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ON THE EXPONENTIAL STABILITY OF NEUTRAL LINEAR SYSTEMS WITH VARIABLE DELAYS
ПРО ЕКСПОНЕНЦІАЛЬНУ СТІЙКІСТЬ НЕЙТРАЛЬНИХ ЛІНІЙНИХ СИСТЕМ ЗІ ЗМІННИМИ ВІДСТАВАННЯМИ ЗА ЧАСОМ

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We investigate the exponential stability of a linear system of neutral-type with variable time lags. With the help of the Newton – Leibniz formula and a Lyapunov – Krasovskii functional, we prove two results of the exponential stability of solutions. The stability criteria are stated in the form of linear matrix inequalities (LMIs). By using MATLAB-Simulink, we give two numerical examples that illustrate the applicability of the assumptions. The obtained results extend and generalize the existing former ones in the related literature.

Досліджено експоненціальну стійкість лінійної системи нейтрального типу зі змінними відставаннями за часом. За допомогою формули Ньютона – Лейбніца та функціоналу Ляпунова – Красовського доведено два результати про экспоненціальну стійкість розв'язків. Критерії стійкості подані у вигляді лінійних матричних нелінійностей. З використанням MATLAB-Simulink наведено два числові приклади, які ілюструють застосовність припущень. Одержані результати розширюють і узагальнюють відомі результати, одержані раніше.

1. Introduction. Stability analysis of linear and nonlinear systems with variable time lags is fundamental to many practical problems and has received considerable attention [1 – 4]. In particular, it is well known that stability of linear systems of neutral-type with variable lags have been an active research topic in last few decades. The main reason for this, neutral systems has been growing commonly because of their successful applications in widespread fields of science and engineering such as circuit theory, bioengineering, population dynamics, telecommunication, automatic control [5] and so on. In addition, we note that neutral systems without or with time varying lags and different models of functional differential equations often occur in many scientific areas such as engineering techniques fields, physics, medicine and etc. (see [4, 6 – 26] and the references therein).

When the relevant literature is examined, especially in the last few years, the problem of stability has been addressed through different approaches for neutral systems. It is seen that the stability method in most of the studies related to the stability of the systems in the control theory is based on linear matrix inequality (LMI) and the classical Lyapunov stability theory. For example, Altun and Tunç [8] obtained some sufficient conditions for the solution of the nonlinear delayed neutral system with periodic coefficient. In [2], by constructing an appropriate Lyapunov – Krasovskii functional combined, new delay-dependent sufficient conditions for the exponential stability of the systems are presented in terms of LMIs. By utilizing free-weight matrices and constructing augmented Lyapunov functionals, some less conservative conditions for asymptotic stability are derived in [19, 27] for systems with lags varying in an interval. Therefore, the current study is worth investigating qualitative properties of solutions of that kind of equations.

Throughout this work, $\Re^n$ denotes n-dimensional Euclidean space with the scalar product $x^T y$ and the Euclidean norm $\|.\|$ for vectors; $D^{n\times r}$ denotes the space of all matrices of $(n \times r)$-dimensions; $*$ denotes the elements below the main diagonal of a symmetric matrix; $C^1\left([0,t],\Re^n\right)$ means the set of all $\Re^n$-valued continuously differentiable functions on $[0,t]; x_t$ is the state at time t defined by $x_t(s) = \{x(t+s)\colon s\epsilon[-h,0]\}$ with $\|x_t\| = \sup_{s\epsilon[-h,0]} \|x(t+s)\|$; A^T means the transpose of the matrix A; B is symmetric if $B = B^T$; C is positive definite $(C > 0)$ if $\langle Cx, x\rangle > 0$ for all $x \neq 0$; the notation $X > Y$, where X and Y are symmetric matrices of same dimensions, means that the matrix $X - Y$ is positive definite; $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimal and maximal eigenvalue of the matrix A, respectively.

2. Problem description. In the current paper, motivated by [2, 5, 14, 15, 25] we consider a neutral linear system with variable time lags:

$$\dot{x}(t) - \sum_{i=1}^{2} B_i \dot{x}(t-\tau_i(t)) = A_0 x(t) + \sum_{i=1}^{2} A_i(t) x(t-\tau_i(t)), \tag{2.1}$$

$$t \geq 0, \quad x(t) = \phi(t), \quad t \in [-H, 0],$$

where $x(t) \in \Re^n$, $H \in \Re$, $H > 0$, $\phi(.) \in C^1\left([-H,0],\Re^n\right)$ is the initial functions, A_0 and $B_i \in$ $\in D^{n\times n}$ are known constant matrices, $A_i(t) \in D^{n\times n}$ are a reel matrix functions, $\tau_i(t)$, $i = 1, 2$, are differentiable variable time lags.

We deal two different cases for the variable time lags as follows:

Case I. The functions τ_i are differentiable such that

$$0 \leq \tau_{1i} \leq \tau_i(t) \leq \tau_{2i}, \qquad \dot{\tau}_i(t) \leq \delta_i < 1, \quad t \geq 0, \quad i = 1, 2, \tag{2.2}$$

where τ_{1i}, τ_{2i}, and δ_i are real constants.

Case II. The functions τ_i are not differentiable or the upper bound of the derivatives of these functions are unknown such that

$$0 \leq \tau_{1i} \leq \tau_i(t) \leq \tau_{2i}, \quad t \geq 0, \quad i = 1, 2. \tag{2.3}$$

Let $H = \{\tau_{21}, \tau_{22}\}$.

The main aim of this work is to do a contribution to the results of [2, 5, 14, 15, 25] and the relevant literature. For example, if we consider the equation studied by Phat et al. [2] and system (2.1), the contribution of this work is clearly seen as the following:

(i) the delayed system studied by Phat et al. [2] has a linear and simple form and includes one variable delay. However, the neutral system (2.1) has a more general form and two variable delays;

(ii) the neutral system (2.1) includes and improve that discussed by Phat et al. [2];

(iii) the results of this paper may be useful for researches working in engineering and sciences. These are some brief contributions of this paper to the related ones.

