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CONTINUABILITY AND BOUNDEDNESS OF SOLUTIONS FOR A KIND OF NONLINEAR DELAY INTEGRO-DIFFERENTIAL EQUATIONS OF THIRD ORDER

НЕПЕРЕРВНІСТЬ ТА ОБМЕЖЕНІСТЬ РОЗВ'ЯЗКІВ ДЕЯКИХ НЕЛІНІЙНИХ ІНТЕГРО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ТРЕТЬОГО ПОРЯДКУ З ЗАПІЗНЕННЯМ

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This paper considers a nonlinear integro-differential equation of third order with delay. We establish sufficient conditions which guarantee the globally existence and boundedness of the solutions of the equation considered. We benefit from the Lyapunov's second method to prove the main result. An example is also given to illustrate the applicability of our result. The result of this paper is new and improves previously known results.

Вивчається нелінійне інтегро-диференціальне рівняння третього порядку з запізненням. Наведено достатні умови глобального існування та обмеженості розв'язків розглянутих рівнянь. Для доведення основного результату використовується другий метод Ляпунова. Також наведено приклад для ілюстрації отриманого результату. Отриманий результат є новим та покращує отримані раніше результати.

1. Introduction. Qualitative properties of delay differential equations of third order have been investigated in the literature by many authors. There have been obtained many interesting results on the stability, boundedness, asymptotic behaviors, periodicity, and etc. of solutions for various nonlinear delay differential equations of third order. In particular, for some works done on the stability and boundedness of solutions to certain nonlinear delay differential equations of third order, the readers can referee to Ademola and Arawomo [1], Bereketoğlu and Karakoç [4], Omeike[12], Oudjedi et al. [13], Remili and Oudjedi [14], Sadek [15], Sinha [16], Tunç [18–23], Zhu [25] and the references therein.

In this paper, we discuss the global existence and boundedness of solutions of the third order nonlinear integro-differential equation with constant delay, r :

$$\left(q(t)\left(p(t)x^{'}\right)^{'}\right)^{'}+a(t)f\left(t,x,x^{'}\right)x^{''}+b(t)g(t,x)x^{'}+c(t)h(x-r)=\int\limits_{0}^{t}C(t,s)x^{'}(s)\,ds, \quad (1)$$

where r is a positive constant, namely, r is a constant delay; p and q are positive and continuously differentiable functions on $\Re^{+}$, $\Re^{+}=[0,\infty)$; $a,b,c\in C^{1}(\Re^{+},(0,\infty))$; $f\in C(\Re^{+}\times R^{2},\Re^{+})$;

$g \in C(\Re^{+} \times \Re, \Re^{+})$; $h \in C^{1}(\Re, \Re)$ and $C(t,s)$ is countinuous for $0 \le t \le s < \infty$. Also $x^{'}$, $x^{''}$ and $x^{'''}$ denote the first, second and third derivatives of the function $x(t)$ with respect to t.

We can write equation (1) in a differential system form as

$$\begin{aligned} x' &= \frac{y}{p(t)}, \\ y' &= \frac{z}{q(t)}, \\ z' &= \int\limits_{0}^{t} C(t,s)\frac{y(s)}{p(s)}\,ds - A(t)z - B(t)y - c(t)h(x) + c(t)\int\limits_{t-r}^{t} \frac{y(s)}{p(s)}\,h^{'}(x(s))ds, \end{aligned} \tag{2}$$

where

$$A(t) = \frac{a(t)}{p(t)q(t)}\, f\left(t, x, \frac{y}{p(t)}\right)$$

and

$$B(t) = \left(\frac{p(t)b(t)g(t,x) - a(t)p^{'}(t)f\left(t, x, \dfrac{y}{p(t)}\right)}{p^{2}(t)}\right)$$

are continuous and differentiable functions.

Besides, it should be better to summarize some papers in the literature on the qualitative behaviors of nonlinear differential equations of third order with delay. Sadek [15] dealt with a nonautonomous third order differential equation with constant delay, r,

$$x^{'''} + a(t)x^{''} + b(t)x^{'} + c(t)h(x(t-r)) = 0.$$

The author investigated the asymptotic stability of the solution $x = 0$. Tunç [18–23] obtained some results on the stability and boundedness of solutions for various nonlinear differential equations of third order with delay. In 2009, Omeike [12] considered the following nonlinear differential equation of third order with a constant delay, r :

$$x^{'''} + a(t)x^{''} + b(t)g(x^{'}) + c(t)h(x(t-r)) = p(t).$$

He studied the asymptotic stability and uniform boundedness of the solutions of this equation, respectively, when $p(t) = 0$ and $p(t) \neq 0$.

