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BOUNDARY-VALUE PROBLEMS FOR THE EVOLUTIONARY SCHRÖDINGER EQUATION. II

КРАЙОВІ ЗАДАЧІ ДЛЯ ЕВОЛЮЦІЙНОГО РІВНЯННЯ ШРЕДІНҐЕРА. II

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A series of results of existence of solutions of boundary-value problems for the Schrödinger equation in the resonance (critical) case is proved. Iterative procedures for the construction of solutions of the corresponding problems in the nonlinear case are proposed.

Доведено низку результатів стосовно існування розв'язків крайових задач для рівняння Шредінгера у резонансному (критичному) випадку. Запропоновано ітеративні алгоритми побудови розв'язків відповідних задач у нелінійному випадку.

Bifurcation conditions of solutions for boundary-value problems of Schrödinger equation. In this part, the conditions of solvability and bifurcation of linear solutions of the boundary-value problem for the evolutionary Schrödinger equation are given.

Statement of the problem. Investigate the conditions for the bifurcation of the boundary-value problems of the evolutionary Schrödinger equation on the finite interval. In the Hilbert space $\mathcal{H}$ the following boundary-value problem is considered:

$$\frac{d\varphi(t)}{dt} = -iH(t)\varphi(t) + \varepsilon H_1(t)\varphi(t) + f(t), \quad t \in J, \tag{1}$$

$$\ell\varphi(\cdot) = \alpha + \varepsilon l_1\varphi(\cdot), \tag{2}$$

where for each $t \in J \subset \mathbb{R}$ the unbounded operator $H(t)$ has the following representation $H(t) = H_0 + V(t)$, where $H_0 = H_0^*$ is the unbounded self-adjoint operator with the domain $D = D(H_0) \subset \mathcal{H}$, mapping $t \to V(t)$ which is strongly continuous, $H_1(t)$ is a linear and bounded operator for all $t \in J$, ℓ, l_1 are linear and bounded operators, which map the solutions (1) in the Hilbert space $\mathcal{H}_1$. Let's define operator-valued function in the same way as in [1]

$$\tilde{V}(t) = e^{itH_0}V(t)e^{-itH_0}.$$

In this case, for $\tilde{V}(t)$ it is fair to use Dyson's representation [1, p. 311] and it is possible to determine the evolutionary operator $\tilde{U}(t,s)$. If

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$$U(t,s) = e^{-itH_0}\tilde{U}(t,s)e^{isH_0},$$

then $\psi_s(t) = U(t,s)\psi$ is a weak (generalized) solution of homogeneous equation with the condition that $\psi_s(s) = \psi$ in the sense, that for each $\eta \in D(H_0)$ the function $(\eta, \psi_s(t))$ is differentiable and

$$\frac{d}{dt}(\eta, \psi_s(t)) = -i(H_0\eta, \psi_s(t)) - i(V(t)\eta, \psi_s(t)), \quad t \in J.$$

Remark 1. It is true to state that if the operator-valued function $t \to [H_0, V(t)]$ is strongly continuous [1, p. 312], then $\psi_s(t)$ is a strong solution.

For simplicity, we assume that D is a dense set in $\mathcal{H}$ and evolutionary operator $U(t,s)$ is bounded and defined on the entire space $\mathcal{H}$ (extended by continuity).

We look for a strong generalized solution for the boundary-value problem (1), (2) for those right parts of $f(t)$ in equation (1), for which unperturbed boundary-value problem $(\varepsilon = 0)$ doesn't contain solutions. We should remark that asymptotic methods for solving equations are powerful methods for the investigation of the boundary-value problem for differential operator equations (see [2, 3] and bibliography).

Linear case (axillary result). For the calculation of the main problem we need to have the conditions for solvability and the possibility to build solutions of unperturbed boundary-value problem

$$\frac{d\varphi(t)}{dt} = -iH(t)\varphi(t) + f(t), \tag{3}$$

$$\ell\varphi(\cdot) = \alpha. \tag{4}$$

Using the results obtained by S. G. Krein [4], any weak solution of the equation (3) can be represented in the following form:

$$\varphi(t,s) = U(t,s)\varphi(s,s) + \int_s^t U(t,\tau)f(\tau)\,d\tau \tag{5}$$

(equality in the sense of scalar product).

Then, putting (5) into the boundary-value problem (4), we obtain operator equation for the element $\varphi(s,s) \in \mathcal{H}$:

$$Q\varphi(s,s) = \alpha - \ell\int_s^{\cdot} U(\cdot,\tau)f(\tau)\,d\tau, \tag{6}$$

where $Q := \ell U(\cdot,s)$ is an operator, obtained by submitting the corresponding linear operator $U(t,s)$ into the equation (4). Let's denote

$$\varphi := \varphi(s,s),$$

$$g := \alpha - \ell\int_s^{\cdot} U(\cdot,\tau)f(\tau)\,d\tau.$$

Then the operator equation (6) we can rewrite in the form

$$Q\varphi = g. \tag{7}$$

We should also describe the construction of the strong Moore – Penrose inverse operator, which is used for the representation of the solutions [5, 6] of the operator equation (7).

We should distinguish between three types of solutions:

(i) Classical generalized solutions.

If operator Q is normally solvable $(R(Q) = \overline{R(Q)})$, then the element g belongs to the sets of values $(g \in R(Q))$ of the operator Q if and only $\mathcal{P}_{N(Q^*)} g = 0$ [2]; $\mathcal{P}_{N(Q^*)}$ is an orthoprojector onto the cokernel of the operator Q. In this case, there is Moore – Penrose pseudoinverse operator Q^+ and the set of solutions of the equation (7) has the form

$$\varphi = Q^+ g + \mathcal{P}_{N(Q)} c \quad \forall\, c \in \mathcal{H},$$

where $\mathcal{P}_{N(Q)}$ is an orthoprojector onto the kernel of the operator Q.

(ii) Strong generalized solutions.

Consider the case, when a set of values of the operator Q is not a closed set, what means that $R(Q) \neq \overline{R(Q)}$. We show that in this case Q can be extended to operator $\overline{Q}$ in such way that operator $\overline{Q}$ is normally solvable [6].

Given that the operator Q is linear and bounded, the expansions of the spaces in the direct sums are possible:

$$\mathcal{H} = N(Q) \oplus X, \qquad \mathcal{H}_1 = \overline{R(Q)} \oplus Y.$$

Here, $X = N(Q)^\perp$, $Y = \overline{R(Q)}^\perp$. It is possible to state that operators of orthogonal projections $\mathcal{P}_{N(Q)}$, $\mathcal{P}_X$ and $\mathcal{P}_{\overline{R(Q)}}$, $\mathcal{P}_Y$ on the corresponding subspaces exist. With $\mathcal{H}_2$ we denote quotient space of the space $\mathcal{H}$ with kernel $N(Q)$ $(\mathcal{H}_2 = \mathcal{H}/N(Q))$. As it is known from [7, 8], there is a continuous bijection $p\colon X \to \mathcal{H}_2$ and projection $j\colon \mathcal{H} \to \mathcal{H}_2$. The triple $(\mathcal{H}, \mathcal{H}_2, j)$ is a local trivial bundle with a typical layer $\mathcal{P}_{N(Q)}\mathcal{H}$. Now we should define the operator

$$\mathcal{Q} = \mathcal{P}_{\overline{R(Q)}} Q j^{-1} p\colon X \to R(Q) \subset \overline{R(Q)}.$$