Before state the exponentially results, the following basic definition and lemmas are needed.

Definition 2.1 [20]. *System (2.1) is said to be exponential stable with convergence rate α if there exist two positive constants α and λ such that*

$$\|x(t)\| \le \lambda e^{-\alpha t}, \quad t \ge 0.$$

Lemma 2.1 [25]. *For any symmetric positive-definite matrix $S \in D^{n\times n}$, a scalar $\gamma > 0$ and a vector function $g\colon [0,\gamma] \to \Re^n$ if the integrations in the following inequality are well defined, then we have*

$$\gamma \int_0^\gamma g^T(s) S g(s) ds \ge \left[\int_0^\gamma g(s)ds\right]^T S \left[\int_0^\gamma g(s)ds\right].$$

Lemma 2.2. *Let $F \in D^{n\times n}$ be a symmetric positive-definite matrix and $a, b \in \Re^n$. Then, we have*

$$\left|a^T b\right| \le a^T F a + b^T F^{-1} b.$$

3. Exponential stability. We now state the exponentially stability results of this paper such that they are proved by the aid of LMIs.

Theorem 3.1. *Suppose that the assumptions of Case I and the inequality $\left\|\sum_{i=1}^2 B_i\right\| < 1$ hold. In addition, let $P,\ Q,\ R,\ S,\ U$ be symmetric positive-definite matrices and the matrices $M_i,\ i = 1, 2, \ldots, 5,$ are given such that the following LMI holds:*

$$\eta = \begin{bmatrix}
\eta_{1,1} & \eta_{1,2} & \eta_{1,3} & \eta_{1,4} & \eta_{1,5} & \eta_{1,6} & \eta_{1,7} & \eta_{1,8} & \eta_{1,9} & \eta_{1,10} \\
* & \eta_{2,2} & 0 & 0 & 0 & \eta_{2,6} & -M_2A_2(t) & M_2 & -M_2B_1 & -M_2B_2 \\
* & * & \eta_{3,3} & 0 & 0 & -M_2A_1(t) & \eta_{3,7} & M_2 & -M_2B_1 & -M_2B_2 \\
* & * & * & \eta_{4,4} & 0 & \eta_{4,6} & -M_2A_2(t) & M_2 & -M_2B_1 & -M_2B_2 \\
* & * & * & * & \eta_{5,5} & -M_2A_1(t) & \eta_{5,7} & M_2 & -M_2B_1 & -M_2B_2 \\
* & * & * & * & * & \eta_{6,6} & \eta_{6,7} & \eta_{6,8} & \eta_{6,9} & \eta_{6,10} \\
* & * & * & * & * & * & \eta_{7,7} & \eta_{7,8} & \eta_{7,9} & \eta_{7,10} \\
* & * & * & * & * & * & * & \eta_{8,8} & \eta_{8,9} & \eta_{8,10} \\
* & * & * & * & * & * & * & * & \eta_{9,9} & \eta_{9,10} \\
* & * & * & * & * & * & * & * & * & \eta_{10,10}
\end{bmatrix} < 0. \tag{3.1}$$

Then, the zero solution of system (2.1) with (2.2) is α-exponential stable for $\alpha > 0,$ where

$$\eta_{1,1} = A_0^T P + PA_0 + 2\alpha P + 2Q - 2M_1 A_0 - \sum_{k=1}^{2}\sum_{i=1}^{2} e^{-2\alpha\tau_{ki}} S,$$

$$\eta_{1,2} = e^{-2\alpha\tau_{11}} S - M_2 A_0, \qquad \eta_{1,3} = e^{-2\alpha\tau_{12}} S - M_2 A_0, \qquad \eta_{1,4} = e^{-2\alpha\tau_{21}} S - M_2 A_0,$$

$$\eta_{1,5} = e^{-2\alpha\tau_{22}} S - M_2 A_0, \qquad \eta_{1,6} = (P - M_1) A_1(t) - M_3 A_0,$$

$$\eta_{1,7} = (P - M_1) A_2(t) - M_3 A_0, \qquad \eta_{1,8} = M_1 - M_4 A_0, \qquad \eta_{1,9} = (P - M_1) B_1 - M_5 A_0,$$

$$\eta_{1,10} = (P - M_1) B_2 - M_5 A_0, \qquad \eta_{2,2} = -e^{-2\alpha\tau_{11}} S - e^{-2\alpha\tau_{21}} U,$$

$$\eta_{2,6} = \eta_{4,6} = e^{-2\alpha\tau_{21}} U - M_2 A_1(t), \qquad \eta_{3,3} = -e^{-2\alpha\tau_{12}} S - e^{-2\alpha\tau_{22}} U,$$

$$\eta_{3,7} = \eta_{5,7} = e^{-2\alpha\tau_{22}} U - M_2 A_2(t), \qquad \eta_{4,4} = -e^{-2\alpha\tau_{21}} (S + U), \qquad \eta_{5,5} = -e^{-2\alpha\tau_{22}} (S + U),$$

$$\eta_{6,6} = -e^{-2\alpha\tau_{21}} [(1 - \delta_1) Q + 2U] - 2M_3 A_1(t), \qquad \eta_{6,7} = -M_3 (A_1(t) + A_2(t)),$$

$$\eta_{6,8} = M_3 - M_4 A_1(t), \quad \eta_{6,9} = -(M_3 B_1 + M_5 A_1(t), \quad \eta_{6,10} = \eta_{7,10} = -(M_3 B_2 + M_5 A_2(t)),$$

