UDC 517.9

ON THE FORCED IMPULSIVE OSCILLATORY NONLINEAR NEUTRAL SYSTEMS OF THE SECOND ORDER*

ПРО КОЛИВАЛЬНІ НЕЛІНІЙНІ НЕЙТРАЛЬНІ СИСТЕМИ ДРУГОГО РОДУ ПРИ ІМПУЛЬСНІЙ ДІЇ

A. K. Tripathy

Dep. Math., Sambalpur Univ.
Sambalpur 768019, India
e-mail: arun_tripathy70@rediffmail.com

S. S. Santra

Dep. Math., Sambalpur Univ.
Sambalpur 768019, India;
Dep. Math., JIS College Eng.,
Kalyani 741235, India
e-mail: shyam01.math@gmail.com

We study the oscillatory and nonoscillatory behavior of solutions for a class of forced impulsive nonlinear neutral differential systems of the form

$$\begin{cases} \big(r(t)\big(y(t)+p(t)y(t-\tau)\big)'\big)' + q(t)G\big(y(t-\sigma)\big) = f(t), & t \neq \tau_k, \quad k \in \mathbb{N}, \\ \Delta\big(r(\tau_k)\big(y(\tau_k)+p(\tau_k)y(\tau_k-\tau)\big)'\big) + h(\tau_k)G\big(y(\tau_k-\sigma)\big) = g(\tau_k), & k \in \mathbb{N}, \end{cases}$$

for various ranges of values of $p(t)$. Sufficient conditions for the existence of positive bounded solutions of this system are also obtained.

Вивчено осцилюючу та неосцилюючу поведінку розв'язків для одного класу нелінійних нейтральних диференціальних систем із імпульсною дією вигляду

$$\begin{cases} \big(r(t)\big(y(t)+p(t)y(t-\tau)\big)'\big)' + q(t)G\big(y(t-\sigma)\big) = f(t), & t \neq \tau_k, \quad k \in \mathbb{N}, \\ \Delta\big(r(\tau_k)\big(y(\tau_k)+p(\tau_k)y(\tau_k-\tau)\big)'\big) + h(\tau_k)G\big(y(\tau_k-\sigma)\big) = g(\tau_k), & k \in \mathbb{N}, \end{cases}$$

для різних областей значень $p(t)$. Також одержано достатні умови існування додатних обмежених розв'язків цієї системи.

1. Introduction. Consider a second order forced impulsive differential system of the form

$$\begin{cases} \big(r(t)\big(y(t)+p(t)y(t-\tau)\big)'\big)' + q(t)G\big(y(t-\sigma)\big) = f(t), & t \neq \tau_k, \quad k \in \mathbb{N}, \\ \Delta\big(r(\tau_k)\big(y(\tau_k)+p(\tau_k)y(\tau_k-\tau)\big)'\big) + h(\tau_k)G\big(y(\tau_k-\sigma)\big) = g(\tau_k), & k \in \mathbb{N}, \end{cases} \tag{E}$$

where $\tau > 0$, $\sigma \geq 0$ are real constants, $G \in C(\mathbb{R}, \mathbb{R})$ is nondecreasing such that $xG(x) > 0$ for $x \neq 0$, $q, r, h \in C(\mathbb{R}_+, \mathbb{R}_+)$, $p \in PC(\mathbb{R}_+, \mathbb{R})$, $p(\tau_k)$, $r(\tau_k)$, $q(\tau_k)$ and $h(\tau_k)$ are constants $(k \in \mathbb{N})$, τ_k for $k \in \mathbb{N}$ with $\tau_1 < \tau_2 < \ldots < \tau_k < \ldots$ and $\lim_{k\to\infty} \tau_k = \infty$ are fixed moments of impulsive effect, and $f, g \in C(\mathbb{R}, \mathbb{R})$. For (E), Δ is the difference operator defined by

$$\Delta\big(r(\tau_k)(z'(\tau_k))\big) = r(\tau_k+0)z'(\tau_k+0) - r(\tau_k-0)z'(\tau_k-0),$$
$$y(\tau_k-0) = y(\tau_k) \qquad \text{and} \qquad y(\tau_k-\tau-0) = y(\tau_k-\tau), \quad k \in \mathbb{N}.$$

The objective of this work is to establish the sufficient conditions for oscillation and nonoscillation of solutions of the impulsive system (E) for various ranges of $p(t)$. Here, we are concerned with the oscillating system which remains oscillating after being perturbed by the instantaneous change of state.

Impulsive differential equations are now recognized as an excellent source of models to simulate processes and phenomena observed in theoretical physics, chemical technology, population dynamics, industrial robotic, economics, rhythmical beating, merging of solutions and noncontinuity of solutions. Moreover, the theory of impulsive differential equations is emerging as an important area of investigation, since it is much richer than the corresponding theory of differential equations without impulse effect. Due to the wide range application of this theory to the real world problems, a good number of interests has been given to this study. We refer the readers to the monographs [1 – 6], where a number of properties of their solutions are discussed and the references cited there in. Our aim in this work is to discuss some oscillation properties and existence of positive bounded solutions of the impulsive system (E).