It is easy to see, that the operator is defined in this way is linear, injective, and continuous. By using the process of completion [9] by norm $\|x\|_{\overline{X}} = \|\mathcal{Q}x\|_F$, where $F = \overline{R(Q)}$, we obtain the new space $\overline{X}$ and the expanded operator $\overline{\mathcal{Q}}$. Then

$$\overline{\mathcal{Q}}\colon \overline{X} \to \overline{R(Q)}, \quad X \subset \overline{X}$$

and so the operator, which is built in this way, performs homeomorphism between the spaces $\overline{X}$ and $\overline{R(Q)}$. Consider the extended operator $\overline{Q} = \overline{\mathcal{Q}}\mathcal{P}_{\overline{X}}\colon \overline{\mathcal{H}} \to \mathcal{H}_1$, where

$$\overline{\mathcal{H}} = N(Q) \oplus \overline{X},$$

$$\mathcal{H}_1 = R(\overline{Q}) \oplus Y.$$

It is clear that $\overline{Q}x = Qx,\ x \in \mathcal{H}$ and the operator $\overline{Q}$ would be normally solvable (in this case $R(\overline{Q}) = \overline{R(\overline{Q})}$), that is why [2] is pseudoinvertible with pseudoinverse $\overline{Q}^{+}$, which is called *strong pseudoinverse* to the operator Q. This construction is easy to display by the following diagram:

$$
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{Q} & \mathcal{H}_1 \\
j\big\downarrow & & \big\downarrow I \\
\mathcal{H}_2 & & \mathcal{H}_1 \\
p^{-1}\big\downarrow & & \big\downarrow \mathcal{P}_{\overline{R(Q)}} \\
X & \xrightarrow{Q} & R(Q) \subset \overline{R(Q)} \\
\bigcap & & \bigcap \\
\overline{\mathcal{H}} & \xrightarrow{\overline{Q}} & \mathcal{H}_1
\end{array} .
$$

Then the set of solutions of the equation (7) would have such form:

$$\varphi = \overline{Q}^{+} g + \mathcal{P}_{N(\overline{Q})} c \quad \forall\, c \in \overline{\mathcal{H}}.$$

(iii) Generalized pseudosolutions.

Consider the case, when $g \notin \overline{R(Q)}$, which is equivalent to the condition $\mathcal{P}_{N(Q^*)} g \neq 0$. In this case, there are elements from $\overline{\mathcal{H}}$, that minimize norm $\|\overline{Q}\varphi - g\|_{\mathcal{H}}$ for $\varphi \in \overline{\mathcal{H}}$,

$$\varphi = \overline{Q}^{+} g + \mathcal{P}_{N(\overline{Q})} c, \quad c \in \overline{\mathcal{H}}.$$

These elements are called *generalized (strong) pseudosolutions* of the equation (7).

By using all the above we can formulate the following statement [5].

Theorem 1. *Let the boundary problem (3), (4), be defined in the Hilbert spaces.*

I. Strong generalized solutions exist if and only if

$$\mathcal{P}_{N(\overline{Q}^*)} \alpha - \mathcal{P}_{N(\overline{Q}^*)} \ell \int_s^{\cdot} U(\cdot, \tau) f(\tau)\, d\tau = 0, \tag{8}$$

if

$$\alpha - \ell \int_s^{\cdot} U(\cdot, \tau) f(\tau)\, d\tau \in R(Q),$$

then the solutions would be classical generalized.

II. Generalized pseudosolutions exist if and only if

$$\mathcal{P}_{N(\overline{Q}^*)} \alpha - \mathcal{P}_{N(\overline{Q}^*)} \ell \int_s^{\cdot} U(\cdot, \tau) f(\tau)\, d\tau \neq 0. \tag{9}$$

III. If (8) or (9) holds, then generalized solutions (strong or pseudosolutions) of the boundary problem (3), (4) look as following:

$$\varphi(t, s, c) = U(t, s) \mathcal{P}_{N(\overline{Q})} c + U(t, s) \overline{Q}^{+} \alpha + \left(\overline{G[f]}\right)(t, s), \tag{10}$$

where

$$\left(\overline{G[f]}\right)(t,s) = \int\limits_s^t U(t,\tau)f(\tau)\,d\tau - U(t,s)\overline{Q}^+\ell\int\limits_s^{\cdot} U(\cdot,\tau)f(\tau)\,d\tau$$

is generalized Green's operator of the boundary problem (3), (4), c *is the arbitrary element of the space* $\overline{\mathcal{H}}$.

Example. One of the examples of the considered above tasks can be the same one as defined in [1, p. 318]. Let $H_0 = -\Delta$ on $L_2(\mathbb{R}^3)$, and functions $V_1(t)$ and $V_2(t)$ are continuously-differentiable with the values in the space $L_2(\mathbb{R}^3)$ and $L_\infty(\mathbb{R}^3)$ respectively, $V(t) = V_1(t)+V_2(t)$. Then the proved in [1, p. 318] theorem X.71 gives us the possibility to use the defined theory for this task. Particularly, as it follows from theorem X.71 [1, p. 318], the evolutionary operator $U(t,s)$ on $L_2(\mathbb{R}^3)$ exist, such that $\varphi_s(t) = U(t,s)\varphi$ is strongly differentiable for any $\varphi \in D(H_0)$ and satisfies the equation

$$\frac{d}{dt}\varphi_s(t) = -iH(t)\varphi_s(t), \quad \varphi_s(s) = \varphi. \tag{11}$$

Now let consider generalized boundary condition

$$l\varphi_s(\cdot) = \alpha. \tag{12}$$

In the case when $\alpha = 0$, we get homogeneous boundary problem. As before let us define operator $Q = lU(\cdot,s)$.

Theorem 2. *Boundary value problem (11), (12), which is considered in the Hilbert space* $\mathcal{H} = L_2(\mathbb{R}^3)$, *has:*

I. Strong generalized solutions if and only if

$$\mathcal{P}_{N(\overline{Q}^*)}\alpha = 0; \tag{13}$$

if $\alpha \in R(Q)$, *then the solutions would be classical generalized solutions;*

II. Generalized pseudosolutions if and only if

$$\mathcal{P}_{N(\overline{Q}^*)}\alpha \neq 0; \tag{14}$$

III. If the conditions (13) or (14) hold, then the solutions of the boundary-value problem (11), (12) have the form

$$\varphi(t,s,c) = U(t,s)\mathcal{P}_{N(\overline{Q})}c + U(t,s)\overline{Q}^+\alpha,$$

where c *is an arbitrary element of the space* $\overline{\mathcal{H}}$.

Let us consider two-point boundary-value condition

$$l\varphi_s(\cdot) := \varphi_s(T) - \varphi_s(s) = \alpha. \tag{15}$$

In the case $\alpha = 0$, we obtain the problem about periodic solutions. Subsetting it ($\varphi_s(t) = U(t,s)\varphi$) into the equation (15), gives the operator equation

$$(I - U(T,s))\varphi = -\alpha. \tag{16}$$

Due to the fact, that in general case the operator $U(T,s)$ is nonexpanding [10, 11], then the standard classical procedure for solvability of operator equation (16) can not be used. However, it is possible to use the previous theorem. As operator Q, in this case, we should choose the operator $Q = I - U(T,s)$. In this way, we obtain corollary [5].

Corollary 1. *Boundary-value problem (11), (15), which is considered in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^3)$, has:*

I. Strong generalized solutions if and only if

$$\mathcal{P}_{N\left(\overline{(I-U(T,s))}^*\right)}\alpha = 0, \tag{17}$$

if $\alpha \in R(I - U(T,s))$, then solutions would be classical generalized;

II. Generalized pseudosolutions if and only if

$$\mathcal{P}_{N\left(\overline{(I-U(T,s))}^*\right)}\alpha \neq 0; \tag{18}$$

III. If the conditions (17) or (18) hold then the solutions of the boundary-value problem (11), (15) have the form

$$\varphi(t,s,c) = U(t,s)\mathcal{P}_{N\left(\overline{I-U(T,s)}\right)}c -$$

$$- U(t,s)\overline{(I-U(T,s))}^{+}\alpha,$$

where c is an arbitrary element of the space $\overline{\mathcal{H}}$.