$$\eta_{7,7} = -e^{-2\alpha\tau_{22}} [(1 - \delta_2) Q + 2U] - 2M_3 A_2(t), \qquad \eta_{7,8} = M_3 - M_4 A_2(t),$$

$$\eta_{7,9} = -(M_3 B_1 + M_5 A_2(t)), \qquad \eta_{8,8} = 2M_4 + 2R + \sum_{k=1}^{2}\sum_{i=1}^{2} \tau_{ki}^2 S + \sum_{i=1}^{2} (\tau_{2i} - \tau_{1i})^2 U,$$

$$\eta_{8,9} = M_5 - M_4 B_1, \qquad \eta_{8,10} = M_5 - M_4 B_2, \qquad \eta_{9,9} = -(1 - \delta_1) e^{-2\alpha\tau_{21}} R - 2M_5 B_1,$$

$$\eta_{9,10} = -M_5 (B_1 + B_2), \qquad \eta_{10,10} = -(1 - \delta_2) e^{-2\alpha\tau_{22}} R - 2M_5 B_2.$$

Proof. Define a Lyapunov – Krasovskii functional by

$$W(t, x_t) = \sum_{k=1}^{5} W_k, \tag{3.2}$$

where

$$W_1 = x^T(t) P x(t),$$

$$W_2 = \sum_{i=1}^{2} \int_{t-\tau_i(t)}^{t} e^{2\alpha(s-t)} x^T(s) Q x(s) ds,$$

$$W_3 = \sum_{i=1}^{2} \int_{t-\tau_i(t)}^{t} e^{2\alpha(s-t)} \dot{x}^T(s) R \dot{x}(s) ds,$$

$$W_4 = \sum_{k=1}^{2}\sum_{i=1}^{2} \tau_{ki} \int_{-\tau_{ki}}^{0} \int_{t+s}^{t} e^{2\alpha(\sigma-t)} \dot{x}^T(\sigma) S \dot{x}(\sigma) d\sigma ds,$$

$$W_5=\sum_{i=1}^{2}(\tau_{2i}-\tau_{1i})\int\limits_{-\tau_{2i}}^{-\tau_{1i}}\int\limits_{t+s}^{t}e^{2\alpha(\sigma-t)}\dot{x}^T(\sigma)U\dot{x}(\sigma)d\sigma ds.$$

It is now easy to verify that

$$\lambda_1\|x(t)\|^2\le W(t,x_t)\le\lambda_2\|x(t)\|^2, \tag{3.3}$$

where

$$\lambda_1=\lambda_{\min}(P),$$

$$\lambda_2=\lambda_{\max}(P)+\sum_{i=1}^{2}\tau_{2i}\lambda_{\max}(Q+R)+2\sum_{i=1}^{2}\tau_{2i}^2\lambda_{max}(S)+\sum_{i=1}^{2}(\tau_{2i}-\tau_{1i})^2\lambda_{\max}(U).$$

By the derivative of the functional W along the solutions of system (2.1), we get

$$\dot{W}_1=2x^T(t)P\dot{x}(t)=$$

$$=x^T(t)\left[A_0^TP+PA_0\right]x(t)+2x^T(t)P\sum_{i=1}^{2}A_i(t)x(t-\tau_i(t))+2x^T(t)P\sum_{i=1}^{2}B_i\dot{x}(t-\tau_i(t)),$$

$$\dot{W}_2=2x^T(t)Qx(t)-\sum_{i=1}^{2}(1-\dot{\tau}_i(t))e^{-2\alpha\tau_i(t)}x^T(t-\tau_i(t))Qx(t-\tau_i(t))-2\alpha W_2,$$

$$\dot{W}_3=2\dot{x}^T(t)R\dot{x}(t)-\sum_{i=1}^{2}(1-\dot{\tau}_i(t))e^{-2\alpha\tau_i(t)}\dot{x}^T(t-\tau_i(t))R\dot{x}(t-\tau_i(t))-2\alpha W_3,$$

$$\dot{W}_4=\sum_{k=1}^{2}\sum_{i=1}^{2}\tau_{ki}^2\dot{x}^T(t)S\dot{x}(t)-\sum_{k=1}^{2}\sum_{i=1}^{2}\tau_{ki}\int\limits_{t-\tau_{ki}}^{t}e^{2\alpha(s-t)}\dot{x}^T(s)S\dot{x}(s)ds-2\alpha W_4\le$$

$$\le\sum_{k=1}^{2}\sum_{i=1}^{2}\tau_{ki}^2\dot{x}^T(t)S\dot{x}(t)-\sum_{k=1}^{2}\sum_{i=1}^{2}\tau_{ki}e^{-2\alpha\tau_{ki}}\int\limits_{t-\tau_{ki}}^{t}\dot{x}^T(s)S\dot{x}(s)ds-2\alpha W_4,$$

$$\dot{W}_5=\sum_{i=1}^{2}(\tau_{2i}-\tau_{1i})^2\dot{x}^T(t)U\dot{x}(t)-\sum_{i=1}^{2}(\tau_{2i}-\tau_{1i})\int\limits_{t-\tau_{2i}}^{t-\tau_{1i}}e^{2\alpha(s-t)}\dot{x}^T(s)U\dot{x}(s)ds-2\alpha W_5\le$$

$$\le\sum_{i=1}^{2}(\tau_{2i}-\tau_{1i})^2\dot{x}^T(t)U\dot{x}(t)-\sum_{i=1}^{2}(\tau_{2i}-\tau_{1i})e^{-2\alpha\tau_{2i}}\int\limits_{t-\tau_{2i}}^{t-\tau_{1i}}\dot{x}^T(s)U\dot{x}(s)ds-2\alpha W_5.$$

