(q_K-\tau_2-i_K\tau_1-s)^{i_K}}{i_K!} \times \\ \times \sigma(q_K-\tau_2-i_K\tau_1-s)dq_K \ldots dq_1. \end{aligned} \tag{3.3}$$

The last integral on the right-hand side is equal to

$$\sum_{i_{K-1},i_K=0}^{\infty} B_1^{i_{K-1}} B_2 B_1^{i_K} \sigma(q_{K-1} - 2\tau_2 - (i_{K-1} + i_K)\tau_1 - s)\times$$

$$\times \int\limits_{s+\tau_2+i_K\tau_1}^{q_{K-1}-\tau_2-i_{K-1}\tau_1} \frac{(q_{K-1} - \tau_2 - i_{K-1}\tau_1 - q_K)^{i_{K-1}}}{i_{K-1}!}\times$$

$$\times \frac{(q_K - \tau_2 - i_K\tau_1 - s)^{i_K}}{i_K!}\, dq_K =$$

$$= \sum_{i_{K-1},i_K=0}^{\infty} B_1^{i_{K-1}} B_2 B_1^{i_K} \frac{(q_{K-1} - 2\tau_2 - (i_{K-1} + i_K)\tau_1 - s)^{i_{K-1}+i_K+1}}{(i_{K-1} + i_K + 1)!}\times$$

$$\times\, \sigma(q_{K-1} - 2\tau_2 - (i_{K-1} + i_K)\tau_1 - s) =$$

$$= \sum_{j_{K-1}=0}^{\infty} \frac{(q_{K-1} - 2\tau_2 - j_{K-1}\tau_1 - s)^{j_{K-1}+1}}{(j_{K-1} + 1)!} \sigma(q_{K-1} - 2\tau_2 - j_{K-1}\tau_1 - s)\times$$

$$\times \sum_{j_K=0}^{j_{K-1}} B_1^{j_{K-1}-j_K} B_2 B_1^{j_K},$$

where we used the substitution $q_K = s + \tau_2 + i_K\tau_1 + \xi(q_{K-1} - 2\tau_2 - (i_{K-1} + i_K)\tau_1 - s)$ leading to a multiple of Euler beta function, and then we changed $i_{K-1} + i_K \to j_{K-1}$, $i_K \to j_K$.

Consequently, the last double integral on the right-hand side of (3.3) is equal to

$$\sum_{i_{K-2},j_{K-1}=0}^{\infty} B_1^{i_{K-2}} B_2 \sum_{j_K=0}^{j_{K-1}} B_1^{j_{K-1}-j_K} B_2 B_1^{j_K} \sigma(q_{K-2} - 3\tau_2 - (i_{K-2} + j_{K-1})\tau_1 - s)\times$$

$$\times \int\limits_{s+2\tau_2+j_{K-1}\tau_1}^{q_{K-2}-\tau_2-i_{K-2}\tau_1} \frac{(q_{K-2} - \tau_2 - i_{K-2}\tau_1 - q_{K-1})^{i_{K-2}}}{i_{K-2}!}\times$$

$$\times \frac{(q_{K-1} - 2\tau_2 - j_{K-1}\tau_1 - s)^{j_{K-1}+1}}{(j_{K-1} + 1)!}\, dq_{K-1} =$$

$$= \sum_{i_{K-2},j_{K-1}=0}^{\infty} B_1^{i_{K-2}} B_2 \sum_{j_K=0}^{j_{K-1}} B_1^{j_{K-1}-j_K} B_2 B_1^{j_K}\times$$

$$\times \frac{(q_{K-2} - 3\tau_2 - (i_{K-2} + j_{K-1})\tau_1 - s)^{i_{K-2}+j_{K-1}+2}}{(i_{K-2} + j_{K-1} + 2)!}\times$$

$$\times\, \sigma(q_{K-2} - 3\tau_2 - (i_{K-2} + j_{K-1})\tau_1 - s) =$$

$$= \sum_{j_{K-2}=0}^{\infty} \frac{(q_{K-2} - 3\tau_2 - j_{K-2}\tau_1 - s)^{j_{K-2}+2}}{(j_{K-2}+2)!} \sigma(q_{K-2} - 3\tau_2 - j_{K-2}\tau_1 - s) \times$$