Remark 2. Theorem 1 holds also in the non-stationary case when operator $H(t)$ depends on time. Examples, that are considered above, explain this case.

Remark 3. Consider this boundary-problem at the whole axis $t \in J = \mathbb{R}$. Then we obtain the boundary-value problem with conditions on the infinity. There are a few examples of the boundary conditions:

(a) $l\varphi_0(\cdot) = \varphi_0(+\infty) - \varphi_0(-\infty) = \alpha$;

(b) $l\varphi_0(\cdot) = \varphi_0(+\infty) - A\varphi_0(-\infty) = \alpha$,

where $\mathcal{H} = \mathcal{H}_1$, operator A is a linear and bounded from the Hilbert space $\mathcal{H}$ into itself, $A \in \mathcal{L}(\mathcal{H})$;

(c) $l\varphi_0(\cdot) = A_1\varphi_0(+\infty) - A_2\varphi_0(-\infty) = \alpha$,

$A_1,\ A_2$ are linear, bounded, and mapped from the Hilbert space $\mathcal{H}$ into the Hilbert space $\mathcal{H}_1$, $A_1, A_2 \in \mathcal{L}(\mathcal{H}, \mathcal{H}_1)$. The questions of a solution existence for evolutionary Schrödinger equation with the conditions at infinity is the actual task with the application in physics (please see the previous part of the paper).

Further, let's obtain the conditions for the bifurcation of solutions for the respective boundary-value problems.

Bifurcation of solutions. Assume that the boundary-value problem (3), (4) doesn't have strong generalized solutions, that means that the condition (9) holds. Let us find the conditions for perturbations $H_1(t),\ l_1$, when perturbed boundary problem (1), (2) would have strong generalized conditions. For this purpose, we would use an operator

$$B_0 = \mathcal{P}_{N(\overline{Q}^*)}\left(l_1U(\cdot,s) - \ell\int\limits_s^{\cdot} U(\cdot,\tau)H_1(\tau)U(\tau,s)\,d\tau\right)\mathcal{P}_{N(\overline{Q})}.$$

We would search for solutions of the boundary-value problem (1), (2) in the form of the series by degrees of the small parameter ε:

$$\varphi(t,\varepsilon) = \sum_{i=-1}^{+\infty} \varepsilon^i \varphi_i(t). \tag{19}$$

Subsetting series (19) in the boundary-value problem (1), (2) and equating coefficients under corresponding powers ε. The problem of the obtaining coefficient $\varphi_{-1}(t)$ at ε^{-1} series (19) goes to this boundary-value problem:

$$\frac{d\varphi_{-1}(t)}{dt} = -iH(t)\varphi_{-1}(t), \tag{20}$$

$$\ell\varphi_{-1}(\cdot) = 0. \tag{21}$$

The set of solutions of operator boundary-value problem (20), (21) would have a form

$$\varphi_{-1}(t,s,c_{-1}) = U(t,s)\mathcal{P}_{N(Q)}c_{-1}, \quad t \in J,$$

for arbitrary element $c_{-1} \in \mathcal{H}$, which is defined on the next step of the iteration process. The problem of defining the coefficient $\varphi_0(t)$ at ε^0 series (19) goes to such boundary-value problem:

$$\frac{d\varphi_0(t)}{dt} = -iH(t)\varphi_0(t) + H_1(t)\varphi_{-1}(t,s,c_{-1}) + f(t), \tag{22}$$

$$\ell\varphi_0(\cdot) = \alpha + l_1\varphi_{-1}(\cdot,s,c_{-1}). \tag{23}$$

If the condition (8) holds, then the criteria for the solvability of the boundary-value problem (22), (23) looks like this:

$$\mathcal{P}_{N(\overline{Q}^*)}\Bigg\{\alpha + l_1\varphi_{-1}(\cdot,s,c_{-1}) - \\ - \ell\int_s^{\cdot} U(\cdot,\tau)\big(H_1(\tau)\varphi_{-1}(\tau,s,c_{-1}) + f(\tau)\big)\,d\tau\Bigg\} = 0.$$

From it, we finally obtain the operator equation

$$B_0 c_{-1} = \mathcal{P}_{N(\overline{Q}^*)}\left(-\alpha + \ell\int_s^{\cdot} U(\cdot,\tau)f(\tau)\,d\tau\right). \tag{24}$$

From here we assume $\mathcal{P}_{N(\overline{B}_0^*)}\mathcal{P}_{N(\overline{Q}^*)} = 0$. Here $\mathcal{P}_{N(\overline{B}_0^*)}$, $\mathcal{P}_{N(\overline{Q}^*)}$ are orthoprojectors on cokernel of operators $\overline{B}_0$, $\overline{Q}$ respectively. Then operator equation (24) would be solvable. The set of strong generalized solutions (24)) would have the form:

$$c_{-1} = \overline{B}_0^+\mathcal{P}_{N(\overline{Q}^*)}\left(-\alpha + \ell\int_s^{\cdot} U(\cdot,\tau)f(\tau)\,d\tau\right) + \mathcal{P}_{N(B_0)}c_\rho$$

for the arbitrary element $c_\rho \in \mathcal{H}$. For simplicity, we rewrite this equality in the following way

$$c_{-1} = \overline{c}_{-1} + \mathcal{P}_{N(B_0)} c_\rho,$$

where

$$\overline{c}_{-1} = \overline{B}_0^{+} \mathcal{P}_{N(\overline{Q}^*)} \left(-\alpha + \ell \int_s^{\cdot} U(\cdot, \tau) f(\tau)\, d\tau \right).$$

Then the solutions set of the boundary-value problem (22), (23) has such representation

$$\varphi_{-1}(t, s, c_\rho) = \overline{\varphi}_{-1}(t, s, \overline{c}_{-1}) + \overline{X}_{-1}(t, s) \mathcal{P}_{N(B_0)} c_\rho,$$

where

$$\overline{\varphi}_{-1}(t, s, \overline{c}_{-1}) = U(t, s) \mathcal{P}_{N(Q)} \overline{c}_{-1},$$

$$\overline{X}_{-1}(t, s) = U(t, s) \mathcal{P}_{N(Q)}.$$

By using the set (10) and the linearity of the generalized Green's operator, the set of the solution of the boundary-value problem (22), (23) can be represented in such form:

$$\begin{aligned} \varphi_0(t, s, c_0) = {} & U(t, s) \mathcal{P}_{N(Q)} c_0 + \\ & + U(t, s) \overline{Q}^{+} \left\{ \alpha + l_1 \overline{\varphi}_{-1}(\cdot, s, \overline{c}_{-1}) \right\} + \\ & + \overline{G \left[H_1(\cdot) \overline{\varphi}_{-1}(\cdot, s, \overline{c}_{-1}) + f(\cdot) \right]}(t, s) + \\ & + \left(U(t, s) \overline{Q}^{+} \ell \overline{X}_{-1}(\cdot, s) + \overline{G \left[H_1(\cdot) \overline{X}_{-1}(\cdot, s) \right]}(t, s) \right) \mathcal{P}_{N(B_0)} c_\rho, \end{aligned}$$

where element $c_0 \in \mathcal{H}$ would be defined on the next step of the iteration. Doing in the same way further, we obtain the theorem [5].

Theorem 3. *Assume that such condition holds:* $\mathcal{P}_{N(\overline{B}_0^*)} \mathcal{P}_{N(\overline{Q}^*)} = 0$.