In [7], Tripathy has considered the impulsive system

$$\begin{cases} \big(y(t)+p(t)y(t-\tau)\big)' + q(t)G\big(y(t-\sigma)\big) = 0, & t \neq \tau_k, \quad k \in \mathbb{N}, \\ \Delta\big(y(\tau_k)+p(\tau_k)y(\tau_k-\tau)\big) + q(\tau_k)G\big(y(\tau_k-\sigma)\big) = 0, & k \in \mathbb{N}, \end{cases} \tag{E$_1$}$$

and studied the oscillatory character of the solutions of the system. For all ranges of $p(t)$, he has established the oscillation criteria for the impulsive system (E_1) which is highly nonlinear but, G could be linear, sublinear or superlinear also. In [8], Tripathy and Santra have studied the characterization of the impulsive system

$$\begin{cases} \big(y(t)-ry(t-\tau)\big)' + qy(t-\sigma) = 0, & t \neq \tau_k, \quad k \in \mathbb{N}, \\ \Delta\big(y(\tau_k)-ry(\tau_k-\tau)\big) + py(\tau_k-\sigma) = 0, & k \in \mathbb{N}, \end{cases} \tag{E$_2$}$$

and linearized oscillation of the system

$$\begin{cases} \big(y(t)-r(t)g(y(t-\tau))\big)' + q(t)f\big(y(t-\sigma)\big) = 0, & t \neq \tau_k, \quad k \in \mathbb{N}, \\ \Delta\big(y(\tau_k)-r(\tau_k)g(y(\tau_k-\tau))\big) + p(\tau_k)f\big(y(\tau_k-\sigma)\big) = 0, & k \in \mathbb{N}. \end{cases} \tag{E$_3$}$$

They have established the conditions for oscillation of the system (E_2) using the pulsatile constant and hence the linearized oscillation results carried out for (E_3) by using its limiting equations (E_2). In another work [9], Tripathy and Santra have established sufficient conditions for oscillation of all solutions and existence of nonoscillating solutions to impulsive equation

$$\begin{cases} \Big(r(t)\big(y(t)+p(t)y(t-\tau)\big)'\Big)' + q(t)G\big(y(t-\sigma)\big) = 0, & t \neq \tau_k, \quad k \in \mathbb{N}, \\ \Delta\Big(r(\tau_k)\big(y(\tau_k)+p(\tau_k)y(\tau_k-\tau)\big)'\Big) + q(\tau_k)G\big(y(\tau_k-\sigma)\big) = 0, & k \in \mathbb{N}, \end{cases} \tag{E$_4$}$$

* This work was supported by the Department of Science and Technology, New Delhi, India, through the letter No. DST/INSPIRE Fellowship/2014/140 (dated Sept. 15, 2014).

for various ranges of $p(t)$. However nothing is known about (E). If $f(t) \equiv 0$ then (E) is reduces to (E_4) (see, for example, [7, 9]). If $f(t) \not\equiv 0$ then (E) is more general than (E_4). Of course, the method which is employed for (E) and (E_4), we find small difference in the technique. Altogether, (E) can be handled by method of (E_4). We may note that this type of work is very rare in the literature signifying that the impulse of the differential equation follows a discrete type equation. In this direction, we refer the reader to some of the related works [10–33] and the references cited there in.

A function $y\colon [-\rho, +\infty) \to \mathbb{R}$ is said to be a solution of (E) with initial function $\varnothing \in C([-\rho, 0], \mathbb{R})$, if $y(t) = \varnothing(t)$ for $t \in [-\rho, 0]$, $y \in PC(\mathbb{R}_+, \mathbb{R})$, $z(t) = y(t) + p(t)y(t-\tau)$ and $r(t)z'(t)$ are continuously differentiable for $t \in \mathbb{R}_+$, and $y(t)$ satisfies (E) for all sufficiently large $t \geq 0$, where $\rho = \max\{\tau, \sigma\}$ and $PC(\mathbb{R}_+, \mathbb{R})$ is the set of all functions $U\colon \mathbb{R}_+ \to \mathbb{R}$ which are continuous for $t \in \mathbb{R}_+$, $t \neq \tau_k$, $k \in \mathbb{N}$, continuous from the left side for $t \in \mathbb{R}_+$, and have discontinuity of the first kind at the points $\tau_k \in \mathbb{R}_+$, $k \in \mathbb{N}$.

A nontrivial solution $y(t)$ of (E) is said to be nonoscillatory, if it is either eventually positive or eventually negative; Otherwise, it is called oscillatory.

A solution $y(t)$ of (E) is said to be regular, if it is defined on some interval $[T_y, +\infty) \subset [t_0, +\infty)$ and

$$\sup\left\{|y(t)|\colon t \geq T_y\right\} > 0$$

for every $T_y \geq T$. A regular solution $y(t)$ of (E) is said to be eventually positive (eventually negative), if there exists $t_1 > 0$ such that $y(t) > 0$ $(y(t) < 0)$ for $t \geq t_1$.

2. Sufficient Conditions for Oscillation. In this section, we discuss the oscillatory behaviour of solutions of the impulsive system (E). In the sequel, we use the following assumption:

(A_0) Suppose there exists $F \in C(\mathbb{R}, \mathbb{R})$ such that $\big(r(t)F'(t)\big) \in C(\mathbb{R}, \mathbb{R})$ and $F(t)$ is a solution of $\big(r(t)F'(t)\big)' = f(t)$ and $\Delta\big(r(\tau_k)F'(\tau_k)\big) = g(\tau_k)$. In addition, we assume that $F(t)$ changes sign with $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty$.

Theorem 2.1. *Let* $0 \leq p(t) \leq a < \infty$, $t \in \mathbb{R}_+$, *and* (A_0) *hold. Assume that:*

(A_1) there exists $\lambda > 0$ *such that* $G(u) + G(v) \geq \lambda G(u+v)$ *for* $u, v > 0$*;*

(A_2) $G(uv) \leq G(u)G(v)$ *for* $u, v \in \mathbb{R}_+$*;*

(A_3) $G(-u) = -G(u)$ *for* $u, v \in \mathbb{R}_+$*;*

(A_4) $F^+(t) = \max\{F(t), 0\}$ *and* $F^-(t) = \max\{-F(t), 0\}$*;*

(A_5) $\displaystyle\int_0^\infty \frac{d\eta}{r(\eta)} + \sum_{k=1}^\infty \frac{1}{r(\tau_k)} = \infty$*;*

