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NOTE ON THE OSCILLATION OF SECOND-ORDER QUASILINEAR NEUTRAL DYNAMIC EQUATIONS ON TIME SCALES

ПРО КОЛИВАННЯ КВАЗІЛІНІЙНИХ НЕЙТРАЛЬНИХ ДИНАМІЧНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ НА ЧАСОВИХ МАСШТАБАХ

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By using a new method, we improve some results from [*Saker S. H.* Oscillation criteria for a second-order quasilinear neutral functional dynamic equation on time scales // Nonlin. Oscillations. – 2011. – **13**, № 3. – P. 407 – 428].

З використанням нового методу покращено деякі результати, одержані в роботі [*Saker S. H.* Oscillation criteria for a second-order quasilinear neutral functional dynamic equation on time scales // Нелін. коливання. – 2010. – **13**, № 3. – С. 379–399].

1. Introduction. In 2011, Saker [1] established some sufficient conditions for the oscillation of the second-order quasilinear neutral functional dynamic equation

$$\left(p(t)\left(\left(y(t)+r(t)y(\tau(t))\right)^{\Delta}\right)^{\gamma}\right)^{\Delta}+f(t,y(\delta(t)))=0,\quad t\in[t_0,\infty)_{\mathbb{T}},\tag{1.1}$$

for which is assumed the following hypotheses:

(h$_1$) $\gamma>0$ is the quotient of odd positive integers, r and p are real-valued rd-continuous positive functions defined on $\mathbb{T}$, $\tau,\delta\colon[t_0,\infty)_{\mathbb{T}}\to\mathbb{T}$, $\tau(t)\le t$, and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\delta(t)=\infty$;

(h$_2$) $0\le r(t)<1$;

(h$_3$) $f(t,u)\colon\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that $uf(t,u)>0$ for all $u\ne0$ and there exists a positive rd-continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)|\ge q(t)|u^{\beta}|$, where $\beta>0$ is a ratio of odd positive integers.

Under the condition

$$\int\limits_{t_0}^{\infty}\frac{1}{p^{\frac{1}{\gamma}}(t)}\Delta t<\infty\tag{1.2}$$

and the assumptions

$$\delta(t) \le \tau(t) \le t, \quad \tau^{\Delta}(t) \ge 0, \quad r^{\Delta}(t) \ge 0, \tag{1.3}$$

Saker [1] obtained some new oscillation criteria for (1.1); see [1] (Section 3). In the last section of the paper [1], the author posed a problem: *How to present oscillation criteria for (1.1) when condition (1.3) does not hold?*

By a solution of (1.1), we mean a nontrivial real-valued function y satisfying (1.1) for $t \in \mathbb{T}$. We recall that a solution y of (1.1) is said to be oscillatory on $[t_0, \infty)_{\mathbb{T}}$ if it is neither eventually positive nor eventually negative; otherwise, the solution is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory. Our attention is restricted to those solutions y of (1.1) which are not eventually identically zero.

Our aim in this paper is to give an answer for the problem posed by [1].

In what follows, all functional inequalities considered in this note are assumed to hold eventually, that is, they are satisfied for all t large enough.

2. Main results. Note that [1] (Eq. (3.7)) plays an important role in the obtained results of [1] (Section 3). Hence, we will change it in order to renew results of [1]. Now we give the following. We let

$$x(t) := y(t) + r(t)y(\tau(t)), \qquad P(t) := \int_t^{\infty} \frac{1}{p^{\frac{1}{\gamma}}(s)} \Delta s, \qquad 1 - r(t)\, \frac{P(\tau(t))}{P(t)} > 0.$$

Lemma 1. *Let (1.2) hold, $\delta(t) \le t$, and y be an eventually positive solution of (1.1). Assume further that $\left(p\left(x^{\Delta}\right)^{\gamma}\right)^{\Delta}(t) < 0,\ x^{\Delta}(t) < 0,\ x(t) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Then*

$$\left(p\left(x^{\Delta}\right)^{\gamma}\right)^{\Delta}(t) + q(t)\left(1 - r(\delta(t))\, \frac{P(\tau(\delta(t)))}{P(\delta(t))}\right)^{\beta} x^{\beta}(t) \le 0. \tag{2.1}$$

Proof. From $\left(p\left(x^{\Delta}\right)^{\gamma}\right)^{\Delta}(t) < 0$, we have

$$x^{\Delta}(s) \le \frac{p^{\frac{1}{\gamma}}(t)}{p^{\frac{1}{\gamma}}(s)}\, x^{\Delta}(t), \quad s \ge t.$$

Integrating this from t to ℓ, we obtain

$$x(\ell) \le x(t) + p^{\frac{1}{\gamma}}(t) x^{\Delta}(t) \int_t^{\ell} \frac{1}{p^{\frac{1}{\gamma}}(s)}\, \Delta s.$$

Letting $\ell \to \infty$, we have

$$x(t) \ge -P(t) p^{\frac{1}{\gamma}}(t) x^{\Delta}(t).$$

Hence

$$\left(\frac{x}{P}\right)^{\Delta}(t) = \frac{x^{\Delta}(t)P(t) - x(t)P^{\Delta}(t)}{P(t)P(\sigma(t))} = \frac{x^{\Delta}(t)P(t) + \dfrac{x(t)}{p^{\frac{1}{\gamma}}(t)}}{P(t)P(\sigma(t))} \ge 0,$$

which yields

$$y(t) = x(t) - r(t)y(\tau(t)) \geq x(t) - r(t)x(\tau(t)) \geq \left(1 - r(t)\,\frac{P(\tau(t))}{P(t)}\right) x(t).$$

Thus, from (1.1), we have

$$\left(p\left(x^{\Delta}\right)^{\gamma}\right)^{\Delta}(t) + q(t)\left(1 - r(\delta(t))\,\frac{P(\tau(\delta(t)))}{P(\delta(t))}\right)^{\beta} x^{\beta}(\delta(t)) \leq 0,$$

which follows from $x^{\Delta}(t) < 0$ and $\delta(t) \leq t$ that (2.1) holds. The proof is complete.

Following ideas of [1] (Theorem 3.1) and Lemma 1 in this note, we can renew [1] (Eq. (3.7)) by the following:

$$\int\limits_{T}^{\infty}\left(\frac{1}{p(s)}\int\limits_{T}^{s} g_{*}(u)P^{\beta}(u)\Delta u\right)^{\frac{1}{\gamma}}\Delta s = \infty, \tag{2.2}$$

where

$$g_{*}(u) := q(u)\left(1 - r(\delta(u))\,\frac{P(\tau(\delta(u)))}{P(\delta(u))}\right)^{\beta}.$$

Therefore, replacing [1] (Eq. (3.7)) and (1.3) with (2.2) and $\delta(t) \leq t$ in this paper, we can renew [1] (Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4, Theorem 3.5). The details are left to the reader.

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