By the assumption (2.2), it follows that

$$\dot{W}_2\le 2x^T(t)Qx(t)-\sum_{i=1}^{2}(1-\delta_i)e^{-2\alpha\tau_{2i}}x^T(t-\tau_i(t))Qx(t-\tau_i(t))-2\alpha W_2$$

and

$$\dot{W}_3 \le 2\dot{x}^T(t)R\dot{x}(t) - \sum_{i=1}^{2}(1-\delta_i)e^{-2\alpha\tau_{2i}}\dot{x}^T(t-\tau_i(t))R\dot{x}(t-\tau_i(t)) - 2\alpha W_3.$$

Applying the Newton – Leibniz formula and Lemma 2.1, we derive

$$\sum_{k=1}^{2}\sum_{i=1}^{2}\int\limits_{t-\tau_{ki}}^{t}\dot{x}(s)ds = 4x(t) - \sum_{k=1}^{2}\sum_{i=1}^{2}x(t-\tau_{ki})$$

and

$$-\sum_{k=1}^{2}\sum_{i=1}^{2}\tau_{ki}\int\limits_{t-\tau_{ki}}^{t}\dot{x}^T(s)S\dot{x}(s)ds \le -\sum_{k=1}^{2}\sum_{i=1}^{2}\left[\int\limits_{t-\tau_{ki}}^{t}\dot{x}(s)ds\right]^T S\left[\int\limits_{t-\tau_{ki}}^{t}\dot{x}(s)ds\right] =$$

$$= -\sum_{k=1}^{2}\sum_{i=1}^{2}[x(t)-x(t-\tau_{ki})]^T S[x(t)-x(t-\tau_{ki})].$$

We note that

$$\sum_{i=1}^{2}\int\limits_{t-\tau_{2i}}^{t-\tau_{1i}}\dot{x}^T(s)U\dot{x}(s)ds = \sum_{i=1}^{2}\int\limits_{t-\tau_{2i}}^{t-\tau_i(t)}\dot{x}^T(s)U\dot{x}(s)ds + \sum_{i=1}^{2}\int\limits_{t-\tau_i(t)}^{t-\tau_{1i}}\dot{x}^T(s)U\dot{x}(s)ds.$$

By the aid of Lemma 2.1 we have

$$\sum_{i=1}^{2}[\tau_{2i}-\tau_i(t)]\int\limits_{t-\tau_{2i}}^{t-\tau_i(t)}\dot{x}^T(s)U\dot{x}(s)ds \ge$$

$$\ge \sum_{i=1}^{2}\left[\int\limits_{t-\tau_{2i}}^{t-\tau_i(t)}\dot{x}(s)ds\right]^T U\left[\int\limits_{t-\tau_{2i}}^{t-\tau_i(t)}\dot{x}(s)ds\right] =$$

$$= \sum_{i=1}^{2}[x(t-\tau_i(t))-x(t-\tau_{2i})]^T U\sum_{i=1}^{2}[x(t-\tau_i(t))-x(t-\tau_{2i})].$$

Since $\tau_{2i}-\tau_i(t) \le \tau_{2i}-\tau_{1i}$, then

$$\sum_{i=1}^{2}[\tau_{2i}-\tau_{1i}]\int\limits_{t-\tau_{2i}}^{t-\tau_i(t)}\dot{x}^T(s)U\dot{x}(s)ds \ge$$

$$\ge \sum_{i=1}^{2}[x(t-\tau_i(t))-x(t-\tau_{2i})]^T U\sum_{i=1}^{2}[x(t-\tau_i(t))-x(t-\tau_{2i})]$$

and

$$-\sum_{i=1}^{2}[\tau_{2i}-\tau_{1i}]\int\limits_{t-\tau_{2i}}^{t-\tau_i(t)}\dot{x}^T(s)U\dot{x}(s)ds \le$$

$$\le -\sum_{i=1}^{2}[x(t-\tau_i(t))-x(t-\tau_{2i})]^T U\sum_{i=1}^{2}[x(t-\tau_i(t))-x(t-\tau_{2i})].$$

Similarly, we derive that

$$-\sum_{i=1}^{2}[\tau_{2i}-\tau_{1i}]\int\limits_{t-\tau_i(t)}^{t-\tau_{1i}}\dot{x}^T(s)U\dot{x}(s)ds \le$$

$$\le -\sum_{i=1}^{2}[x(t-\tau_{1i})-x(t-\tau_i(t))]^T U\sum_{i=1}^{2}[x(t-\tau_{1i})-x(t-\tau_i(t))].$$

Gathering up these results, we have

$$\begin{aligned}\dot{W}(\cdot)+2\alpha W(\cdot) \le{}& x^T(t)[A_0^TP+PA_0+2\alpha P+2Q]x(t)+\\ &+2x^T(t)P\sum_{i=1}^{2}A_i(t)x(t-\tau_i(t))+2x^T(t)P\sum_{i=1}^{2}B_i\dot{x}(t-\tau_i(t))-\\ &-\sum_{i=1}^{2}(1-\delta_i)e^{-2\alpha\tau_{2i}}x^T(t-\tau_i(t))Qx(t-\tau_i(t))-\\ &-\sum_{i=1}^{2}(1-\delta_i)e^{-2\alpha\tau_{2i}}\dot{x}^T(t-\tau_i(t))R\dot{x}(t-\tau_i(t))+\\ &+\dot{x}^T(t)\left[2R+\sum_{k=1}^{2}\sum_{i=1}^{2}\tau_{ki}^2S+\sum_{i=1}^{2}(\tau_{2i}-\tau_{1i})^2U\right]\dot{x}(t)-\\ &-\sum_{k=1}^{2}\sum_{i=1}^{2}e^{-2\alpha\tau_{ki}}[x(t)-x(t-\tau_{ki})]^TS[x(t)-x(t-\tau_{ki})]-\\ &-\sum_{i=1}^{2}e^{-2\alpha\tau_{2i}}[x(t-\tau_i(t))-x(t-\tau_{2i})]^TU[x(t-\tau_i(t))-x(t-\tau_{2i})]-\\ &-\sum_{i=1}^{2}e^{-2\alpha\tau_{2i}}[x(t-\tau_{1i})-x(t-\tau_i(t))]^TU[x(t-\tau_{1i})-x(t-\tau_i(t))]. \quad (3.4)\end{aligned}$$