$$\times \sum_{j_{K-1}=0}^{\infty} B_1^{j_{K-2}-j_{K-1}} B_2 \sum_{j_K=0}^{j_{K-1}} B_1^{j_{K-1}-j_K} B_2 B_1^{j_K}$$

for $j_{K-2} = i_{K-2} + j_{K-1}$. Analogously for other multiple integrals. Finally, the right-hand side of (3.3) is equal to

$$\sum_{i_0, j_1=0}^{\infty} B_1^{i_0} B_2 \sum_{j_2=0}^{j_1} B_1^{j_1-j_2} B_2 \ldots \sum_{j_K=0}^{j_{K-1}} B_1^{j_{K-1}-j_K} B_2 B_1^{j_K} \sigma(t - K\tau_2 - (i_0 + j_1)\tau_1 - s) \times$$

$$\times \int\limits_{s+K\tau_2+j_1\tau_1}^{t-i_0\tau_1} \frac{(t - i_0\tau_1 - q_1)^{i_0}}{i_0!} \frac{(q_1 - K\tau_2 - j_1\tau_1 - s)^{j_1+K-1}}{(j_1+K-1)!} dq_1 =$$

$$= \sum_{i_0, j_1=0}^{\infty} B_1^{i_0} B_2 \sum_{j_2=0}^{j_1} B_1^{j_1-j_2} B_2 \ldots \sum_{j_K=0}^{j_{K-1}} B_1^{j_{K-1}-j_K} B_2 B_1^{j_K} \times$$

$$\times \frac{(t - K\tau_2 - (i_0+j_1)\tau_1 - s)^{i_0+j_1+K}}{(i_0+j_1+K)!} \sigma(t - K\tau_2 - (i_0+j_1)\tau_1 - s) =$$

$$= \sum_{j_0=0}^{\left\lfloor \frac{t-K\tau_2-s}{\tau_1} \right\rfloor} \frac{(t - K\tau_2 - j_0\tau_1 - s)^{j_0+K}}{(j_0+K)!} \times$$

$$\times \sum_{j_1=0}^{j_0} B_1^{j_0-j_1} B_2 \sum_{j_2=0}^{j_1} B_1^{j_1-j_2} B_2 \ldots \sum_{j_K=0}^{j_{K-1}} B_1^{j_{K-1}-j_K} B_2 B_1^{j_K},$$

where $j_0 = i_0 + j_1$. Now changing $j_0 \to i_0$, $j_0 - j_1 \to i_1, \ldots, j_{K-1} - j_K \to i_K$ we get

$$X_K(t,s) = \sum_{i_0=0}^{\left\lfloor \frac{t-K\tau_2-s}{\tau_1} \right\rfloor} \frac{(t - K\tau_2 - i_0\tau_1 - s)^{i_0+K}}{(i_0+K)!} \sum_{i_1=0}^{i_0} B_1^{i_1} B_2 \sum_{i_2=0}^{i_0-i_1} B_1^{i_2} B_2 \ldots$$

$$\ldots \sum_{i_K=0}^{i_0-(i_1+\ldots+i_{K-1})} B_1^{i_K} B_2 B_1^{i_0-(i_1+\ldots+i_K)} = \sum_{i_0=0}^{\left\lfloor \frac{t-K\tau_2-s}{\tau_1} \right\rfloor} \frac{(t - K\tau_2 - i_0\tau_1 - s)^{i_0+K}}{(i_0+K)!} \times$$

$$\times \sum_{\substack{i_1,\ldots,i_K \geq 0 \\ i_1+\ldots+i_K \leq i_0}} B_1^{i_1} B_2 B_1^{i_2} B_2 \ldots B_1^{i_K} B_2 B_1^{i_0-(i_1+\ldots+i_K)}$$

for each $K \in \mathbb{N}$ such that $1 \leq K \leq k$ and $s + k\tau_2 \leq t < s + (k+1)\tau_2$. So, we obtain the following result.