Recently, Oudjedi et al. [13] gave criteria for the stability of solutions to the following third order nonlinear delay differential equation:

$$\left(q(t)\left(p(t)x^{'}\right)^{'}\right)^{'} + a(t)x^{''} + b(t)x^{'} + c(t)f(x(t-r)) = 0.$$

However, it is worth mentioning that in spite of the existence of many results on the stability and boundedness of nonlinear differential equations of third order with and without delay, to the best of our knowledge there is no result on the globally existence and boundedness of solutions of nonlinear differential and integro-differential equations of third order with and without delay. To the best of our knowledge, this paper is the first attempt on the subject in the literature to integro-differential equations. The motivation of this paper comes from the mentioned papers and the papers of Baxley [3], Changian et al. [5], Constantin [6], Graef and Tunç [8, 9], Napoles Valdes [11], Tidke and Dhakne [17]. Throughout all the mentioned papers the Lyapunov's second method is used as a basic tool to verify the results therein. The aim of this paper is to give certain sufficient conditions to ensure the global existence and boundedness of solutions of Eq. (1). We prove a result on the topic by defining an appropriate new Lyapunov functional. However, we should state that it is always difficult to find a suitable Lyapunov functional for higher order differential and integro-differential equations, which verifies the assumptions of the Lyapunov's stability theorems (see Ahmad and Rama Mohana Rao [2], Èl'sgol'ts [7], Krasovskii [10], and Yoshizawa[24]). The result obtained in this paper complements the previous ones in the literature on the qualitative properties of ordinary and functional differential equations. Besides, it improves the existing results on the third order nonlinear ordinary and functional differential equations in the literature to functional integro-differential equations of third order, and it may be useful for researchers working on the qualitative behaviors of solutions of integro-differential equations of higher order with and without delay. This is the novelty and originality of this paper.

We assume that there are positive constants a_0, a_1, δ_0, δ_1, m, n, L, M, and N such that the following conditions hold:

(A_1) $0 < m \le q(t) \le p(t) \le M$, $-L < p^{'}(t) \le q^{'}(t) \le 0$ and $p^{''}(t) \ge 0$,

(A_2) $h(0) = 0$, $\dfrac{h(x)}{x} \ge \delta_0$, $x \ne 0$, $|h'(x)| \le \delta_1$,

(A_3) $0 < a_0 < a(t) < a_1$, $a^{'}(t) \le 0$,

(A_4) $0 < n \le c(t) \le b(t) \le N$, $-N < b^{'}(t) \le c^{'}(t) \le 0$,

(A_5) $0 < f_0 \le f(t,x,y) \le f_1$, $f_t(t,x,y) \le 0$,

(A_6) $0 < g_0 \le g(t,x) \le g_1, g_t(t,x) \le 0$.

2. Main result. Our main result is the following theorem.

Theorem 1. *Suppose that conditions (A_1) – (A_6) hold. Then all solutions of system (2) are continuable and bounded provided that there exists α satisfying*

$$\frac{M}{a_0 f_0} < \alpha < \frac{g_0}{M\delta_1} \le 1$$

such that

$$d = \frac{LM^2 a_1 f_1}{m^3} < n(g_0 - \alpha M \delta_1)$$

and

$$r < \min \left(\frac{2m\left(c_0 - \left(\frac{1}{2}\int_0^t |C(t,s)|\,ds + \frac{1}{m^2}\int_t^\infty |C(u,t)|du\right)\right)}{2m\lambda + N\delta_1}, \right.$$

$$\left. \frac{2m\left(c_1 - \frac{\alpha}{2}\int_0^t |C(t,s)|ds\right)}{\alpha N \delta_1} \right),$$

where

$$c_0 = \frac{1}{M}\left(n\left(g_0 - \alpha M \delta_1\right) - d\right), \quad c_1 = \frac{1}{M}\left(\frac{\alpha a_0 f_0}{M} - 1\right), \quad \int\limits_0^t |C(t,s)|ds < \infty$$

such that

$$\int\limits_0^t |C(t,s)|ds < \frac{2c_1}{\alpha},$$

and

$$\int\limits_t^\infty |C(u,t)|du < \infty$$

and

$$\frac{1}{2}\int\limits_0^t |C(t,s)|ds + \frac{1}{m^2}\int\limits_t^\infty |C(u,t)|du < c_0.$$