By using the equality

$$\dot{x}(t)-\sum_{i=1}^{2}B_i\dot{x}(t-\tau_i(t))-A_0x(t)-\sum_{i=1}^{2}A_i(t)x(t-\tau_i(t))=0,$$

we have

$$2x^T(t)M_1\dot{x}(t) - 2x^T(t)M_1\sum_{i=1}^{2}B_i\dot{x}(t-\tau_i(t)) -$$

$$-2x^T(t)M_1A_0x(t) - 2x^T(t)M_1\sum_{i=1}^{2}A_i(t)x(t-\tau_i(t)) = 0,$$

$$\sum_{k=1}^{2}\sum_{i=1}^{2}2x^T(t-\tau_{ki})M_2\dot{x}(t) - \sum_{k=1}^{2}\sum_{i=1}^{2}2x^T(t-\tau_{ki})M_2\sum_{i=1}^{2}B_i\dot{x}(t-\tau_i(t)) -$$

$$-\sum_{k=1}^{2}\sum_{i=1}^{2}2x^T(t-\tau_{ki})M_2A_0x(t) - \sum_{k=1}^{2}\sum_{i=1}^{2}2x^T(t-\tau_{ki})M_2\sum_{i=1}^{2}A_i(t)x(t-\tau_i(t)) = 0,$$

$$\sum_{i=1}^{2}2x^T(t-\tau_i(t))M_3\dot{x}(t) - \sum_{i=1}^{2}2x^T(t-\tau_i(t))M_3\sum_{i=1}^{2}B_i\dot{x}(t-\tau_i(t)) -$$

$$-\sum_{i=1}^{2}2x^T(t-\tau_i(t))M_3A_0x(t) - \sum_{i=1}^{2}2x^T(t-\tau_i(t))M_3\sum_{i=1}^{2}A_i(t)x(t-\tau_i(t)) = 0,$$

$$2\dot{x}^T(t)M_4\dot{x}(t) - 2\dot{x}^T(t)M_4\sum_{i=1}^{2}B_i\dot{x}(t-\tau_i(t)) -$$

$$-2\dot{x}^T(t)M_4A_0x(t) - 2\dot{x}^T(t)M_4\sum_{i=1}^{2}A_i(t)x(t-\tau_i(t)) = 0,$$

$$\sum_{i=1}^{2}2\dot{x}^T(t-\tau_i(t))M_5\dot{x}(t) - \sum_{i=1}^{2}2\dot{x}^T(t-\tau_i(t))M_5\sum_{i=1}^{2}B_i\dot{x}(t-\tau_i(t)) -$$

$$-\sum_{i=1}^{2}2\dot{x}^T(t-\tau_i(t))M_5A_0x(t) - \sum_{i=1}^{2}2\dot{x}^T(t-\tau_i(t))M_5\sum_{i=1}^{2}A_i(t)x(t-\tau_i(t)) = 0. \quad (3.5)$$

In view of (3.5) and (3.4), we deduce that

$$\dot{W}(\cdot) + 2\alpha W(\cdot) \le x^T(t)\left[A_0^TP + PA_0 + 2\alpha P + 2Q - 2M_1A_0 - \sum_{k=1}^{2}\sum_{i=1}^{2}e^{-2\alpha\tau_{ki}}S\right]x(t) +$$

$$+2x^T(t)\sum_{k=1}^{2}\sum_{i=1}^{2}\left(e^{-2\alpha\tau_{ki}}S - M_2A_0\right)x(t-\tau_{ki}) +$$

$$+2x^T(t)\sum_{i=1}^{2}\left[P - M_1A_1(t) - M_3A_0\right]x(t-\tau_i(t)) +$$

$$+ 2x^T(t)(M_1 - M_4A_0)\dot{x}(t) + 2x^T(t)(P - M_1)\sum_{i=1}^{2} B_i\dot{x}(t - \tau_i(t)) -$$

$$- 2x^T(t)M_5A_0\sum_{i=1}^{2}\dot{x}(t - \tau_i(t)) -$$

$$- \sum_{k=1}^{2}\sum_{i=1}^{2} x^T(t - \tau_{ki})\left(e^{-2\alpha\tau_{ki}}S + e^{-2\alpha\tau_{2i}}U\right)x(t - \tau_{ki}) +$$

$$+ 2\sum_{k=1}^{2}\sum_{i=1}^{2} e^{-2\alpha\tau_{2i}}x^T(t - \tau_{ki})Ux(t - \tau_i(t)) -$$

$$- \sum_{k=1}^{2}\sum_{i=1}^{2} 2x^T(t - \tau_{ki})M_2\sum_{i=1}^{2} A_i(t)x(t - \tau_i(t)) +$$

$$+ 2\sum_{k=1}^{2}\sum_{i=1}^{2} x^T(t - \tau_{ki})M_2\dot{x}(t) - 2\sum_{k=1}^{2}\sum_{i=1}^{2} x^T(t - \tau_{ki})M_2\sum_{i=1}^{2} B_i\dot{x}(t - \tau_i(t)) -$$

$$- \sum_{i=1}^{2}\left(e^{-2\alpha\tau_{2i}}x^T(t - \tau_i(t)[(1 - \delta_i)Q + 2U]x(t - \tau_i(t))\right) -$$

$$- 2\sum_{i=1}^{2} x^T(t - \tau_i(t))M_3\sum_{i=1}^{2} A_i(t)x(t - \tau_i(t)) +$$