If unperturbed operator-valued boundary problem (3), (4) doesn't have strong generalized solutions, then the operator boundary problem (1), (2) has ρ *— parametric set of strong generalized solutions in the form of series*

$$\varphi(t, s, \varepsilon, c_\rho) = \sum_{i=-1}^{\infty} \varepsilon^i \left[\overline{\varphi}_i(t, s, \overline{c}_i) + \overline{X}_i(t, s) \mathcal{P}_{N(B_0)} c_\rho \right] \quad \text{for any} \quad c_\rho \in \mathcal{H},$$

absolutely convergent for the small fixed parameter $\varepsilon \in (0, \varepsilon_*]$*; here*

$$\overline{\varphi}_{-1}(t, s, \overline{c}_{-1}) = U(t, s) \mathcal{P}_{N(Q)} \overline{c}_{-1},$$

$$\begin{aligned} \overline{\varphi}_0(t, s, \overline{c}_0) = {} & U(t, s) \mathcal{P}_{N(Q)} \overline{c}_0 + \\ & + U(t, s) \overline{Q}^{+} \left\{ \alpha + l_1 \overline{\varphi}_{-1}(\cdot, s, \overline{c}_{-1}) \right\} + \\ & + \overline{G \left[H_1(\cdot) \overline{\varphi}_{-1}(\cdot, s, \overline{c}_{-1}) + f(\cdot) \right]}(t, s), \end{aligned}$$

$$\overline{\varphi}_i(t,s,\overline{c}_i) = U(t,s)\mathcal{P}_{N(Q)}\overline{c}_i + U(t,s)\overline{Q}^+ l_1 \overline{\varphi}_{i-1}(\cdot,s,\overline{c}_{i-1}) + \\ + \overline{G\left[H_1(\cdot)\overline{\varphi}_{i-1}(\cdot,s,\overline{c}_{i-1})\right]}(t,s), \quad i \in \mathbb{N};$$

$$\overline{c}_i = \overline{B}_0^+ \mathcal{P}_Y \Bigg\{ l \int_s^{\cdot} U(\cdot,\tau)H_1(\tau)\Big\{U(\tau,s)\overline{Q}^+ l_1\overline{\varphi}_{i-1}(\cdot,s,\overline{c}_0) + \\ + \overline{G\left[H_1(\cdot)\overline{\varphi}_{i-1}(\cdot,s,\overline{c}_0)\right]}(\tau,s)\Big\}d\tau - \\ - l_1(U(\cdot,s)\overline{Q}^+ l_1\overline{\varphi}_{i-1}(\cdot,s,\overline{c}_0) + \overline{G\left[H_1(\cdot)\overline{\varphi}_{i-1}(\cdot,s,\overline{c}_0)\right]}(\cdot,s)) \Bigg\},$$

$$\mathcal{F}_i = \overline{B}_0^+ \mathcal{P}_Y \Bigg\{ l \int_s^{\cdot} U(\cdot,\tau)H_1(\tau)\Big\{U(\tau,s)\overline{Q}^+ l_1 \overline{X}_{i-1}(\cdot,s) + \\ + \overline{G\left[H_1(\cdot)\overline{X}_{i-1}(\cdot,s)\right]}(\tau,s)\Big\}d\tau - \\ - l_1\Big(U(\cdot,s)\overline{Q}^+ l_1\overline{X}_{i-1}(\cdot,s) + \overline{G\left[H_1(\cdot)\overline{X}_{i-1}(\cdot,s)\right]}(t,s)\Big) + I \Bigg\},$$

$$\overline{X}_i(t,s) = U(t,s)\mathcal{P}_{N(Q)}\mathcal{F}_i + \\ + U(t,s)\overline{Q}^+ l_1 \overline{X}_{i-1}(\cdot,s) + \\ + \overline{G\left[H_1(\cdot)\overline{X}_{i-1}(\cdot,s)\right]}(t,s).$$

Boundary-value problems for a nonlinear nonstationary Schrödinger's equation. We establish the conditions of normal and generalized solvability of boundary-value problems for Schrödinger's equation.

In the Hilbert space $\mathcal{H}$ we consider only a nonlinear Schrödinger's differential equation

$$\frac{d\varphi(t,\varepsilon)}{dt} = -iH(t)\varphi(t,\varepsilon) + \varepsilon Z(\varphi(t,\varepsilon),t,\varepsilon) + f(t), \quad t \in J, \tag{25}$$

with operator boundary condition

$$\ell\varphi(\cdot,\varepsilon) = \alpha + \varepsilon J(\varphi(\cdot,\varepsilon),\varepsilon), \tag{26}$$

where $J \subset \mathbb{R}$ is a finite interval. For every t the unbounded operator $H(t)$ has a form $H(t) = H_0 + V(t)$ with self-adjoint operator $H_0 = H_0^*$ on $D = D(H_0) \subset \mathcal{H}$ and strongly continuous mapping $t \to V(t)$. Operator ℓ is linear and bounded that maps the set of solutions (25) into the Hilbert space $\mathcal{H}_1$, α is an arbitrary element of the space $\mathcal{H}_1$. It is necessary to find such solution $\varphi(t,\varepsilon)$ of the boundary-value problem (25), (26), that converts to one of solutions of the following boundary-value problem:

$$\frac{d\varphi_0(t)}{dt} = -iH(t)\varphi_0(t) + f(t), \quad t \in J, \tag{27}$$

$$\ell\varphi_0(\cdot) = \alpha \tag{28}$$

for $\varepsilon = 0$. Operator-functions $Z(\varphi(t,\varepsilon),t,\varepsilon)$, $J(\varphi(t,\varepsilon),\varepsilon)$ hold such restrictions in the neighborhood of generating solution $\varphi_0(t)$ on a set of variables

$$Z(\cdot,\cdot,\cdot) \in C^1\left[\|\varphi - \varphi_0\| \le q\right] \times C(J,\mathcal{H}) \times C[0,\varepsilon_0],$$

$$J(\cdot,\cdot) \in C^1\left[\|\varphi - \varphi_0\| \le q\right] \times C[0,\varepsilon_0],$$

where q is positive constant.

Necessary and sufficient conditions for the solutions existence. Firstly, we need to find the necessary condition for the existence of a strong generalized solution $\varphi(t,s,\varepsilon)$ of the boundary-value problem (25), (26), which at $\varepsilon = 0$ turns into a generating solution $\varphi_0(t,s,c)$ of the form (10). We assume that the boundary-value problem (27), (28) has strong generalized solutions, which means that the condition (8) is satisfied [12].

Theorem 4 (necessary condition)**.** *Assume the boundary-value problem (25), (26) has strong generalized solution $\varphi(t,s,\varepsilon)$, which for $\varepsilon = 0$ turns in one of solutions of generating boundary-value-problem $\varphi_0(t,s,c^0)$ (10) with element $c = c^0$. Then element $c^0 \in \mathcal{H}$ satisfies the operator equation for generating elements*

$$F(c) = \mathcal{P}_{N(\overline{Q}^*)}\Bigg\{J(\varphi_0(\cdot,s,c),0) -$$

$$- \ell\int_s^{\cdot} U(\cdot,\tau)Z(\varphi_0(\tau,s,c),\tau,0)\,d\tau\Bigg\} = 0. \tag{29}$$

Remark 4. For obtaining results from Theorem 4 from non-linearities $Z(\varphi(t,\varepsilon),t,\varepsilon)$, $J(\varphi(t,s,\varepsilon),\varepsilon)$ it is sufficient to demand only continuity in the neighborhood of the generating solution.