Proof. To prove the theorem, we define a Lyapunov functional $V(t) = V(t, x(t), y(t), z(t))$ by

$$V(t) = V(t, x(t), y(t), z(t)) = e^{-\frac{\theta(t)}{\mu}} U(t, x(t), y(t), z(t)), \tag{3}$$

where

$$\theta(t) = \int\limits_0^t D(s)ds = \frac{\alpha M}{2}\int\limits_0^t \left(\frac{2a_1 f_1 p'^2(s) - p(s) N g_1 p'(s)}{p^3(s)} - a_2 c'(s)\right) ds \leq$$

$$\leq \alpha a_1 M f_1 \int\limits_0^t \left(-\frac{p'(s)}{p^3(s)}\right)\left(-p'(s)\right) ds + \frac{\alpha M N g_1}{2m} + \frac{\alpha a_2 M N}{2} \leq$$

$$\leq \frac{\alpha a_1 L M^2 f_1}{m^3} + \frac{\alpha M N g_1}{2m} + \frac{\alpha a_2 M N}{2} < \infty, \quad a_2 = \frac{N g_1}{nm} + \frac{L a_1 f_1}{nm^2},$$

and

$$U(t) = U(t, x(t), y(t), z(t)) = p(t)c(t)H(x) + \alpha q(t)B(t)\frac{y^2}{2} +$$

$$+ \alpha q(t)c(t)h(x)y + \frac{1}{2}\left(A(t)q(t)y^2 + \alpha z^2 + 2yz\right) +$$

$$+ \lambda \int_{-r}^{0}\int_{t+s}^{t} y^2(u)duds + \int_{0}^{t}\int_{t}^{\infty} |C(u,s)|\,\frac{y^2(s)}{p^2(s)}\,du\,ds \tag{4}$$

such that $H(x) = \int_0^x h(u)\,du$. Also μ and λ are positive constants which will be determined later. From the definition of $U(t)$ in (4), we observe that our Lyapunov functional can be rewritten as follows:

$$U(t) = U_1 + U_2 + \lambda \int_{-r}^{0}\int_{t+s}^{t} y^2(u)duds + \int_{0}^{t}\int_{t}^{\infty} |C(u,s)|\frac{y^2(s)}{p^2(s)}\,du\,ds,$$

where

$$U_1 = p(t)c(t)H(x) + \alpha q(t)B(t)\,\frac{y^2}{2} + \alpha q(t)c(t)h(x)y,$$

$$U_2 = \frac{1}{2}\left(A(t)q(t)y^2 + \alpha z^2 + 2yz\right).$$

First consider

$$U_1 = p(t)c(t)H(x) + \alpha q(t)B(t)\,\frac{y^2}{2} + \alpha q(t)c(t)h(x)y.$$

By noting the assumptions of the theorem and since $c(t)g(t,x) \le p(t)B(t)$, we have

$$U_1 = p(t)c(t)H(x) + \alpha q(t)B(t)\frac{y^2}{2} + \alpha q(t)c(t)h(x)y =$$

$$= p(t)c(t)H(x) + \frac{\alpha}{2}\,q(t)B(t)\left(y + \frac{c(t)h(x)}{B(t)}\right)^2 - \frac{\alpha q(t)c^2(t)h^2(x)}{2B(t)} \ge$$

$$\ge p(t)c(t)\int_0^x \left(1 - \frac{\alpha q(t)c(t)}{p(t)B(t)}\,h^{'}(u)\right)h(u)\,du \ge$$

$$\ge mn\int_0^x \left(1 - \frac{\alpha M\delta_1}{g_0}\right)h(u)du \ge \delta_2\delta_0\int_0^x u\,du \ge \frac{\delta_2\delta_0}{2}x^2,$$

where $\delta_2 = mn\left(1 - \dfrac{\alpha M\delta_1}{g_0}\right)$.