$$+ 2\sum_{i=1}^{2} x^T(t - \tau_i(t))(M_3 - M_4A_i(t))\dot{x}(t) -$$

$$- 2\sum_{i=1}^{2} x^T(t - \tau_i(t))M_3\sum_{i=1}^{2} B_i\dot{x}(t - \tau_i(t)) -$$

$$- 2\sum_{i=1}^{2} \dot{x}^T(t - \tau_i(t))M_5\sum_{i=1}^{2} A_i(t)x(t - \tau_i(t)) +$$

$$+ \dot{x}(t)\left[2M_4 + 2R + \sum_{k=1}^{2}\sum_{i=1}^{2}\tau_{ki}^2 S + \sum_{i=1}^{2}(\tau_{2i} - \tau_{1i})^2 U\right]\dot{x}(t) +$$

$$+ 2\sum_{i=1}^{2} \dot{x}^T(t - \tau_i(t))M_5\dot{x}(t) - 2\sum_{i=1}^{2} B_i\dot{x}(t - \tau_i(t))M_4\dot{x}(t) -$$

$$- \sum_{i=1}^{2}(1 - \delta_i)e^{-2\alpha\tau_{2i}}\dot{x}^T(t - \tau_i(t)R\dot{x}(t - \tau_i(t)) -$$

$$-2\sum_{i=1}^{2}\dot{x}^T(t-\tau_i(t))M_5\sum_{i=1}^{2}B_i\dot{x}(t-\tau_i(t))=\xi^T(t)\eta\xi(t),$$

where

$$\begin{aligned}\xi(t)=\big[&x(t),x(t-\tau_{11}),x(t-\tau_{12}),x(t-\tau_{21}),x(t-\tau_{22}),\\&x(t-\tau_1(t)),x(t-\tau_2(t)),\dot{x}(t),\dot{x}(t-\tau_1(t)),\dot{x}(t-\tau_2(t))\big].\end{aligned}$$

If the condition (3.1) holds, then

$$\dot{W}(t,x_t)\le -2\alpha W(t,x_t),\quad t\ge 0. \tag{3.6}$$

Integrating both sides of (3.6) from 0 to $t,$ we have

$$W(t,x_t)\le W(0,\phi)e^{-2\alpha t},\quad t\ge 0.$$

Furthermore, taking into account the inequality (3.3), we obtain

$$\lambda_1\|x(t,\phi)\|^2\le W(t,x_t)\le W(0,\phi(0))e^{-2\alpha t}\le \lambda_2 e^{-2\alpha t}\|\phi\|^2.$$

Hence, it follows that

$$\|x(t,\phi)\|\le\sqrt{\frac{\lambda_2}{\lambda_1}}\,e^{-\alpha t}\|\phi\|,\quad t\ge 0.$$

This ends the proof of the theorem.

Theorem 3.2. *Suppose that the assumptions of Case II and the inequality* $\left\|\sum_{i=1}^{2}B_i\right\|<1$ *hold. Further, we assume that there exist symmetric positive-definite matrices* $P,$ $S,$ U *and the matrices* $M_i,$ $i=1,2,\ldots,5,$ *such that the following LMI holds:*

$$\xi=\begin{bmatrix}\xi_{1,1}&\xi_{1,2}&\xi_{1,3}&\xi_{1,4}&\xi_{1,5}&\xi_{1,6}&\xi_{1,7}&\xi_{1,8}&\xi_{1,9}&\xi_{1,10}\\ *&\xi_{2,2}&0&0&0&\xi_{2,6}&-M_2A_2(t)&M_2&-M_2B_1&-M_2B_2\\ *&*&\xi_{3,3}&0&0&-M_2A_1(t)&\xi_{3,7}&M_2&-M_2B_1&-M_2B_2\\ *&*&*&\xi_{4,4}&0&\xi_{4,6}&-M_2A_2(t)&M_2&-M_2B_1&-M_2B_2\\ *&*&*&*&\xi_{5,5}&-M_2A_1(t)&\eta_{5,7}&M_2&-M_2B_1&-M_2B_2\\ *&*&*&*&*&\xi_{6,6}&\xi_{6,7}&\xi_{6,8}&\xi_{6,9}&\xi_{6,10}\\ *&*&*&*&*&*&\xi_{7,7}&\xi_{7,8}&\xi_{7,9}&\xi_{7,10}\\ *&*&*&*&*&*&*&\xi_{8,8}&\xi_{8,9}&\xi_{8,10}\\ *&*&*&*&*&*&*&*&-2M_5B_1&\xi_{9,10}\\ *&*&*&*&*&*&*&*&*&-2M_5B_2\end{bmatrix}<0, \tag{3.7}$$

where

$$\xi_{1,1}=A_0^TP+PA_0+2\alpha P-2M_1A_0-\sum_{k=1}^{2}\sum_{i=1}^{2}e^{-2\alpha\tau_{ki}}S,\qquad \xi_{1,2}=e^{-2\alpha\tau_{11}}S-M_2A_0,$$

$$\xi_{1,3} = e^{-2\alpha\tau_{12}}S - M_2A_0, \qquad \xi_{1,4} = e^{-2\alpha\tau_{21}}S - M_2A_0, \qquad \xi_{1,5} = e^{-2\alpha\tau_{22}}S - M_2A_0,$$

$$\xi_{1,6} = (P - M_1)A_1(t) - M_3A_0, \xi_{1,7} = (P - M_1)A_2(t) - M_3A_0, \qquad \xi_{1,8} = M_1 - M_4A_0,$$

$$\xi_{1,9} = (P - M_1)B_1 - M_5A_0, \qquad \xi_{1,10} = (P - M_1)B_2 - M_5A_0,$$