**Proposition 3.1.** *The solution of the initial value problem (3.1), (1.2) has the form*

$$x(t) = \begin{cases} \varphi(t), & \gamma \le t < 0, \\ X(t,0)\varphi(0) + B_1 \displaystyle\int_0^{\tau_1} X(t,s)\varphi(s-\tau_1)ds + & \\ +B_2 \displaystyle\int_0^{\tau_2} X(t,s)\varphi(s-\tau_2)]ds + \int_0^t X(t,s)f(s)ds, & 0 \le t, \end{cases} \tag{3.4}$$

*where*

$$X(t,s) = \sum_{K=0}^{\left\lfloor \frac{t-s}{\tau_2} \right\rfloor} \sum_{i_0=0}^{\left\lfloor \frac{t-K\tau_2-s}{\tau_1} \right\rfloor} \frac{(t-K\tau_2-i_0\tau_1-s)^{i_0+K}}{(i_0+K)!} \times$$
$$\times \sum_{\substack{i_1,\dots,i_K \ge 0 \\ i_1+\dots+i_K \le i_0}} B_1^{i_1} B_2 B_1^{i_2} B_2 \dots B_1^{i_K} B_2 B_1^{i_0-(i_1+\dots+i_K)}.$$

***Proof.*** From the previous arguments it follows that the solution has the form

$$x(t) = \begin{cases} \varphi(t), & \gamma \le t < 0, \\ X(t,0)\varphi(0) + \displaystyle\int_0^t X(t,s)[B_1\psi(s-\tau_1) + & \\ +B_2\psi(s-\tau_2)]ds + \displaystyle\int_0^t X(t,s)f(s)ds, & 0 \le t, \end{cases}$$

with

$$X(t,s) = \sum_{K=0}^{\left\lfloor \frac{t-s}{\tau_2} \right\rfloor} X_K(t,s)$$

for $X_K(t,s)$ defined as $X_0(t,s) := X_{g_1}^{B_1}(t,s)$ of (3.2) and by (3.3) for $K = 1, \dots, \left\lfloor \dfrac{t-s}{\tau_2} \right\rfloor$. Note that $X(t,s) = \Theta$ whenever $t < s$. That gives formula (3.4).

In particular, we obtain the known result (see [10]).

***Corollary 3.1.*** *If $B_1B_2 = B_2B_1$, then the solution of (3.1), (1.2) has the form (3.4) where*

$$X(t,s) = \sum_{\substack{i,j \ge 0 \\ i\tau_1+j\tau_2 \le t-s}} B_1^i B_2^j \frac{(t-i\tau_1-j\tau_2-s)^{i+j}}{i!j!}.$$

***Proof.*** Applying

$$\sum_{\substack{i_1,\dots,i_K \ge 0 \\ i_1+\dots+i_K \le i_0}} 1 = \frac{(i_0+K)!}{i_0!K!}$$

the statement follows immediately.

**4. Nonexistence of blow-up solutions.** The results of Section 2 may be applied, e.g., in stability or controllability theory. In this section we illustrate the results by proving a nonexistence of blow-up solutions.

First we recall the following estimation from [9].

**Lemma 1.** *Let $s \in \mathbb{R}$, $n \in \mathbb{N}$, $B_i \in C([s,\infty), L(\mathbb{R}^N,\mathbb{R}^N))$ and $g_i \in G^s$ for $i = 1,\ldots,n$. Then*

$$\|X^{B_1,\ldots,B_n}_{g_1,\ldots,g_n}(t,s)\| \le \exp\left\{\int\limits_s^t \sum_{i=1}^n \|B_i(q)\|dq\right\}$$

*for any $t \ge s$.*

**Theorem 4.1.** *Let $n \in \mathbb{N}$, $A, B_i \in C([0,\infty), L(\mathbb{R}^N,\mathbb{R}^N))$, $g_i \in G^0$ for $i = 1,\ldots,n$, $\gamma :=$ $:= \min\{g_1(0),\ldots,g_n(0)\}$, $f \in C([0,\infty)\times\mathbb{R}^{(n+1)N},\mathbb{R}^N)$ and $\varphi \in C([\gamma,0],\mathbb{R}^N)$ be given functions. Let $x\colon [\gamma,b) \to \mathbb{R}^N$ with $0 < b \le \infty$ be a continuous solution of the equation*