On the other hand, using assumptions (A_1), (A_3) and (A_5), since $\alpha > \dfrac{M}{a_0 f_0}$ we get

$$U_2 = \frac{1}{2}\left(A(t)q(t)y^2 + \alpha z^2 + 2yz\right) = \frac{\alpha}{2}\left(z + \frac{y}{\alpha}\right)^2 + \frac{1}{2}y^2\left(A(t)q(t) - \frac{1}{\alpha}\right) \geq$$

$$\geq \frac{\alpha}{2}\left(z + \frac{y}{\alpha}\right)^2 + \frac{1}{2}y^2\left(\frac{a_0 f_0}{M} - \frac{1}{\alpha}\right) \geq \delta_3 y^2 + \delta_4 z^2,$$

where δ_3 and δ_4 are sufficiently small positive constants. Thus, taking into consideration the above discussion, it follows that

$$U(t) \geq \frac{\delta_2\delta_0}{2}x^2 + \delta_3 y^2 + \delta_4 z^2 + \lambda\int\limits_{-r}^{0}\int\limits_{t+s}^{t} y^2(u)\,du\,ds + \int\limits_{0}^{t}\int\limits_{t}^{\infty}|C(u,s)|\,\frac{y^2(s)}{p^2(s)}\,du\,ds.$$

Hence, it is evident, from the terms contained in the last inequality, that there exists a sufficiently small positive constant δ_5 such that

$$U(t) \geq \delta_5\left(x^2 + y^2 + z^2\right), \tag{5}$$

where $\delta_5 = \min\left\{\dfrac{\delta_2\delta_0}{2}, \delta_3, \delta_4\right\}$. This implies that $V(t) \geq 0$.

Let $(x(t), y(t), z(t))$ be a solution of (2). Calculating the time derivative of the functional $U(t)$, along the trajectories of system (2), we obtain

$$U'(t) = (p(t)c(t))'H(x) + \frac{\alpha q'(t)B(t)}{2}\,y^2 + \alpha(q(t)c(t))'h(x)y +$$

$$+\left(\frac{\alpha q(t)B'(t)}{2} + \frac{A(t)q'(t)}{2} + G(t)\right)y^2 +$$

$$+\left(\frac{1}{q(t)} - \alpha A(t)\right)z^2 + (y + \alpha z)c(t)\int\limits_{t-r}^{t}\frac{y(s)}{p(s)}h'(x(s))\,ds +$$

$$+(y + \alpha z)\int\limits_{0}^{t}C(t,s)\,\frac{y(s)}{p(s)}\,ds + \lambda r y^2 - \lambda\int\limits_{t-r}^{t}y^2(s)\,ds +$$

$$+\frac{y^2}{p^2(t)}\int\limits_{t}^{\infty}|C(u,t)|du - \int\limits_{0}^{t}|C(t,s)|\,\frac{y^2(s)}{p^2(s)}\,du,$$

where

$$G(t) = \frac{A'(t)q(t)}{2} + \alpha\,\frac{q(t)c(t)}{p(t)}\,h'(x) - B(t),$$

and since $(q(t)c(t))^{'} = q^{'}(t)c(t) + q(t)c^{'}(t)$, we obtain the following:

$$\frac{\alpha q^{'}(t)B(t)}{2}\, y^2 = \frac{\alpha (q(t)c(t))^{'}B(t)}{2c(t)}\, y^2 - \frac{\alpha q(t)c^{'}(t)B(t)}{2c(t)}\, y^2,$$

consequently, we have

$$\begin{aligned} U^{'}(t) &= (p(t)c(t))^{'}H(x) + \frac{\alpha (q(t)c(t))^{'}B(t)}{2c(t)}\, y^2 + \alpha (q(t)c(t))^{'}h(x)y + \\ &+ \left(\frac{\alpha q(t)B^{'}(t)}{2} - \frac{\alpha q(t)c^{'}(t)B(t)}{2c(t)} + \frac{A(t)q^{'}(t)}{2} + G(t)\right) y^2 + \\ &+ \left(\frac{1}{q(t)} - \alpha A(t)\right) z^2 + (y+\alpha z)c(t)\int\limits_{t-r}^{t} \frac{y(s)}{p(s)} h^{'}(x(s))\, ds + \\ &+ (y+\alpha z)\int\limits_{0}^{t} C(t,s)\frac{y(s)}{p(s)}\, ds + \lambda r y^2 - \lambda \int\limits_{t-r}^{t} y^2(s)\, ds + \\ &+ \frac{y^2}{p^2(t)} \int\limits_{t}^{\infty} |C(u,t)| du - \int\limits_{0}^{t} |C(t,s)|\, \frac{y^2(s)}{p^2(s)}\, du. \end{aligned} \tag{6}$$