(A_6) $\displaystyle\int_T^\infty Q(\eta)G\big(F^+(\eta-\sigma)\big)d\eta + \sum_{k=1}^\infty H_k G(F^+(\tau_k - \sigma)) = \infty,\ T > 0,$

and

(A_7) $\displaystyle\int_T^\infty Q(\eta)G\big(F^-(\eta-\sigma)\big)d\eta + \sum_{k=1}^\infty H_k G(F^-(\tau_k - \sigma)) = \infty,\ T > 0,$

hold, where $Q(t) = \min\{q(t), q(t-\tau)\}$, $t \geq \tau$, *and* $H_k = \min\big\{h(\tau_k), h(\tau_k - \tau)\big\}$, $k \in \mathbb{N}$. *Then every regular solution of the system (E) oscillates.*

Proof. Let $y(t)$ be a regular solution of (E). For the sake of contradiction, let the regular solution be nonoscillatory. So there exists $t_0 > \rho$ such that $y(t) > 0$, $y(t-\tau) > 0$ and $y(t-\sigma) > 0$ for $t \geq t_0$. Setting

$$\begin{aligned} z(t) &= y(t) + p(t)y(t-\tau), \quad t \neq \tau_k, \quad k \in \mathbb{N}, \\ z(\tau_k) &= y(\tau_k) + p(\tau_k)y(\tau_k - \tau), \quad k \in \mathbb{N}, \end{aligned} \tag{2.1}$$

and

$$w(t)=z(t)-F(t), \qquad w(\tau_k)=z(\tau_k)-F(\tau_k) \tag{2.2}$$

due to (A_0), it follows from (E) that

$$\big(r(t)w'(t)\big)'=-q(t)G\big(y(t-\sigma)\big)\le 0, \quad t\ne\tau_k, \quad k\in\mathbb{N}, \tag{2.3}$$

$$\Delta\big(r(\tau_k)w(\tau_k)\big)'=-h(\tau_k)G\big(y(\tau_k-\sigma)\big)\le 0, \quad k\in\mathbb{N}, \tag{2.4}$$

for $t\ge t_1>t_0+\sigma$. Consequently, $\big(r(t)w'(t)\big)$ is nonincreasing and $w'(t)$, $w(t)$ are of either eventually positive or eventually negative on $[t_2,\infty)$, $t_2>t_1$. Since $z(t)>0$, then $w(t)<0$ for $t\ge t_2$ implies that $F(t)>0$ for $t\ge t_2$ which is absurd. Hence, $w(t)>0$ for $t\ge t_2$. In what follows, we consider the cases $\big(r(t)w'(t)\big)<0$ or >0 for $t\ge t_2$. Let the former hold for $t\ge t_2$. So, there exist $C>0$ and $t_3>t_2$ such that $\big(r(t)w'(t)\big)\le -C$ for $t\ge t_3$. Ultimately, $\big(r(\tau_k)w'(\tau_k)\big)\le -C$. Integrating the relation $w'(t)\le -\dfrac{C}{r(t)}$, $t\ge t_3$, from t_3 to $t(>t_3)$, we obtain

$$w(t)-w(t_3)-\sum_{t_3\le\tau_k<t}w'(\tau_k)\le -C\int\limits_{t_3}^{t}\frac{d\eta}{r(\eta)},$$

that is,

$$w(t)\le w(t_3)-C\left[\int\limits_{t_3}^{t}\frac{d\eta}{r(\eta)}+\sum_{t_3\le\tau_k<t}\frac{1}{r(\tau_k)}\right]\to-\infty \quad \text{as} \quad t\to\infty,$$

a contradiction to the fact that $w(t)>0$ for $t\ge t_2$. Hence, $\big(r(t)w'(t)\big)>0$ for $t\ge t_2$. Ultimately, $z(t)>F(t)$ and hence $z(t)>\max\{0,F(t)\}=F^+(t)$ for $t\ge t_2$. Due to (2.1) – (2.3) becomes

$$0=\big(r(t)w'(t)\big)'+q(t)G\big(y(t-\sigma)\big)+G(a)\big[\big(r(t-\tau)w'(t-\tau)\big)'+q(t-\tau)G\big(y(t-\tau-\sigma)\big)\big]$$

for $t\ge t_2$ and because of (A_1) and (A_2), we find that

$$0\ge\big(r(t)w'(t)\big)'+G(a)\big(r(t-\tau)w'(t-\tau)\big)'+Q(t)\big[G\big(y(t-\sigma)\big)+G\big(ay(t-\tau-\sigma)\big)\big]\ge$$
$$\ge\big(r(t)w'(t)\big)'+G(a)\big(r(t-\tau)w'(t-\tau)\big)'+\lambda Q(t)G\big(z(t-\sigma)\big) \tag{2.5}$$

for $t\ge t_3>t_2+\sigma$. Similarly, from (2.4), we obtain

$$0\ge\Delta\big(r(\tau_k)w'(\tau_k)\big)+G(a)\Delta\big(r(\tau_k-\tau)w'(\tau_k-\tau)\big)+\lambda H_kG\big(z(\tau_k-\sigma)\big) \tag{2.6}$$

for $k\in\mathbb{N}$. Integrating (2.5) from t_3 to $+\infty$, we get

$$\lambda\int\limits_{t_3}^{\infty}Q(\eta)G\big(z(\eta-\sigma)\big)d\eta\le-\big[r(\eta)w'(\eta)+G(a)(r(\eta-\tau)w'(\eta-\tau))\big]_{t_3}^{\infty}+$$
$$+\sum_{t_3\le\tau_k<\infty}\Delta\big[r(\tau_k)w'(\tau_k)+G(a)(r(\tau_k-\tau)w'(\tau_k-\tau))\big]\le$$
$$\le-\big[r(\eta)w'(\eta)+G(a)\big(r(\eta-\tau)w'(\eta-\tau)\big)\big]_{t_3}^{\infty}-$$
$$-\lambda\sum_{t_3\le\tau_k<\infty}H_kG\big(z(\tau_k-\sigma)\big)$$

due to (2.6). Since $\lim_{t\to\infty}\big(r(t)w'(t)\big)$ exists, then the above inequality becomes

$$\lambda\left[\int\limits_{t_3}^{\infty} Q(\eta)G\big(z(\eta-\sigma)\big)d\eta + \sum_{t_3\le\tau_k<\infty} H_k G\big(z(\tau_k-\sigma)\big)\right] < \infty,$$

that is,

$$\lambda\left[\int\limits_{t_3}^{\infty} Q(\eta)G\big(F^+(\eta-\sigma)\big)d\eta + \sum_{t_3\le\tau_k<\infty} H_k G\big(F^+(\tau_k-\sigma)\big)\right] < \infty$$

which contradicts (A_6).