$$\xi_{2,2} = -e^{-2\alpha\tau_{11}}S - e^{-2\alpha\tau_{21}}U, \qquad \xi_{2,6} = \xi_{4,6} = e^{-2\alpha\tau_{21}}U - M_2A_1(t),$$

$$\xi_{3,3} = -e^{-2\alpha\tau_{12}}S - e^{-2\alpha\tau_{22}}U, \qquad \xi_{3,7} = \xi_{5,7} = e^{-2\alpha\tau_{22}}U - M_2A_2(t),$$

$$\xi_{4,4} = -e^{-2\alpha\tau_{21}}(S + U), \qquad \xi_{5,5} = -e^{-2\alpha\tau_{22}}(S + U), \qquad \xi_{6,6} = -e^{-2\alpha\tau_{21}}U - 2M_3A_1(t),$$

$$\xi_{6,7} = -M_3(A_1(t) + A_2(t)), \qquad \xi_{6,8} = M_3 - M_4A_1(t), \qquad \xi_{6,9} = -(M_3B_1 + M_5A_1(t)),$$

$$\xi_{6,10} = \xi_{7,10} = -(M_3B_2 + M_5A_2(t)), \qquad \xi_{7,7} = -e^{-2\alpha\tau_{22}}U - 2M_3A_2(t),$$

$$\xi_{7,8} = M_3 - M_4A_2(t), \qquad \xi_{7,9} = -(M_3B_1 + M_5A_2(t)),$$

$$\xi_{8,8} = 2M_4 + \sum_{k=1}^{2}\sum_{i=1}^{2}\tau_{ki}^2 S + \sum_{i=1}^{2}(\tau_{2i} - \tau_{1i})^2 U, \qquad \xi_{8,9} = M_5 - M_4B_1,$$

$$\xi_{8,10} = M_5 - M_4B_2, \qquad \xi_{9,10} = -M_5(B_1 + B_2),$$

$$\mu_1 = \lambda_{\min}(P), \qquad \mu_2 = \lambda_{\max}(P) + 2\sum_{i=1}^{2}\tau_{2i}^2\lambda_{\max}(S) + \sum_{i=1}^{2}(\tau_{2i} - \tau_{1i})^2\lambda_{\max}(U).$$

Then, the zero solution of system (2.1) with (2.3) is α*-exponential stable for* $\alpha > 0$.

Proof. In the light assumptions of Theorem 3.2, we use the auxiliary functional given by (3.2) provided that $Q = R = 0$. Hence, if LMI (3.7) holds, then we easily obtain

$$\dot{W}(t, x_t) \leq -2\alpha W(t, x_t), \quad t \geq 0. \tag{3.8}$$

Integrating both sides of (3.8) from 0 to t, we obtain

$$W(t, x_t) \leq W(0, \phi(0))e^{-2\alpha t}, \quad t \geq 0.$$

As before done, we derive that

$$\mu_1\|x(t)\|^2 \leq W(t, x_t) \leq \mu_2\|x(t)\|^2, \quad t \geq 0. \tag{3.9}$$

Taking into account inequality (3.9), since the functional $W(t, x_t)$ is decreasing, we have

$$\mu_1\|x(t, \phi)\|^2 \leq W(t, x_t) \leq W(0, \phi(0))e^{-2\alpha t} \leq \mu_2 e^{-2\alpha t}\|\phi\|^2.$$

Hence, it follows that

$$\|x(t, \phi)\| \leq \sqrt{\frac{\mu_2}{\mu_1}}e^{-\alpha t}\|\phi\|, \quad t \geq 0.$$

This ends the proof of Theorem 3.2.

4. Applications and simulation results. We give the following two examples with numerical simulations to show the applicability of the results of this paper.

Example 4.1. For the case $n = 2$ and $i, k = 1$, as a special case of system (2.1), we consider the following linear neutral system with a variable time lag:

$$\frac{d}{dt}\left[x(t) - B_1 x(t - \tau_1(t))\right] = A_0 x(t) + A_1(t) x(t - \tau_1(t)), \quad t \geq 0, \tag{4.1}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad A_0 = \begin{bmatrix} -5 & 0 \\ 0 & -3 \end{bmatrix}, \qquad A_1(t) = \begin{bmatrix} -0.0025 & 0 \\ 0 & -0.0045 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.01025 & 0 \\ 0 & 0.01012 \end{bmatrix},$$

and

$$0.4 \leq \tau_1(t) = 0.4 + 0.1 \sin^2(t) \leq 0.5.$$

Let

$$P = \begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}, \qquad Q = \begin{bmatrix} 16.5 & 0 \\ 0 & 0.15 \end{bmatrix}, \qquad R = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix},$$

$$S = \begin{bmatrix} 0.00102 & 0 \\ 0 & 0.00121 \end{bmatrix}, \qquad U = \begin{bmatrix} 0.12035 & 0 \\ 0 & 0.10365 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} -0.152 & 0 \\ 0 & -0.275 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0.0102 & 0 \\ 0 & -0.0203 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.004 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} -0.65 & 0.5 \\ 0.02 & -0.04 \end{bmatrix}, \qquad M_5 = \begin{bmatrix} -0.0125 & 0 \\ 0 & 0.03 \end{bmatrix}.$$

Hence, for the this special case, the eigenvalues of matrix in (3.7) are found as -21.9939, -8.2830, -5.2892, -7.5237, -0.4119, -0.2117, -0.0002, -0.0034, -0.0503, -0.0576, -0.0651 and -0.0666, respectively. Further, the solution $\|x(t, \phi)\|$ of the system (4.1) satisfies

$$\|x(t, \phi)\| \leq 1.5948 e^{-0.6t} \|\phi\|, \quad t \geq 0.$$

Hence, it is followed that all the assumptions of Theorem 3.1 are satisfied. This discussion implies that the zero solution of system (4.1) is exponentially stable.