Now, we verify

$$F(t,x,y) = (p(t)c(t))^{'}H(x) + \frac{\alpha (q(t)c(t))^{'}B(t)}{2c(t)}\, y^2 + \alpha (q(t)c(t))^{'}h(x)y \le 0,$$

for all x, y and $t \ge 0$. The last estimate can be written as

$$\begin{aligned} F(t,x,y) &= (q(t)c(t))^{'}\left(\frac{(p(t)c(t))^{'}}{(q(t)c(t))^{'}} H(x) + \frac{\alpha B(t)}{2c(t)}\, y^2 + \alpha h(x) y \right) = \\ &= (q(t)c(t))^{'}\left(\frac{(p(t)c(t))^{'}}{(q(t)c(t))^{'}} H(x) + \frac{\alpha B(t)}{2c(t)} \left(y + \frac{\alpha c(t)h(x)}{B(t)}\right)^2 - \frac{\alpha c(t) h^2(x)}{2B(t)} \right), \end{aligned}$$

also by assumption (A_1) and (A_4) we obtain $\dfrac{(p(t)c(t))^{'}}{(q(t)c(t))^{'}} \ge 1$, this requires

$$F(t,x,y) \le (q(t)c(t))^{'} \int\limits_{0}^{x} \left(1 - \frac{\alpha c(t)}{B(t)}\, h^{'}(u)\right) h(u)\, du.$$

From $(\mathrm{A}_1)-(\mathrm{A}_6)$ we get $\dfrac{c(t)}{B(t)} \le \dfrac{M}{g_0}$. In the same way, we have

$$F(t,x,y) \le (q(t)c(t))' \int_0^x \left(1 - \frac{\alpha M \delta_1}{g_0}\right) h(u)du \le (q(t)c(t))' \frac{\delta_2 \delta_0}{2mn} x^2 \le 0.$$

Furthermore, using the assumptions of the theorem, we get

$$B(t) \le \frac{Ng_1}{m} + \frac{La_1 f_1}{m^2}$$

and

$$\begin{aligned} B'(t) = {} & \frac{b'(t)g(t,x) + b(t)g_t(t,x)}{p(t)} - \frac{p'(t)b(t)g(t,x)}{p^2(t)} - \\ & - \frac{a'(t)p'(t)f\left(t,x,\dfrac{y}{p(t)}\right)}{p^2(t)} - \frac{a(t)p''(t)f\left(t,x,\dfrac{y}{p(t)}\right)}{p^2(t)} - \\ & - \frac{a(t)p'(t)f_t\left(t,x,\dfrac{y}{p(t)}\right)}{p^2(t)} + \frac{2a(t)p'^2(t)f\left(t,x,\dfrac{y}{p(t)}\right)}{p^3(t)} \le \\ \le {} & \frac{2a_1 f_1 p'^2(t) - Ng_1 p(t)p'(t)}{p^3(t)}. \end{aligned}$$

Hence, it is easily seen that

$$\begin{aligned} \frac{\alpha q(t)B'(t)}{2} - \frac{\alpha q(t)c'(t)B(t)}{2c(t)} + \frac{A(t)q'(t)}{2} & \le \frac{\alpha q(t)}{2}\left(B'(t) - \frac{c'(t)B(t)}{c(t)}\right) \le \\ & \le \frac{\alpha M}{2}\left(\frac{2a_1 f_1 p'^2(t) - Ng_1 p(t)p'(t)}{p^3(t)} - a_2 c'(t)\right) = \\ & = D(t) \end{aligned}$$

and

$$\begin{aligned} G(t) = {} & \frac{A'(t)q(t)}{2} + \alpha\frac{q(t)c(t)}{p(t)}h'(x) - B(t) \le \\ & \le -\frac{(p(t)c(t))' a(t) f\left(t,x,\dfrac{y}{p(t)}\right)}{2p^2(t)q(t)} + \frac{b(t)}{p(t)}\left(\alpha\,\frac{c(t)}{b(t)}\,q(t)h'(x) - g(t,x)\right) \le \\ & \le \frac{LMa_1 f_1}{m^3} + \frac{n}{M}\,(\alpha M\delta_1 - g_0) = \frac{1}{M}\,(d + n\,(\alpha M \delta_1 - g_0)) = -c_0 < 0. \end{aligned}$$