If $y(t)<0$ for $t\ge t_0$, then we set $x(t)=-y(t)$ for $t\ge t_0$ in (E) and we obtain that

$$\begin{cases} \big(r(t)(x(t)+p(t)x(t-\tau))'\big)' + q(t)G\big(x(t-\sigma)\big) = \widetilde{f}(t), & t\ne\tau_k, \quad k\in\mathbb{N}, \\ \Delta\big(r(\tau_k)(x(\tau_k)+p(\tau_k)x(\tau_k-\tau))'\big) + h(\tau_k)G\big(x(\tau_k-\sigma)\big) = \widetilde{g}(\tau_k), & k\in\mathbb{N}, \end{cases} \qquad (\widetilde{\mathrm{E}})$$

where $\widetilde{f}(t)=-f(t),\ \widetilde{g}(\tau_k)=-g(\tau_k)$ due to (A_3). Let $\widetilde{F}(t)=-F(t)$. Then

$$-\infty < \liminf_{t\to\infty}\widetilde{F}(t) < 0 < \limsup_{t\to\infty}\widetilde{F}(t) < \infty$$

and $\big(r(t)\widetilde{F}'(t)\big)'=\widetilde{f}(t),\ \Delta\big(r(\tau_k)\widetilde{F}'(\tau_k)\big)=\widetilde{g}(\tau_k)$ hold. Proceeding as above for $(\widetilde{\mathrm{E}})$, we can find a contradiction to (A_7). Thus, the proof of the theorem is complete.

Theorem 2.2. *Let $-1\le p(t)\le 0,\ t\in\mathbb{R}_+$. Assume that ($A_0$) and ($A_3$)–($A_5$) hold. If any one of the following conditions:*

(A_8) $\int_T^{\infty} q(\eta)G\big(F^+(\eta-\sigma)\big)d\eta + \sum_{k=1}^{\infty} h(\tau_k)G\big(F^+(\tau_k-\sigma)\big)=\infty,\ T>0$;

(A_9) $\int_T^{\infty} q(\eta)G\big(F^-(\eta+\tau-\sigma)\big)d\eta + \sum_{k=1}^{\infty} h(\tau_k)G\big(F^-(\tau_k+\tau-\sigma)\big)=\infty,\ T>0$;

(A_{10}) $\int_T^{\infty} q(\eta)G\big(F^-(\eta-\sigma)\big)d\eta + \sum_{k=1}^{\infty} h(\tau_k)G\big(F^-(\tau_k-\sigma)\big)=\infty,\ T>0$;

(A_{11}) $\int_T^{\infty} q(\eta)G\big(F^+(\eta+\tau-\sigma)\big)d\eta + \sum_{k=1}^{\infty} h(\tau_k)G\big(F^+(\tau_k+\tau-\sigma)\big)=\infty,\ T>0,$

hold, then every regular solution of (E) oscillates.

Proof. On the contrary, we proceed as in the proof of the Theorem 2.1 to conclude that $w(t)$ and $\big(r(t)w'(t)\big)$ are of either eventually positive or eventually negative on $[t_2,\infty)$. Assume that $w'(t)<0$ for $t\ge t_2$. Then as in Theorem 2.1, we find that $w(t)<0$ and $\lim_{t\to\infty}w(t)=-\infty$. So, there exists $t_3>t_2$ such that $z(t)<F(t)$ for $t\ge t_3$. If $z(t)>0$, then $F(t)>0$ which is not possible. Hence, $z(t)<0$ and $z(t)<F(t)$ for $t\ge t_3$. On the other hand, $z(t)<0$ for $t\ge t_3$ implies that

$$y(t)\le -p(t)y(t-\tau)\le y(t-\tau)\le y(t-2\tau)\le\ldots\le y(t_3), \quad t\ne\tau_k,$$

and also

$$y(\tau_k)\le y(\tau_k-\tau)\le\ldots\le y(t_3), \quad t\ne\tau_k, \quad k\in\mathbb{N},$$

that is, $y(t)$ is bounded on $[t_3,\infty)$. Consequently, $\lim_{t\to\infty}w(t)$ exists, a contradiction. Therefore, $w'(t)>0$ for $t\ge t_2$. Here we consider the cases: $w(t)<0$, $\big(r(t)w'(t)\big)>0$ and $w(t)>0$, $\big(r(t)w'(t)\big)>0$ on $[t_3,\infty),\ t_3>t_2$. With the former case $w(t)<0$, we get $z(t)<F(t)$ of course