To obtain a sufficient condition for the existence of a solution, we replace the variables in the boundary-value problem (25), (26)

$$\varphi(t,s,\varepsilon) = \varphi_0(t,s,c^0) + \psi(t,s,\varepsilon),$$

where $\varphi_0(t,s,c^0)$ is generated solution (10) with the element c^0, which satisfies the operator equation for the generated elements (29). Among the new variables, we search for the strong generalized solution of the boundary-value problem

$$\frac{d\psi(t,\varepsilon)}{dt} = -iH(t)\psi(t,\varepsilon) + \varepsilon Z(\varphi_0(t,s,c^0) + \psi(t,\varepsilon),t,\varepsilon), \tag{30}$$

$$\ell\psi(\cdot,\varepsilon) = \varepsilon J\left(\varphi_0(\cdot,s,c^0) + \psi(\cdot,\varepsilon),\varepsilon\right), \tag{31}$$

which in the case of $\varepsilon = 0$ turns to zero solution. The solvability of the boundary-value problem (30), (31) is equivalent to the solvability of the boundary-value problem (25), (26). By using the continuous differentiation of the non-linearity in the neighborhood of the generating solution, we highlight left-hand side as ψ and the terms of zero-order as ε. Then we have the following:

$$Z\left(\varphi_0(t,s,c^0)+\psi(t,s,\varepsilon),t,\varepsilon\right)=Z\left(\varphi_0(t,s,c^0),t,0\right)+$$

$$+A_1(t)\psi(t,s,\varepsilon)+\mathcal{R}(\psi(t,s,\varepsilon),t,\varepsilon),$$

$$J(\varphi_0\left(\cdot,s,c^0\right)+\psi(\cdot,s,\varepsilon),\varepsilon)=J\left(\varphi_0\left(\cdot,s,c^0\right),0\right)+$$

$$+l_1\psi(\cdot,s,\varepsilon)+\mathcal{R}_1(\psi(\cdot,s,\varepsilon),\varepsilon),$$

where

$$A_1(t)=A_1(t,c^0)=Z^{(1)}_{\varphi}(v,t,\varepsilon)\Big|_{v=\varphi_0(t,s,c^0),\varepsilon=0},$$

$$l_1=J^{(1)}(\varphi_0,0),$$

are Fréchet derivatives in the point $\varphi=\varphi_0(t,s,c^0)$, $\varepsilon=0$, and in case of higher orders $\mathcal{R}(\psi,t,\varepsilon),\mathcal{R}_1(\psi,\varepsilon)$ such equalities hold

$$\mathcal{R}(0,t,0)=0,\qquad \mathcal{R}^{(1)}_{\psi}(0,t,0)=0,$$

$$\mathcal{R}_1(0,0)=0,\qquad \mathcal{R}^{(1)}_{1\psi}(0,0)=0.$$

Then, given the replacement, we consider the boundary-value problem

$$\frac{d\psi(t,\varepsilon)}{dt}=-iH(t)\psi(t,\varepsilon)+$$

$$+\varepsilon\left\{Z\left(\varphi_0(t,s,c^0),t,0\right)+A_1(t)\psi(t,\varepsilon)+\mathcal{R}(\psi(t,\varepsilon),t,\varepsilon)\right\},$$

$$\ell\psi(\cdot,\varepsilon)=\varepsilon\left\{J\left(\varphi_0\left(\cdot,s,c^0\right),0\right)+l_1\psi(\cdot,\varepsilon)+\mathcal{R}_1(\psi(\cdot,\varepsilon),\varepsilon)\right\},$$

which has strong generalized solution

$$\psi(t,s,c)=U(t,s)\mathcal{P}_{N(\overline{Q})}c+\overline{\psi}\,(t,s,\varepsilon)\,,\quad c\in\overline{\mathcal{H}},$$

$$\overline{\psi}(t,s,\varepsilon)=\varepsilon U(t,s)\overline{Q}^{+}\left\{J\left(\varphi_0\left(\cdot,s,c^0\right),0\right)+l_1\psi(\cdot,s,\varepsilon)+\mathcal{R}_1\left(\psi\left(\cdot,s,\varepsilon\right),\varepsilon\right)\right\}+$$

$$+\varepsilon\overline{G\left[Z(\varphi_0\left(\cdot,s,c^0\right),\cdot,0)+A_1(\cdot)\psi(\cdot,s,\varepsilon)+\mathcal{R}(\psi(\cdot,s,\varepsilon),\cdot,\varepsilon)\right]}(t,s),$$

if the following condition holds:

$$\mathcal{P}_{N(\overline{Q}^*)}\Bigg(\left\{J\left(\varphi_0\left(\cdot,s,c^0\right),0\right)+l_1\psi\left(\cdot,s,\varepsilon\right)+\mathcal{R}_1\left(\psi(\cdot,s,\varepsilon),\varepsilon\right)\right\}-$$

$$- \ell \int_s^{\cdot} U(\cdot,\tau)\left\{Z\left(\varphi_0\left(\tau,s,c^0\right),\tau,0\right) + A_1(\tau)\psi(\tau,s,\varepsilon) + \mathcal{R}\left(\psi(\tau,s,\varepsilon),\tau,\varepsilon\right)\right\} d\tau \Bigg) = 0.$$

Replacing the linear part of the last expression defined above $\psi(t,s,\varepsilon)$ with the representation $U(t,s)\mathcal{P}_{N(\overline{Q})}c + \overline{\psi}(t,s,\varepsilon)$ and given the condition (29), gives us the operator equation for $c \in \overline{\mathcal{H}}$:

$$B_0 c = \mathcal{P}_{N(\overline{Q}^*)}\ell \int_s^{\cdot} U(\cdot,\tau)\left\{A_1(\tau)\overline{\psi}(\tau,s,\varepsilon) + \mathcal{R}\left(\psi(\tau,s,\varepsilon),\tau,\varepsilon\right)\right\} d\tau -$$

$$- \mathcal{P}_{N(\overline{Q}^*)}\left\{l_1\overline{\psi}(\cdot,s,\varepsilon) + \mathcal{R}_1\left(\psi\left(\cdot,s,\varepsilon\right),\varepsilon\right)\right\},$$

where operator B_0 is defined as follows:

$$B_0 := \mathcal{P}_{N(\overline{Q}^*)}\left(l_1 U(\cdot,s) - \ell \int_s^{\cdot} U(\cdot,\tau)A_1(\tau)U(\tau,s)d\right)\mathcal{P}_{N(Q)}.$$

By using all mentioned above we can obtain the statement [12].

Theorem 5. *Suppose that the following condition holds for the operator B_0:*

$$\mathcal{P}_{N(\overline{B}_0^*)}\mathcal{P}_{N(\overline{Q}^*)} = 0.$$

Then for an arbitrary element $c = c^0 \in \overline{\mathcal{H}}$, which satisfies the equation for generating elements (29), at least one strong generalized solution of the boundary-value problem (25), (26) exists. This solution can be found using the iterative process

$$\overline{\psi}_{k+1}(t,s,\varepsilon) = \varepsilon U(t,s)\overline{Q}^+ J\left(\varphi_0\left(\cdot,s,c^0\right) + \psi_k(\cdot,s,\varepsilon),\varepsilon\right) +$$

$$+ \varepsilon\overline{G\left[Z\left(\varphi_0\left(\cdot,s,c^0\right) + \psi_k(\cdot,s,\varepsilon),\cdot,\varepsilon\right)\right]}(t,s),$$

$$c_k = \overline{B}_0^+\Bigg\{\mathcal{P}_{N(\overline{Q}^*)}\ell \int_s^{\cdot} U(\cdot,\tau)\left\{A_1(\tau)\overline{\psi}_k(\tau,s,\varepsilon) + \mathcal{R}(\psi_k(\tau,s,\varepsilon),\tau,\varepsilon)\right\} d\tau -$$

$$- \mathcal{P}_{N(\overline{Q}^*)}\left\{l_1\overline{\psi}_k\left(\cdot,s,\varepsilon\right) + \mathcal{R}_1\left(\psi_k(\cdot,s,\varepsilon),\varepsilon\right)\right\}\Bigg\},$$

$$\psi_{k+1}(t,s,c) = U(t,s)\mathcal{P}_{N(\overline{Q})}c_k + \overline{\psi}_{k+1}(t,s,\varepsilon),$$

$$\varphi_k(t,s,\varepsilon) = \varphi_0(t,s,c^0) + \psi_k(t,s,\varepsilon), \quad k = 0,1,2,\ldots, \quad \psi_0(t,s,\varepsilon) = 0,$$

$$\varphi(t,s,\varepsilon) = \lim_{k\to\infty}\varphi_k(t,s,\varepsilon).$$

Connection between necessary and sufficient conditions. Let's formulate the result, which connects the necessary and sufficient conditions [12].