We have also that

$$\frac{1}{q(t)} - \alpha A(t) = \frac{1}{q(t)}\,(1 - \alpha q(t)A(t)) \le \frac{1}{M}\left(1 - \frac{\alpha a_0 f_0}{M}\right) = -c_1 < 0.$$

Therefore (6) becomes

$$U^{'}(t) \le (D(t) + \lambda r - c_0)y^2 - c_1 z^2 + (|y| + \alpha|z|)N\int\limits_{t-r}^{t}\left|\frac{y(s)}{p(s)}\right|\left|h^{'}(x(s))\right|ds +$$

$$+ (|y| + \alpha|z|)\int\limits_{0}^{t}|C(t,s)|\left|\frac{y(s)}{p(s)}\right|ds - \lambda\int\limits_{t-r}^{t}y^2(s)\,ds +$$

$$+ \frac{y^2}{p^2(t)}\int\limits_{t}^{\infty}|C(u,t)|\,du - \int\limits_{0}^{t}|C(t,s)|\frac{y^2(s)}{p^2(s)}\,du.$$

Using the Schwartz inequality $|uv| \le \frac{1}{2}\left(u^2 + v^2\right)$, we get

$$U^{'}(t) \le \left(D(t) - c_0 + r\left(\lambda + \frac{N\delta_1}{2m}\right) + \left(\frac{1}{2}\int\limits_{0}^{t}|C(t,s)|\,ds + \frac{1}{m^2}\int\limits_{t}^{\infty}|C(u,t)|\,du\right)\right)y^2 +$$

$$+ \left(\frac{\alpha}{2}\int\limits_{0}^{t}|C(t,s)|\,ds + \frac{\alpha N\delta_1 r}{2m} - c_1\right)z^2 + \left(\frac{N\delta_1(1+\alpha)}{2m} - \lambda\right)\int\limits_{t-r}^{t}y^2(s)\,ds +$$

$$+ \left(\frac{\alpha - 1}{2}\right)\int\limits_{0}^{t}|C(t,s)|\,\frac{y^2(s)}{p^2(s)}\,du. \tag{7}$$

By choosing $\lambda = \dfrac{N\delta_1(1+\alpha)}{2m}$, we have from (7) that

$$U^{'}(t) \le D(t)y^2. \tag{8}$$

It is now clear that the time derivative of $V(t)$ defined by (3) along any solution of system (2) leads that

$$V^{'}(t) = e^{-\frac{\theta(t)}{\mu}}\left(-\frac{D(t)}{\mu}\,U(t,x(t),y(t),z(t)) + \frac{d}{dt}\,U(t,x(t),y(t),z(t))\right).$$

Thus by (5), (8) and taking $\mu = \delta_5$, we obtain

$$V^{'}(t) \le e^{-\frac{\theta(t)}{\mu}}\left(-\frac{D(t)}{\delta_5}\delta_5 y^2 + D(t)y^2\right) = 0.$$

This implies that $V^{'}(t) \le 0$. Since all the functions appearing in equation (1) are continuous, it is obvious that there exists at least a solution of equation (1) defined on $[t_0, t_0+\delta)$ for some $\delta > 0$. We need to show that the solution can be extended to the entire interval $[t_0, \infty)$. We suppose that on the contrary that there is a first time $T < \infty$ such that the solution exist on $[t_0, T)$ and

$$\lim_{t \to T^-} (|x(t)| + |y(t)| + |z(t)) = \infty.$$

Let $(x(t), y(t), z(t))$ be such a solution of system (2) with initial condition (x_0, y_0, z_0). Since $V(t)$ is a positive definite and decreasing functional on the trajectories of system (2), $V^{'}(t) \le 0$, we can say that $V(t)$ is bounded $[t_0, T)$, that is,

$$V(x(T), y(T), z(T)) \le V(t_0, x_0, y_0, z_0) = V_0.$$

Hence, it follows from (3) and (5) that

$$x^2(T) + y^2(T) + z^2(T) \le \frac{V_0}{K},$$

where $K = \delta_5 e^{-\frac{\theta(t)}{\mu}}$. This inequality implies that $|x(t)|$, $|y(t)|$ and $|z(t)|$ are bounded as $t \to T^-$. Thus, we can conclude that $T < \infty$ is not possible, we must have $T = \infty$.

Theorem 1 is proved.