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^n B_i(t)x(g_i(t)) + f(t,x(t),x(g_1(t)),\ldots,x(g_n(t))), \quad t \ge 0, \tag{4.1}$$

*satisfying the initial condition (1.2). If*

$$\|f(t,u_0,\ldots,u_n)\| \le \sum_{i=0}^n R_i(t)\omega_i(\|u_i\|), \quad (t,u_0,\ldots,u_n) \in \mathbb{R}\times\mathbb{R}^{(n+1)N}$$

*where $R_i$, $\omega_i$, $i = 0,\ldots,n$, are continuous nonnegative functions defined on $[0,\infty)$, and $\omega_i$, $i = 0,\ldots,n$, are nondecreasing such that $\omega_0(0) + \ldots + \omega_n(0) > 0$ and*

$$\int\limits_0^\infty \frac{du}{\omega_0(u) + \sum_{i=1}^n \omega_i(2u)} = \infty,$$

*then $\lim_{t\to T^-}\|x(t)\| < \infty$ for all $T \in (0,b)$.*

***Proof.*** Let us suppose in contrary that there exists a smallest $T \in (0,b)$ such that

$$\lim_{t\to T^-}\|x(t)\| = \infty.$$

By Theorem 2.2 and in its notation, $x(t)$ satisfies

$$x(t) = \widetilde{X}(t,0)\varphi(0) + \sum_{i=1}^n \int\limits_0^t \widetilde{X}(t,s)B_i(s)\psi(g_i(s))ds + \int\limits_0^t \widetilde{X}(t,s)F(s)ds \tag{4.2}$$

for $t \ge 0$, where $F(t) = f(t,x(t),x(g_1(t)),\ldots,x(g_n(t)))$.

Now, since

$$\Phi(t) = \mathbb{I} + \int\limits_0^t A(s)\Phi(s)ds, \quad t \ge 0,$$

Gronwall lemma [4] yields

$$\|\Phi(t)\| \leq \exp\left\{\int\limits_0^t \|A(s)\|ds\right\}, \quad t \geq 0,$$

for an induced matrix norm $\|\cdot\|$. Similarly,

$$\Phi^{-1}(t) = \mathbb{I} - \int\limits_0^t \Phi^{-1}(s)A(s)ds, \quad t \geq 0,$$

i.e.,

$$\|\Phi^{-1}(t)\| \leq \exp\left\{\int\limits_0^t \|A(s)\|ds\right\}, \quad t \geq 0.$$

Therefore, along with Lemma 4.1,

$$\|\widetilde{X}(t,s)\| \leq \exp\left\{\int\limits_0^t \|A(q)\|dq + \int\limits_s^t \sum_{i=1}^n \|\widetilde{B}_i(q)\|dq + \int\limits_0^s \|A(q)\|dq\right\} =: M(t,s)$$

for any $0 \leq s \leq t$.

Hence, denoting $\|\varphi\| := \max_{\gamma \leq t \leq 0} \|\varphi(t)\|$, from (4.2) one obtains

$$\|x(t)\| \leq M(t,0)\|\varphi\| + \sum_{i=1}^n \int\limits_0^{\min\{t, g_i^{-1}(0)\}} M(t,s)\|B_i(s)\|\|\varphi\|ds +$$

$$+ \int\limits_0^t M(t,s)\left(R_0(s)\omega_0(\|x(s)\|) + \sum_{i=1}^n R_i(s)\omega_i(\|x(g_i(s))\|)\right)ds$$

for any $t \geq 0$. Let us denote

$$m_1 := \|\varphi\| \max_{0 \leq t \leq T}\left(M(t,0) + \sum_{i=1}^n \int\limits_0^{\min\{t, g_i^{-1}(0)\}} M(t,s)\|B_i(s)\|ds\right),$$

$$m_2 := \left(\max_{\substack{0 \leq s \leq t \\ 0 \leq t \leq T}} M(t,s)\right) \max_{\substack{0 \leq t \leq T \\ i=0,\ldots,n}} R_i(s), \quad \eta(s) := \omega_0(s) + \sum_{i=1}^n \omega(2s).$$