$\lim_{t\to\infty}\big(r(t)w'(t)\big)$ exists. If $z(t)>0$ then $F(t)>0,$ a contradiction. Hence, $z(t)<0.$ Clearly, $-z(t)>-F(t)$ implies that $-z(t)>\max\{0,-F(t)\}=F^-(t).$ Therefore, for $t\ge t_3$

$$-y(t-\tau)\le p(t)y(t-\tau)\le z(t)<-F^-(t),$$

that is, $y(t-\sigma)>F^-(t+\tau-\sigma),\ t\ge t_4>t_3,$ and (2.3), (2.4) reduce to

$$\begin{aligned}&\big(r(t)w'(t)\big)'+q(t)G\big(F^-(t+\tau-\sigma)\big)\le 0,\quad t\ne\tau_k,\quad k\in\mathbb{N},\\&\Delta\big(r(\tau_k)w'(\tau_k)\big)+h(\tau_k)G\big(F^-(\tau_k+\tau-\sigma)\big)\le 0,\quad k\in\mathbb{N},\end{aligned}$$

for $t\ge t_4.$ Integrating the above impulsive system from t_4 to $+\infty,$ we obtain

$$\int\limits_{t_4}^{\infty}q(\eta)G\big(F^-(\eta+\tau-\sigma)\big)d\eta+\sum_{t_4\le\tau_k<\infty}h(\tau_k)G\big(F^-(\tau_k+\tau-\sigma)\big)<\infty$$

which contradicts (A_9). With the later case, it follows that $z(t)>F(t).$ If $z(t)<0,$ then $F(t)<0$ which is absurd. Therefore, $z(t)>0$ and $z(t)\le y(t)$ for $t\ge t_3>t_2.$ In this case, $\lim_{t\to\infty}\big(r(t)w'(t)\big)$ exists. Since, $F^+(t)=\max\{F(t),0\}<z(t)\le y(t)$ for $t\ge t_3,$ then (2.3) and (2.4) can be viewed as

$$\begin{aligned}&\big(r(t)w'(t)\big)'+q(t)G\big(F^+(t-\sigma)\big)\le 0,\quad t\ne\tau_k,\quad k\in\mathbb{N},\\&\Delta\big(r(\tau_k)w'(\tau_k)\big)+h(\tau_k)G\big(F^+(\tau_k-\sigma)\big)\le 0,\quad k\in\mathbb{N}.\end{aligned}$$

Integrating the above impulsive system from t_3 to $+\infty,$ we get

$$\int\limits_{t_3}^{\infty}q(\eta)G\big(F^+(\eta-\sigma)\big)d\eta+\sum_{t_3\le\tau_k<\infty}h(\tau_k)G\big(F^+(\tau_k-\sigma)\big)<\infty$$

a contradiction to (A_8). The case $y(t)<0$ for $t\ge t_0$ is similar. Hence, the proof of the theorem is complete.

Theorem 2.3. *Let $-\infty<-b\le p(t)\le-1,\ t\in\mathbb{R}_+,\ b>0.$ Assume that (A_0), (A_3)–(A_5), (A_8) and A_{10}) hold. Furthermore, assume that*

(A_{12}) $\displaystyle\int_T^\infty q(\eta)G\left(\frac{1}{b}F^-(\eta+\tau-\sigma)\right)d\eta+\sum\nolimits_{k=1}^\infty h(\tau_k)G\left(\frac{1}{b}F^-(\tau_k+\tau-\sigma)\right)=\infty,\ T>0,$

and

(A_{13}) $\displaystyle\int_T^\infty q(\eta)G\left(\frac{1}{b}F^+(\eta+\tau-\sigma)\right)d\eta+\sum\nolimits_{k=1}^\infty h(\tau_k)G\left(\frac{1}{b}F^+(\tau_k+\tau-\sigma)\right)=\infty,\ T>0.$

Then every bounded solution of (E) oscillates.

Proof. The proof of the theorem can be followed from the proof of the Theorem 2.2. Hence, the details are omitted.

In the following, we establish sufficient conditions for oscillation of all solution of (E) under the assumption that

(A_{14}) $\displaystyle\int_0^\infty\frac{d\eta}{r(\eta)}+\sum\nolimits_{k=1}^\infty\frac{1}{r(\tau_k)}<\infty.$

Let $R(t) = \int_t^{\infty} \frac{d\eta}{r(\eta)}$. Then $\int_0^{\infty} \frac{d\eta}{r(\eta)} < \infty$ implies that $R(t) \to 0$ as $t \to \infty$, since $R(t)$ is nonincreasing.

Theorem 2.4. *Let $0 \le p(t) \le a < \infty,\ t \in \mathbb{R}_+$. Assume that ($A_0$)–($A_4$), ($A_6$), ($A_7$) and ($A_{14}$) hold. If*

(A_{15}) $\int_T^{\infty} \frac{1}{r(\eta)} \left[\int_{T_1}^{\eta} Q(\zeta) G\big(F^+(\zeta-\sigma)\big) d\zeta + \sum_{k=1}^{\infty} H_k G\big(F^+(\tau_k-\sigma)\big)\right] d\eta = \infty,\ T, T_1 > 0$,

and

(A_{16}) $\int_T^{\infty} \frac{1}{r(\eta)} \left[\int_{T_1}^{\eta} Q(\zeta) G\big(F^-(\zeta-\sigma)\big) d\zeta + \sum_{k=1}^{\infty} H_k G\big(F^-(\tau_k-\sigma)\big)\right] d\eta = \infty,\ T,\ T_1 > 0$,

hold, then every regular solution of (E) is oscillatory, where $Q(t)$ and $H_k,\ k \in \mathbb{N}$, are defined in Theorem 2.1.