Corollary 2. *Assume operator $F(c)$ has Fréchet derivative $F^{(1)}(c)$ for any element c^0 from the Hilbert space $\mathcal{H}$, which satisfies the equation for the generating elements (29). If $F^{(1)}(c)$ has bounded inverse, then the boundary-value problem (25), (26) has only one solution for each c^0.*

Proof. The proof directly comes from the theorem of superposition of differentiable mappings and from equality

$$F^{(1)}(c)[h] = \mathcal{P}_{N(\overline{Q}^*)}\Bigg\{ J^{(1)}(v,\varepsilon)\Big|_{v=\varphi_0,\varepsilon=0}\Big[\varphi_0^{(1)}(\cdot,s,c)[h]\Big] -$$

$$- \ell\int_s^{\cdot} U(\cdot,\tau)\, Z^{(1)}(v,\tau,\varepsilon)\Big|_{v=\varphi_0,\varepsilon=0}\Big[\varphi_0^{(1)}(\tau,s,c)[h]\Big]\,d\tau\Bigg\} = B_0[h].$$

Due to the invertibility of the operator $F^{(1)}(c)$ operator B_0 is also invertible. This is why the equation (29) and the respective boundary-value problem (25), (26) have the only solution for each element $c = c^0$.

Two-point boundary-value problem for the Schrödinger equation with a constant operator. Consider the boundary-value problem for the Schrödinger equation with the constant operator in the Hilbert space $\mathcal{H}_T$:

$$\frac{d\varphi(t)}{dt} = -iH_0\varphi(t) + f(t), \quad t\in[0;w], \tag{32}$$

$$\varphi(0) - \varphi(w) = \alpha \in D, \tag{33}$$

where $\mathcal{H}_T = \mathcal{H}\oplus\mathcal{H},$ $\mathcal{H}$ is Hilbert space and vector-function $f(t)$ is integrable; unbounded operator H_0 for every $t\in[0;w]$ has a form [1, 13]

$$H_0 = i\begin{pmatrix} 0 & T\\ -T & 0\end{pmatrix} = i\begin{pmatrix} T & 0\\ 0 & T\end{pmatrix}\begin{pmatrix} 0 & I\\ -I & 0\end{pmatrix} = i\begin{pmatrix} 0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix} T & 0\\ 0 & T\end{pmatrix}.$$

In the general case operator H_0 can have a form

$$H_0 = iJ\begin{pmatrix} T & 0\\ 0 & T\end{pmatrix} = i\begin{pmatrix} T & 0\\ 0 & T\end{pmatrix}J, \quad J = J^* = J^{-1},$$

where T is a positive self-adjoint operator in the Hilbert space $\mathcal{H}$. Since the operator T is closed, then the domain $D(T)$ of the operator T is the Hilbert space with the scalar product (Tu, Tu). Operator H_0 is a is self-adjoint in the domain $D = D(T)\oplus D(T)$ with the scalar product

$$(\langle u,v\rangle,\langle u,v\rangle)_{\mathcal{H}^T} = (Tu,Tu)_{\mathcal{H}} + (Tv,Tv)_{\mathcal{H}}$$

and an infinitesimal generator of a strongly continuous evolution group:

$$U(t) := U(t,0) = \begin{pmatrix} \cos\, tT & \sin\, tT\\ -\sin\, tT & \cos\, tT\end{pmatrix},$$

$$U^n(t) = \begin{pmatrix} \cos\, ntT & \sin\, ntT \\ -\sin\, ntT & \cos\, ntT \end{pmatrix},$$

$\|U^n(t)\| = 1,\ n \in \mathbb{N}$ (nonexpanding group);

$$\varphi(t) = (\varphi_1(t), \varphi_2(t))^T, \qquad \alpha = (\alpha_1, \alpha_2)^T, \qquad f(t) = (f_1(t), f_2(t))^T.$$

Weak solutions of the equation (32) can be represented in the form

$$\varphi(t) = U(t)c + \int\limits_0^t U(t)U^{-1}(\tau)f(\tau)\, d\tau$$

for an arbitrary element $c \in \mathcal{H}_T$. Substituting it in the condition (33), we obtain that the solvability of the boundary-value problem (32), (33) is equivalent to the solvability of such operator equation:

$$(I - U(w))c = g_1, \tag{34}$$

where

$$g_1 = \alpha + U(w) \int\limits_0^w U^{-1}(\tau)f(\tau)\, d\tau.$$

Consider the case, when the set $I - U(w)$ is closed $R(I - U(w)) = \overline{R(I - U(w))}$. The same as in the papers [11, 13], the solvability of (34) can be defined, by using the operator

$$U_0(w) = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} U^k(w)}{n} = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} U(kw)}{n},$$

which is orthoprojector of the space $\mathcal{H}_T$ on the subspace of $1 \in \sigma(U(w))$. Equation (34) would be solvable if and only if

$$U_0(w)g_1 = 0.$$

Under this condition the solutions of (34) would have the representation

$$c = U_0(w)\overline{c} + \left(\sum_{k=0}^{\infty} (\mu - 1)^k \left\{ \sum_{l=0}^{\infty} \mu^{-l-1}(U(w) - U_0(w))^l \right\}^{k+1} - U_0(w) \right) g_3$$

for $0 < \mu - 1 < \dfrac{1}{\|R_\mu(U(w))\|}$ and for an arbitrary $\overline{c} \in \mathcal{H}_T$. We obtain such result.

Lemma 1. *Suppose the operator $I - U(w)$ has the closed set of values*

$$R(I - U(w)) = \overline{R(I - U(w))}.$$

Then:

I. Generalized solutions of the boundary-value problem (32), (33) exist if and only if the following condition holds:

$$U_0(w)\left(\alpha + \int\limits_0^w U^{-1}(\tau)f(\tau)\, d\tau \right) = 0. \tag{35}$$

II. Under the condition (35) the solutions (32), (33) have the form

$$\varphi(t,\overline{c}) = U(t)U_0(w)\overline{c} + (G[f,\alpha])(t), \tag{36}$$

where

$$(G[f,\alpha])(t) = U(t)\sum_{k=0}^{\infty}(\mu-1)^k\left\{\sum_{l=0}^{\infty}\mu^{-l-1}(U(w)-U_0(w))^l\right\}^{k+1}\times$$
$$\times\left(\alpha+\int\limits_0^w U(w)U^{-1}(\tau)f(\tau)\,d\tau\right)-$$
$$-U(t)U_0(w)\left(\alpha+\int\limits_0^w U(w)U^{-1}(\tau)f(\tau)\,d\tau\right)+$$
$$+\int\limits_0^t U(t)U^{-1}(\tau)f(\tau)\,d\tau,$$

is generalized Green's operator of the boundary-value problem (32), (33) for $0<\mu-1<$ $<1/\|R_\mu(U(w))\|$.