Example. We consider the following third order nonlinear delay integro-differential equation:

$$\begin{aligned}\left(\left(1+\frac{1}{25e^t+1}\right)\left(\left(1+\frac{e^{-t}}{25}\right)x'(t)\right)'\right)' &+ \left(4+\frac{1}{2+t}\right)\left(1+\frac{e^{-t}}{1+y^2}\right)x^{''}(t)+ \\ &+ \left(8+\frac{1}{1+t}\right)\left(1+\frac{1}{1+t+x^2}\right)x^{'}(t)+ \\ &+ \left(8+\frac{1}{2+t}\right)\left(2x(t-0.001)+\frac{x(t-0.001)}{1+x^2(t-0.001)}\right)= \\ &= \int_0^t \frac{t}{(1+50t^2)^2}\,x'(s)\,ds. \end{aligned} \tag{9}$$

When we compare equation (9) with equation (1), it can be seen the existence of the following estimates:

$$p(t) = 1+\frac{e^{-t}}{25}, \quad q(t) = 1+\frac{1}{25e^t+1},$$

$$m = 1 \le q(t) \le p(t) \le \frac{26}{25} = M, \quad -L = \frac{-1}{25} \le p^{'}(t) \le q^{'}(t) \le 0, \quad p^{''}(t) \ge 0,$$

$$a(t) = 4+\frac{1}{2+t}, \quad a_0 = 3 < a(t) < 5 = a_1, \quad a^{'}(t) \le 0,$$

$$b(t) = 8 + \frac{1}{1+t}, \quad c(t) = 8 + \frac{1}{2+t},$$

$$n = 8 \le c(t) \le b(t) \le 9 = N, \quad -N = -9 \le b'(t) \le c'(t) \le 0,$$

$$f\left(t, x, x'\right) = 1 + \frac{e^{-t}}{1+y^2}, \quad f_0 = 1 \le f\left(t, x, x'\right) \le 2 = f_1, \quad f_t\left(t, x, x'\right) \le 0,$$

$$g(t,x) = 1 + \frac{1}{1+t+x^2}, \quad g_0 = 1 \le g(t,x) \le 2 = g_1, \quad g_t(t,x) \le 0,$$

$$h(x) = 2x + \frac{x}{1+x^2}, \quad \frac{h(x)}{x} = 2 + \frac{1}{1+x^2},$$

$$h(0) = 0, \quad \frac{h(x)}{x} \ge 2 = \delta_0, \quad x \ne 0, \quad \left|h'(x)\right| \le 3 = \delta_1,$$

$$\frac{1}{2}\int\limits_0^t |C(t,s)| ds = \quad \frac{1}{2}\int\limits_0^t \frac{t}{(50t^2+1)^2}\, ds = \frac{1}{200} < \infty,$$

$$\frac{1}{2}\int\limits_0^t |C(t,s)| ds + \frac{1}{m^2}\int\limits_t^\infty |C(u,t)|\, du = \frac{1}{2}\int\limits_0^t \frac{t}{\left(50t^2+1\right)^2}\, ds + \int\limits_t^\infty \frac{u}{\left(50u^2+1\right)^2}\, du \le \frac{1}{100}.$$

Thus, all assumptions $(A_1)-(A_6)$ hold. Therefore, we can conclude that all solutions of equation (9) are continuable and bounded.

Also the trajectories of solutions of equation (9) are shown in Fig. 1.

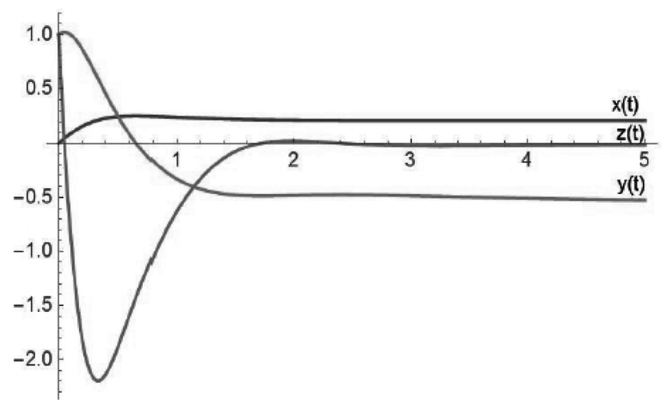

Fig. 1. Time evolution of the states $x(t)$, $y(t)$ and $z(t)$ of equation (9).

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