Proof. Let $y(t)$ be a regular nonoscillatory solution of the impulsive system (E). Proceeding as in Theorem 2.1, we get (2.3) and (2.4) for $t \ge t_1$. In what follows, $\big(r(t)w'(t)\big)$ and $w(t)$ are monotonic functions on $[t_2, \infty),\ t_2 > t_1$. Consider the case when $\big(r(t)w'(t)\big) < 0,\ w(t) > 0$ for $t \ge t_2$. Therefore, for $s \ge t > t_2,\ \big(r(s)w'(s)\big) \le \big(r(t)w'(t)\big)$ implies that $w'(s) \le \frac{r(t)w'(t)}{r(s)}$, that is,

$$w(s) \le w(t) + r(t)w'(t) \int_t^s \frac{d\theta}{r(\theta)}.$$

Since, $\big(r(t)w'(t)\big)$ is nonincreasing, then there exists a constant $C > 0$ such that $\big(r(t)w'(t)\big) \le -C$ for $t \ge t_2$. As a result, $w(s) \le w(t) - C \int_t^s \frac{d\theta}{r(\theta)}$. As $s \to \infty$, it follows that $0 \le w(t) - CR(t)$ for $t \ge t_2$. Clearly, $w(\tau_k) \ge CR(\tau_k),\ k \in \mathbb{N}$. Therefore, $z(t) \ge F(t) + CR(t)$ implies that $z(t) - CR(t) \ge F(t)$. If $z(t) - CR(t) < 0$, then $F(t) < 0$, a contradiction. Hence, $z(t) - CR(t) > 0$ and, hence, $z(t) - CR(t) \ge F^+(t)$, that is, $z(t) \ge CR(t) + F^+(t) \ge F^+(t)$. Also, $z(\tau_k) \ge F^+(\tau_k)$, $k \in \mathbb{N}$. Consequently, (2.5) and (2.6) reduce to

$$\big(r(t)w'(t)\big)' + G(a)\big(r(t-\tau)w'(t-\tau)\big)' + \lambda Q(t) G\big(F^+(t-\sigma)\big) \le 0,$$
$$\Delta\big(r(\tau_k)w'(\tau_k)\big) + G(a)\Delta\big(r(\tau_k-\tau)w'(\tau_k-\tau)\big) + \lambda H_k G\big(F^+(\tau_k-\sigma)\big) \le 0$$

for $t \ge t_3 > t_2,\ t \ne \tau_k,\ k \in \mathbb{N}$. Integrating the above impulsive system from t_3 to $t (> t_3)$, we obtain

$$\big[r(\eta)w'(\eta)\big]_{t_3}^t + G(a)\big[r(\eta-\tau)w'(\eta-\tau)\big]_{t_3}^t - \sum_{t_3 \le \tau_k < t} \Delta\big(r(\tau_k)w'(\tau_k)\big) -$$
$$- G(a) \sum_{t_3 \le \tau_k < t} \Delta\big(r(\tau_k-\tau)w'(\tau_k-\tau)\big) + \lambda \int_{t_3}^t Q(\eta) G\big(F^+(\eta-\sigma)\big) d\eta \le 0,$$

that is,

$$\lambda \left[\int_{t_3}^t Q(\eta) G\big(F^+(\eta-\sigma)\big) d\eta + \sum_{t_3 \le \tau_k < t} H_k G\big(F^+(\tau_k-\sigma)\big)\right] \le$$

$$\leq -\big[\big(r(\eta)w'(\eta)\big) + G(a)\big(r(\eta-\tau)w'(\eta-\tau)\big)\big]_{t_3}^{t} \leq$$
$$\leq -\big[\big(r(t)w'(t)\big) + G(a)\big(r(t-\tau)w'(t-\tau)\big)\big] \leq$$
$$\leq -\big(1+G(a)\big)\big(r(t)w'(t)\big)$$

implies that

$$\frac{\lambda}{\big(1+G(a)\big)}\frac{1}{r(t)}\left[\int\limits_{t_3}^{t} Q(\eta)G\big(F^+(\eta-\sigma)\big)d\eta + \sum_{t_3\leq\tau_k<t} H_k G\big(F^+(\tau_k-\sigma)\big)\right] \leq -w'(t).$$

Further integration of the above inequality, we obtain that

$$\frac{\lambda}{\big(1+G(a)\big)}\int\limits_{t_3}^{u}\frac{1}{r(\eta)}\left[\int\limits_{t_3}^{\eta} Q(\zeta)G\big(F^+(\zeta-\sigma)\big)d\zeta + \sum_{t_3\leq\tau_k<\eta} H_k G\big(F^+(\tau_k-\sigma)\big)\right]d\eta \leq$$
$$\leq -\big[w(\eta)\big]_{t_3}^{u} + \sum_{t_3\leq\tau_k<u}\Delta w(\tau_k) =$$
$$= -\big[w(\eta)\big]_{t_3}^{u} + \sum_{t_3\leq\tau_k<u}[w(\tau_k+0) - w(\tau_k-0)] \leq$$
$$\leq w(t_3) + \sum_{t_3\leq\tau_k<u} w(\tau_k+0).$$

Since $w(t)$ is bounded and monotonic, then it follows that

$$\int\limits_{t_3}^{\infty}\frac{1}{r(\eta)}\left[\int\limits_{t_3}^{\eta} Q(\zeta)G\big(F^+(\zeta-\sigma)\big)d\zeta + \sum_{k=1}^{\infty} H_k G\big(F^+(\tau_k-\sigma)\big)\right]d\eta < \infty,$$

a contradiction to (A_{15}). The rest of the proof follows from the proof Theorem 2.1. Hence the proof of the theorem is complete.

Theorem 2.5. *Let* $-1 \leq p(t) \leq 0,\ t \in \mathbb{R}_+$. *Assume that* ($A_0$), ($A_3$), ($A_4$), ($A_8$)–($A_{11}$) *and* ($A_{14}$) *hold. Furthermore, assume that*

(A_{17}) $\displaystyle\int_T^\infty \frac{1}{r(\eta)}\int_{T_1}^{\eta} q(\zeta)G\big(F^+(\zeta+\tau-\sigma)\big)d\zeta d\eta + R(T)\sum_{k=1}^{\infty} h(\tau_k)G\big(F^+(\tau_k+\tau-\sigma)\big) = \infty$;

(A_{18}) $\displaystyle\int_T^\infty \frac{1}{r(\eta)}\int_{T_1}^{\eta} q(\zeta)G\big(F^-(\zeta+\tau-\sigma)\big)d\zeta d\eta + R(T)\sum_{k=1}^{\infty} h(\tau_k)G\big(F^-(\tau_k+\tau-\sigma)\big) = \infty$;