Let us show how to remove the condition $R(I-U(w))=\overline{R(I-U(w))}$ from the lemma and to make the boundary-value problem (32), (33) always solvable in the certain sense. Lets extend all possible cases in detail and clarify a few aspects of some operators and spaces expansions.

(i) Classical generalized solutions.

If $R(I-U(w))=\overline{R(I-U(w))}$, then [2] $g_1\in R(I-U(w))$ if and only if $\mathcal{P}_{N((I-U(w))^*)}g_1=0$ and the set of solutions (34) has a form [2] $c=G[g_1]+U_0(w)\overline{c},\ \overline{c}\in\mathcal{H}_T,$ where [10]

$$G[g_1]=(I-U(w))^+\,g_1=(I-(U(w)-U_0(w)))^{-1}-U_0(w))g_1$$

is generalized Green's operator (or in the form of the convergent series).

(ii) Strong generalized solutions.

If $R(I-U(w))\neq\overline{R(I-U(w))}$ and element $g_1\in\overline{R(I-U(w))}$, then operator $I-U(w)$ can be extended to the operator $\overline{I-U(w)}$ with the closed set of values.

Now we describe previously built theory in terms of the corresponding spaces and operators [13]. Since the operator $I-U(w)$ is bounded, then such expansion of the space $\mathcal{H}_T$ on the direct sums is fair:

$$\mathcal{H}_T=N(I-U(w))\oplus X,\qquad \mathcal{H}_T=\overline{R(I-U(w))}\oplus Y$$

with $X=N(I-U(w))^{\perp}=\overline{R(I-U(w))}$ and $Y=\overline{R(I-U(w))}^{\perp}=N(I-U(w))$. Denoting $E=\mathcal{H}_T/N(I-U(w))$ the quotient space of the space $\mathcal{H}_T$ by the kernel $N(I-U(w))$, $\mathcal{P}_{\overline{R(I-U(w))}}$ and $\mathcal{P}_{N(I-U(w))}$ are orthoprojectors, which project the space $\mathcal{H}_T$ on the $\overline{R(I-U(w))}$ and $N(I-U(w))$ respectively. Then the operator

$$\mathcal{I}-\mathcal{U}(w)=\mathcal{P}_{\overline{R(I-U(w))}}(I-U(w))j^{-1}p\colon X\to R(I-U(w))\subset\overline{R(I-U(w))}$$

is linear, continuous and injective. Here

$$p\colon X \to E = \mathcal{H}_T/N(I - U(w)), \quad j\colon \mathcal{H}_T \to E$$

is a continuous bijection and projection accordingly. The triple $(\mathcal{H}_T, E, j)$ is a locally trivial bundle with typical fibre $\mathcal{H}_1 = \mathcal{P}_{N(I-U(w))}\mathcal{H}$ [8]. In this case [9, p. 26 – 29] we can define the strong generalized solution of the equation

$$(\mathcal{I} - \mathcal{U}(w))x = g_1, \quad x \in X. \tag{37}$$

Filling the space X by the norm $\|x\|_{\overline{X}} = \|(\mathcal{I} - \mathcal{U}(w))x\|_F$, where $F = \overline{R(I - U(w))}$ [9], we obtain the new space $\overline{X}$. Extended operator

$$\overline{\mathcal{I} - \mathcal{U}(w)}\colon \overline{X} \to \overline{R(I - U(w))}, \quad X \subset \overline{X}$$

is a homeomorphism between $\overline{X}$ and $\overline{R(I - U(w))}$. Then the equation

$$(\overline{\mathcal{I} - \mathcal{U}(w)})\overline{\xi} = g_1$$

has one solution $(\overline{\mathcal{I} - \mathcal{U}(w)})^{-1}g_1$, which is traditionally called *a strong generalized solution* of the equation (37).

Remark 5. We emphasize that such expansions of the spaces and the respective operators exist:

$$\overline{p}\colon \overline{X} \to E, \quad \overline{j}\colon \overline{\mathcal{H}}_T \to E, \quad \overline{\mathcal{P}}_X = \mathcal{P}_{\overline{X}}\colon \overline{\mathcal{H}}_T \to \overline{X}, \quad \overline{G}\colon \overline{R(I - U(w))} \to \overline{X},$$

where

$$\overline{\mathcal{H}}_T = N(I - U(w)) \oplus \overline{X}, \quad \overline{p}(x) = p(x), \quad x \in X, \quad \overline{j}(x) = j(x), \quad x \in \mathcal{H}_T,$$

$$\overline{\mathcal{P}}_X(x) = \mathcal{P}_X(x), \quad x \in \mathcal{H}_T \left(\mathcal{P}_X = \mathcal{P}_X^2 = \mathcal{P}_X^*\right),$$

$$\overline{G}[g_1] = G[g_1], \quad g_1 \in R(I - U(w)).$$

Operator $\overline{I - U(w)} = \left(\overline{\mathcal{I} - \mathcal{U}(w)}\right)\mathcal{P}_{\overline{X}}\colon \overline{\mathcal{H}}_T \to \mathcal{H}_T$ is an extension of the operator $I - U(w)$, $\overline{(I - U(w))}c = (I - U(w))c$ for the arbitrary element $c \in \mathcal{H}_T$.

(iii) Strong pseudosolutions.

Consider element $g_1 \notin \overline{R(I - U(w))}$. This is equivalent to the condition $\mathcal{P}_{N(I-U(w))^*}g_1 \neq 0$. In this case, the elements from $\overline{\mathcal{H}}_T$ exist, which minimize the norm $\left\|\overline{(I - U(w))}\xi - g_1\right\|_{\mathcal{H}_T}$:

$$\xi = \overline{(\mathcal{I} - \mathcal{U}(w))}^{-1}g_1 + \mathcal{P}_{N\left(\overline{I - U(w)}\right)}\overline{c} \quad \forall\, \overline{c} \in \overline{\mathcal{H}}_T.$$

These elements are called *strong pseudosolutions* analogously to [2].

Let us formulate now complete theorem of solvability of two-point boundary-value problem for Schrödinger's equation [13].

Theorem 6. *Define the boundary problem (32), (33).*

I (a) Classical or strong generalized solutions of the boundary problem (32), (33) exist if and only if

$$U_0(w)\left(\alpha+\int\limits_0^w U^{-1}(\tau)f(\tau)\,d\tau\right)=0. \tag{38}$$

If

$$\left(\alpha+\int\limits_0^w U^{-1}(\tau)f(\tau)d\tau\right)\in R(I-U(w)),$$

then the solutions of (32), (33) are classical generalized.

(b) If (38) holds, then the solutions (32), (33) have the form

$$\varphi(t,\overline{c})=U(t)U_0(w)\overline{c}+\left(\overline{G[f,\alpha]}\right)(t),$$

where $\left(\overline{G[f,\alpha]}\right)(t)$ *is an expansion of operator* $(G[f,\alpha])(t)$.