Then the last inequality implies

$$\|x(t)\| \leq m_1 + m_2 \int\limits_0^t \omega_0(\|x(s)\|) + \sum_{i=1}^n \omega_i(\|x(g_i(s))\|)ds =: z(t)$$

for any $0 \le t \le T$. Note that the right-hand side $z(t)$ is nondecreasing, $m_1 \ge \|\varphi\|$ and

$$\|x(g_i(t))\| \le \sup_{0\le s\le t} \|x(g_i(s))\| \le \sup_{0\le s\le g_i^{-1}(0)} \|x(g_i(s))\| + \sup_{g_i^{-1}(0)\le s\le t} \|x(g_i(s))\| \le 2z(t).$$

Consequently,

$$\|x(t)\| \le z(t) \le m_1 + m_2 \int_0^t \omega_0(z(s)) + \sum_{i=1}^n \omega_i(2z(s))ds = m_1 + m_2 \int_0^t \eta(z(s))ds$$

for any $0 \le t \le T$. The well-known Bihari inequality [1] implies

$$\Omega(\|x(t)\|) \le \Omega(z(t)) \le \Omega(m_1) + m_2 t \le \Omega(m_1) + m_2 T, \quad 0 \le t \le T,$$

where $\Omega(z) = \int_0^z \frac{ds}{\eta(s)}$. A contradiction follows from the limit

$$\Omega(\|x(t)\|) \to \int_0^\infty \frac{du}{\eta(u)} = \infty, \quad t \to T^-.$$

## References


1. *Bihari I.A.* A generalization of a lemma of Bellman and its application to uniqueness problem of differential equation // Acta Math. Acad. Sci. Hung. — 1956. — **7**. — P. 81 – 94.
2. *Boichuk A. A., Medved' M., Zhuravliov V. P.* Fredholm boundary-value problems for linear delay systems defined by pairwise permutable matrices // Electron. J. Qual. Theory Different. Equat. — 2015. — **23**. — P. 1 – 9.
3. *Diblík J., Morávková B.* Representation of the solutions of linear discrete systems with constant coefficients and two delays // Abstr. Appl. Anal. — 2014. — **2014**. — Article ID 320476. — P. 1 – 19.
4. *Hartman P.* Ordinary differential equations. — New York: John Wiley & Sons, Inc., 1964.
5. *Khusainov D. Ya., Shuklin G. V.* Linear autonomous time-delay system with permutation matrices solving // Stud. Univ. Žilina, Math. Ser. — 2003. — **17**. — P. 101 – 108.
6. *Medved' M., Pospíšil M.* Sufficient conditions for the asymptotic stability of nonlinear multidelay differential equations with linear parts defined by pairwise permutable matrices // Nonlinear Anal.-Theory Methods Appl. — 2012. — **75**. — P. 3348 – 3363.
7. *Medved' M., Pospíšil M., Škripková L.* Stability and the nonexistence of blowing-up solutions of nonlinear delay systems with linear parts defined by permutable matrices // Nonlinear Anal.-Theory Methods Appl. — 2011. — **74**. — P. 3903 – 3911.
8. *Medved' M., Pospíšil M., Škripková L.* On exponential stability of nonlinear fractional multidelay integro-differential equations defined by pairwise permutable matrices // Appl. Math. and Comput. — 2014. — **227**. — P. 456 – 468.
9. *Pospíšil M.* Representation and stability of solutions of systems of functional differential equations with multiple delays // Electron. J. Qual. Theory Different. Equat. — 2012. — **54**. — P. 1 – 30.
10. *Pospíšil M., Jaroš F.* On the representation of solutions of delayed differential equations via Laplace transform // Electron. J. Qual. Theory Different. Equat. (to appear).


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