(A_{19}) $\displaystyle\int_T^\infty \frac{1}{r(\eta)}\int_{T_1}^{\eta} q(\zeta)G\big(F^+(\zeta-\sigma)\big)d\zeta d\eta + R(T)\sum_{k=1}^{\infty} h(\tau_k)G\big(F^+(\tau_k-\sigma)\big) = \infty$

and

(A_{20}) $\displaystyle\int_T^\infty \frac{1}{r(\eta)}\int_{T_1}^{\eta} q(\zeta)G\big(F^-(\zeta-\sigma)\big)d\zeta d\eta + R(T)\sum_{k=1}^{\infty} h(\tau_k)G\big(F^-(\tau_k-\sigma)\big) = \infty,$

where $T,\ T_1 > 0$. *Then every regular solution of (E) oscillates.*

Proof. For contrary, let $y(t)$ be a regular nonoscillatory solution of (E). Then proceeding as in Theorem 2.2 we obtain that $w(t)$ and $\big(r(t)w'(t)\big)$ are monotonic on $[t_2,\infty)$. If $w(t) < 0$ and $\big(r(t)w'(t)\big) < 0$ for $t \geq t_3 > t_2$, then we use the same type of argument as in Theorem 2.2 to get that $y(t)$ is bounded, that is, $\lim_{t\to\infty} w(t)$ exists. Clearly, $z(t) < 0$ implies that $-z(t) > -F(t)$ and thus $-z(t) > F^-(t)$. Therefore, for $t \geq t_3$

$$-y(t-\tau) \leq p(t)y(t-\tau) \leq z(t) < -F^-(t).$$

Consequently, $y(t-\sigma) > F^-(t+\tau-\sigma),\ t \geq t_4 > t_3$ and (2.3), (2.4) yield

$$\big(r(t)w'(t)\big)' + q(t)G\big(F^-(t+\tau-\sigma)\big) \leq 0, \quad t \neq \tau_k, \quad k \in \mathbb{N},$$
$$\Delta\big(r(\tau_k)w'(\tau_k)\big) + h(\tau_k)G\big(F^-(\tau_k+\tau-\sigma)\big) \leq 0, \quad k \in \mathbb{N},$$

for $t \geq t_4$. Integrating the preceding impulsive system from t_4 to $+\infty$, we obtain

$$\int\limits_{t_4}^{\infty} q(\eta)G\big(F^-(\eta+\tau-\sigma)\big)d\eta + \sum_{t_4 \leq \tau_k < \infty} h(\tau_k)G\big(F^-(\tau_k+\tau-\sigma)\big) < -r(t)w'(t),$$

therefore

$$\frac{1}{r(t)}\left[\int\limits_{t_4}^{\infty} q(\eta)G\big(F^-(\eta+\tau-\sigma)\big)d\eta + \sum_{t_4 \leq \tau_k < \infty} h(\tau_k)G\big(F^-(\tau_k+\tau-\sigma)\big)\right] < -w'(t).$$

Further integration of the last inequality we find

$$\int\limits_{t_4}^{\infty} \frac{1}{r(\eta)}\left[\int\limits_{t_4}^{\infty} q(\zeta)G\big(F^-(\zeta+\tau-\sigma)\big)d\zeta + \sum_{t_4 \leq \tau_k < \infty} h(\tau_k)G\big(F^-(\tau_k+\tau-\sigma)\big)\right]d\eta < \infty$$

which contradicts (A_{18}). If $w(t) > 0$ and $\big(r(t)w'(t)\big) < 0$ for $t \geq t_3$, then following to Theorem 2.4 we find $z(t) \geq F^+(t) + CR(t) \geq F^+(t)$ and $z(t) > 0$, that is, $y(t) \geq F^+(t)$. The rest of the proof can similarly be dealt with the proof of Theorem 2.2. Hence, the theorem is proved.

Theorem 2.6. *Let $-\infty < -b \leq p(t) \leq -1,\ t \in \mathbb{R}_+$. Assume that ($A_0$), ($A_3$), ($A_4$), ($A_8$)–($A_{11}$), ($A_{14}$), ($A_{19}$) and ($A_{20}$) hold. Furthermore, assume that*

(A_{21}) $\int_T^{\infty} \frac{1}{r(\eta)} \int_{T_1}^{\eta} q(\zeta)G\left(\frac{1}{b}F^+(\zeta+\tau-\sigma)\right)d\zeta d\eta + R(T)\sum_{k=1}^{\infty} h(\tau_k)G\left(\frac{1}{b}F^+(\tau_k+\tau - \sigma)\right) = \infty$

and

(A_{22}) $\int_T^{\infty} \frac{1}{r(\eta)} \int_{T_1}^{\eta} q(\zeta)G\left(\frac{1}{b}F^-(\zeta+\tau-\sigma)\right)d\zeta d\eta + R(T)\sum_{k=1}^{\infty} h(\tau_k)G\left(\frac{1}{b}F^-(\tau_k+\tau - \sigma)\right) = \infty,$

where $T,\ T_1 > 0$. Then every bounded solution of (E) oscillates.

Proof. The proof of the theorem can be followed from the proof of the Theorem 2.5. Hence, the details are omitted.

3. Sufficient conditions for nonoscillation. This section deals with the necessary conditions for oscillation to show that the impulsive system (E) admits a positive bounded solution for various ranges of $p(t)$.

Theorem 3.1. *Let* $p \in C(\mathbb{R}_+, [-1, 0])$ *and assume that (A*$_0$*) hold. If*

(A$_{23}$) $\displaystyle\int_0^{\infty} \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)d\zeta + \sum\nolimits_{k=1}^{\infty} h(\tau_k) \right] d\eta < \infty,$

then the system (E) admits a positive bounded solution.