II (a) Strong pseudosolutions of boundary-value problem (32), (33) exist if and only if

$$U_0(w)\left(\alpha+\int\limits_0^w U^{-1}(\tau)f(\tau)\,d\tau\right)\neq 0. \tag{39}$$

(b) Under the condition (39) the strong pseudosolutions (32), (33) have the following view:

$$\varphi(t,\overline{c})=U(t)U_0(w)\overline{c}+\left(\overline{G[f,\alpha]}\right)(t),$$

where

$$\left(\overline{G[f,\alpha]}\right)(t)=U(t)\overline{G}[g_1]+\int\limits_0^t U(t)U^{-1}(\tau)f(\tau)\,d\tau=$$

$$=U(t)\overline{(\mathcal{I}-\mathcal{U}(w))}^{-1}g_1+\int\limits_0^t U(t)U^{-1}(\tau)f(\tau)\,d\tau.$$

Nonlinear case. In the Hilbert space $\mathcal{H}_T$, which is defined above, consider the boundary-value problem

$$\frac{d\varphi(t,\varepsilon)}{dt}=-iH_0\varphi(t,\varepsilon)+\varepsilon Z(\varphi(t,\varepsilon),t,\varepsilon)+f(t), \tag{40}$$

$$\varphi(0,\varepsilon)-\varphi(w,\varepsilon)=\alpha. \tag{41}$$

We need to find the solution $\varphi(t,\varepsilon)$ of the boundary-value problem (40), (41), which turns in one of the solutions of generating equation (32), (33) $\varphi_0(t,\overline{c})$ of the form (36) with $\varepsilon=0$.

To find the necessary condition of the operator-valued function $Z(\varphi, t, \varepsilon),$ we require continuity in the neighborhood of the generating solution

$$Z(\cdot,\cdot,\cdot) \in C[\|\varphi - \varphi_0\| \leq q] \times C([0; w], \mathcal{H}_T) \times C[0, \varepsilon_0],$$

where q is some positive constant.

This problem can be solved by using the operator equation for generating amplitudes [13]:

$$F(\overline{c}) := U_0(w) \int\limits_0^w U^{-1}(\tau) Z(\varphi_0(\tau, \overline{c}), \tau, 0)\, d\tau = 0. \tag{42}$$

Theorem 7 (necessary condition)**.** *Assume a nonlinear boundary-value problem (40), (41) has solution $\varphi(\cdot, \varepsilon)$, which turns in one of the solutions $\varphi_0(t, \overline{c})$ of generated problem (32), (33) with element $\overline{c} = c^0,\ \varphi(t, 0) = \varphi_0(t, c^0)$ for $\varepsilon = 0$. Then this element is a root of the operator equation for generating amplitudes (42).*

Suppose that operator-function $Z(\varphi, t, \varepsilon)$ is strongly differentiable in the neighborhood of generating solution

$$Z(\cdot, t, \varepsilon) \in C^1\left[\|\varphi - \varphi_0\| \leq q\right].$$

Sufficient condition can be obtained by using the following operator:

$$B_0 := \left.\frac{dF(\overline{c})}{d\overline{c}}\right|_{\overline{c}=c_0} = U_0(w) \int\limits_0^w U^{-1}(t) A_1(t) dt \colon \mathcal{H} \to \mathcal{H},$$

where $A_1(t) = Z^1(v, t, \varepsilon)\big|_{v=\varphi_0, \varepsilon=0}$ (derivative in the Fréchet sense).

Theorem 8 (sufficient condition)**.** *Suppose the operator B_0 satisfies the following conditions:*

(i) B_0 has Moore – Penrose pseudoinverse;

(ii) $\mathcal{P}_{N(B_0^*)} U_0(w) = 0.$

Then for any element $c = c^0 \in \mathcal{H}_T$, which satisfies operator equation for generating amplitudes (42) there is at least one strong generalized solution (40), (41).

This solution can be found by using the iterative process

$$\overline{v}_{k+1}(t, \varepsilon) = \varepsilon G\left[Z\left(\varphi_0\left(\tau, c^0\right) + v_k, \tau, \varepsilon\right), \alpha\right](t),$$

$$c_k = -B_0^+ U_0(w) \int\limits_0^w U^{-1}(\tau) \left\{A_1(\tau)\overline{v}_k(\tau, \varepsilon) + \mathcal{R}(v_k(\tau, \varepsilon), \tau, \varepsilon)\right\} d\tau,$$

$$\mathcal{R}\left(v_k(t, \varepsilon), t, \varepsilon\right) = Z\left(\varphi_0(t, c^0) + v_k(t, \varepsilon), t, \varepsilon\right) - Z\left(\varphi_0(t, c^0), t, 0\right) - A_1(t) v_k(t, \varepsilon),$$

$$\mathcal{R}(0, t, 0) = 0, \qquad \mathcal{R}^1_x(0, t, 0) = 0,$$

$$v_{k+1}(t, \varepsilon) = U(t) U_0(w) c_k + \overline{v}_{k+1}(t, \varepsilon),$$

$$\varphi_k(t, \varepsilon) = \varphi_0(t, c^0) + v_k(t, \varepsilon), \quad k = 0, 1, 2, \ldots,$$

$$v_0(t, \varepsilon) = 0, \varphi(t, \varepsilon) = \lim_{k\to\infty} \varphi_k(t, \varepsilon).$$

Corollary 3. *Let the operator $F(\overline{c})$ has Fréchet derivative $F^{(1)}(\overline{c})$ for every element c^0 of the Hilbert space $\mathcal{H}$, which satisfies operator equation for generating amplitudes (42). If $F^{(1)}(\overline{c})$ has bounded inverse, then boundary-value problem (40), (41) has the only solution for every c^0.*

References

1. *Reed M., Simon B.* Methods of modern mathematical physics. Harmonic analysis. Self-adjointness. Vol. 2. – Moscow: Mir, 1978. – 395 p.
2. *Boichuk A. A., Samoilenko A. M.* Generalized inverse operators and Fredholm boundary-value problems. 2nd ed. – Berlin; Boston: De Gruyter, 2016. – 296 p.
3. *Samoilenko A., Petrishin R.* Multifrequency oscillations of nonlinear systems. – Dordrecht: Springer, 2004. – 317 p.
4. *Krein S. G.* Linear differential equations in Banach space. – Moscow: Nauka, 1967. – 464 p.
5. *Boichuk A. A., Pokutnyi A. A.* Theory of bifurcations of the Schrödinger equation // Differ. Equ. – 2017. – **53**, № 7. – P. 882 – 890.
6. *Pokutnyi O. O.* Generalized-inverse operator in the Fréchet, Banach and Hilbert spaces // Bull. Kiev Univ. Ser. Phys. Math. – 2013. – № 4. – P. 158 – 161.
7. *Robertson A., Robertson V.* Topological vector spaces. – Moscow: Mir, 1967. – 257 p.
8. *Atiyah M. F.* K-theory: lectures / Notes by D. W. Anderson, fall, 1964 // New-York, Amsterdam: Benjamin Inc., 1967. – 166+xlix p.
9. *Klyushin D. A., Lyashko S. I., Nomirovskii D. A., Petunin Yu. I., Semenov V. V.* Generalized solutions of operator equations and extreme elements // New York; Dordrecht; Heidelberg; London: Springer Science+Business Media, LLC, 2012. – 212 p.
10. *Biletskyi B. A., Boichuk A. A., Pokutnyi A. A.* Periodic problems of difference equations and ergodic theory // Abstr. Appl. Anal. – 2011. – Article ID 928587. – 12 p.; http://www.hindawi.com/journals/aaa/2011/928587
11. *Pokutnyi A. A.* Periodic solutions of the Hill equation // J. Math. Sci. – 2014. – **197**, № 1. – P. 114 – 121.
12. *Pokutnyi A. A.* Representation of the solutions of boundary-value problems for the schrodinger equation in a hilbert space // J. Math. Sci. – 2015. – **205**, № 6. – P. 821 – 831.
13. *Boichuk A. A., Pokutnyi A. A.* Solutions of the Schrodinger equation in a Hilbert space // Bound. Value Probl. – 2014. – **2014**, № 4. – 9 p.; http://www.boundaryvalueproblems.com/content/2014/1/4

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