We would like to mention that the graph given by Fig. 1 shows behaviors of the solutions of the system (4.1), which have been solved by MATLAB-Simulink.

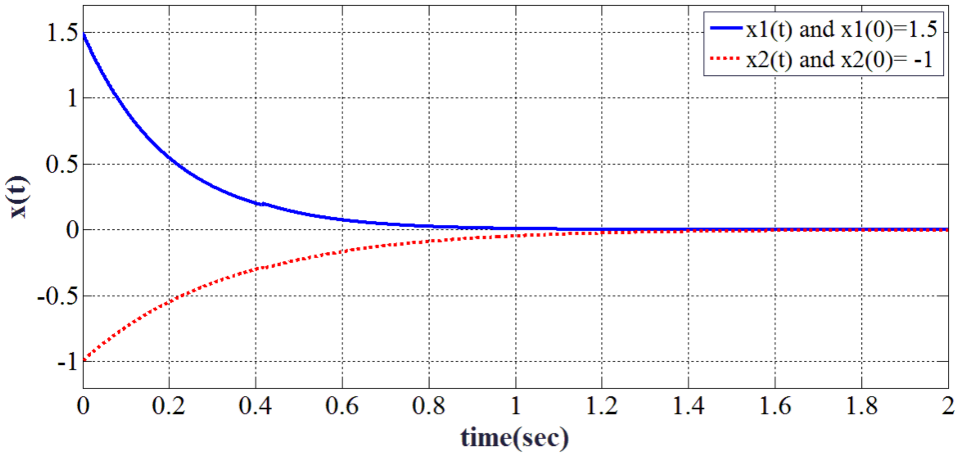

Fig. 1. The orbits of solutions of system (4.1).

Example 4.2. We now consider a particular case of neutral system (4.1) with

$$A_0 = \begin{bmatrix} -4.35 & 0 \\ 0 & -4.38 \end{bmatrix}, \qquad A_1(t) = \begin{bmatrix} -0.0025 & 0 \\ 0 & -0.0045 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0.01025 & 0 \\ 0 & 0.01012 \end{bmatrix}$$

and

$$\tau_1(t) = 0.1 + 0.3\sin^2(t) \le 0.4, \quad \text{if} \quad t \in I = \bigcup_{k\ge 0} [2k\Pi, (2k+1)\Pi],$$

$$\tau_1(t) = 0, \quad t \in \Re^+/I.$$

We note that the delay function $\tau_1(t)$ here is not differentiable. Next, let us choose

$$P = \begin{bmatrix} 5 & 4.5 \\ 4.5 & 5.5 \end{bmatrix}, \qquad S = \begin{bmatrix} 0.102 & 0 \\ 0 & 0.121 \end{bmatrix}, \qquad U = \begin{bmatrix} 0.12035 & 0 \\ 0 & 0.10365 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} -0.52 & 0 \\ 0 & -0.75 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0.0102 & 0 \\ 0 & -0.0203 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.004 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} -0.65 & -0.75 \\ 0.02 & -0.04 \end{bmatrix}, \qquad M_5 = \begin{bmatrix} -0.01015 & 0.012 \\ 0.10015 & 0.03 \end{bmatrix}.$$

For the this special case, the eigenvalues of matrix in (3.7) are calculated as -79.3938, -12.0819, -0.1159, -0.1159, -0.2597, -0.2405, -0.0006, -0.0006, -0.0448, -0.0503, -0.1600 and -0.1568, respectively. Moreover, the solution of the give system satisfies

$$\|x(t,\phi)\| \le 3.6342e^{-0.6t}\|\phi\|, \quad t \ge 0.$$

At the end, we conclude that all the assumptions of Theorem 3.2 are satisfied. This discussion implies that the zero solution of the give system is exponentially stable.

We would also like to mention that the graph given by Fig. 2 shows behaviors of the solutions of the system in Example 4.2, which have been solved by MATLAB-Simulink.

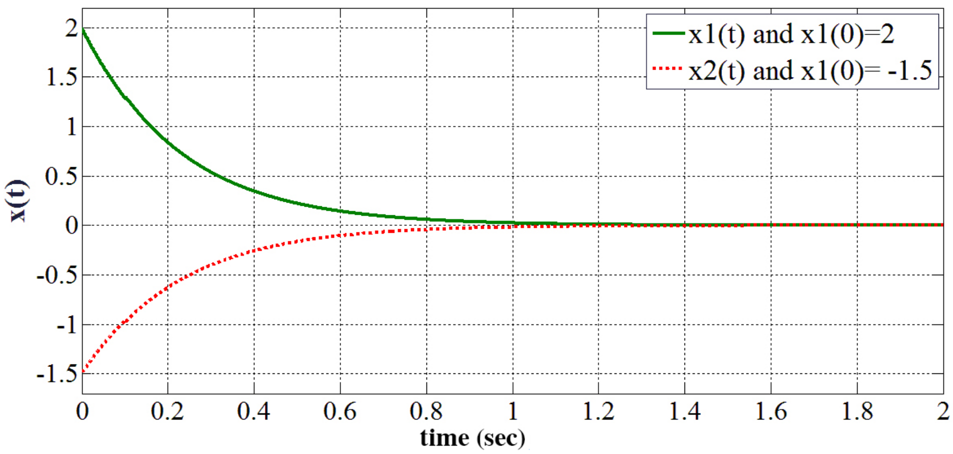

Fig. 2. The orbits of solutions of the given equation.

5. Conclusion. In this work, we derive some new sufficient conditions to guarantee the exponentially stability a linear system of neutral-type with variable time lags. The stability criteria are by the aid of an auxiliary functional and LMIs. Benefited from by MATLAB-Simulink, two numerical examples are presented to verify applicability of idea of this paper.

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