Proof. (i) Let $-1 < -b \leq p(t) \leq 0,\ t \in \mathbb{R}_+,$ and $b > 0.$ Due to (A$_{23}$), it is possible to find a $T > \rho$ such that

$$\int\limits_T^t \frac{1}{r(\eta)} \left[\int\limits_\eta^\infty q(\zeta)d\zeta + \sum_{k=1}^{\infty} h(\tau_k) \right] d\eta < \frac{1-b}{10G(1)}.$$

We consider the set

$$M = \left\{ y : y \in C([T-\rho, +\infty), \mathbb{R}),\ \ y(t) = 0 \ \text{ for } \ t \in [T-\rho, T] \ \text{ and } \ \frac{1-b}{20} \leq y(t) \leq 1 \right\}$$

and define $\Phi : M \to C([T-\rho, +\infty), \mathbb{R})$ by the formula

$$(\Phi y)(t) = \begin{cases} 0, & t \in [T-\rho, T), \\ -p(t)y(t-\tau) + \displaystyle\int_T^t \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta) G\big(y(\zeta-\sigma)\big) d\zeta + \right. & \\ \qquad \left. + \sum\nolimits_{k=1}^{\infty} h(\tau_k) G\big(y(\tau_k - \sigma)\big) \right] d\eta + F(t) + \dfrac{1-b}{10}, & t \geq T, \end{cases}$$

where $F(t)$ be such that $|F(t)| \leq \dfrac{1-b}{20}$. For every $y \in M$,

$$(\Phi y)(t) \leq -p(t)y(t-\tau) + G(1) \int\limits_T^t \frac{1}{r(\eta)} \left[\int\limits_\eta^\infty q(\zeta)d\zeta + \sum_{k=1}^{\infty} h(\tau_k) \right] d\eta + \frac{1-b}{20} + \frac{1-b}{10} \leq$$

$$\leq b + \frac{1-b}{10} + \frac{1-b}{20} + \frac{1-b}{10} \leq \frac{1+3b}{4} < 1,$$

and

$$(\Phi y)(t) \geq F(t) + \frac{1-b}{10} \leq -\frac{1-b}{20} + \frac{1-b}{10} = \frac{1-b}{20}$$

implies that $(\Phi y)(t) \in M$. Define $u_n : [T-\rho, +\infty) \to \mathbb{R}$ by the recursive formula

$$u_n(t) = (\Phi u_{n-1})(t), \quad n \geq 1,$$

with the initial condition

$$u_0(t) = \begin{cases} 0, & t \in [T-\rho, T), \\ \dfrac{1-p}{20}, & t \geq T. \end{cases}$$

Inductively it is easy to verify that

$$\frac{1-b}{20} \le u_{n-1}(t) \le u_n(t) \le 1$$

for $t \ge T$. Therefore for $t \ge T-\rho$, $\lim_{n\to\infty} u_n(t)$ exists. Let $\lim_{n\to\infty} u_n(t) = u(t)$ for $t \ge T-\rho$. By the Lebesgue's dominated convergence theorem $u \in M$ and $(\Phi u)(t) = u(t)$, where $u(t)$ is a solution of the impulsive system (E) on $[T-\rho, \infty)$ such that $u(t) > 0$. Hence, (A_{18}) is necessary.

(ii) If $p(t) \equiv -1$, $t \in \mathbb{R}_+$, we choose $-1 < p_0 < 0$ such that $p_0 \ne -\dfrac{1}{2}$. In this case, we can apply the above method. Here, we note that

$$\int_T^t \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)d\zeta + \sum_{k=1}^\infty h(\tau_k) \right] d\eta < \frac{1+2p_0}{10G(-p_0)} \qquad \text{and} \qquad -\frac{1+2p_0}{40} \le F(t) \le \frac{1+2p_0}{20}.$$

We set

$$M = \left\{ y : y \in C([T-\rho, +\infty), \mathbb{R}),\ \ y(t) = 0 \ \text{ for } \ t \in [T-\rho, T] \ \text{ and } \ \frac{7+2p_0}{40} \le y(t) \le -p_0 \right\}.$$

Also, we define $\Phi : M \to C([T-\rho, +\infty), \mathbb{R})$ by

$$(\Phi y)(t) = \begin{cases} 0, & t \in [T-\rho, T), \\ y(t-\tau) + \displaystyle\int_T^t \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta) G\big(y(\zeta-\sigma)\big) d\zeta + \right. & \\ \qquad \left. + \displaystyle\sum_{k=1}^\infty h(\tau_k) G\big(y(\tau_k - \sigma)\big) \right] d\eta + F(t) + \dfrac{2+p_0}{10}, & t \ge T. \end{cases}$$

This completes the proof of the theorem.

Theorem 3.2. *Let $p \in C[\mathbb{R}_+, [0,1)]$. Let G be Lipchitzian on the interval of the form $[a,b]$, $0 < a < b < \infty$. If (A_0) and (A_{23}) hold, then the impulsive system (E) admits a positive bounded solution.*

Proof. Let $0 \le p(t) \le a < 1$. It is possible to find $t_1 > 0$ such that

$$\int_{t_1}^\infty \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)d\zeta + \sum_{k=1}^\infty h(\tau_k) \right] d\eta < \frac{1-a}{5K},$$

where $K = \max\{K_1, G(1)\}$, K_1 is the Lipschitz constant on $\left[\dfrac{3}{5}(1-a), 1\right]$. Let $F(t)$ be such that $|F(t)| < \dfrac{1-a}{10}$ for $t \ge t_2$. For $t_3 > \max\{t_1, t_2\}$, we set $X = BC([T, \infty), \mathbb{R})$, the space of real valued continuous functions on $[t_3, \infty]$. Clearly, X is a Banach space with respect to sup norm defined by

$$\|x\| = \sup\big\{|x(t)| : t \ge t_3\big\}.$$

Let's define

$$S = \left\{ u \in X : \frac{3}{5}(1-a) \le u(t) \le 1,\ \ t \ge t_3 \right\}.$$

We notice that S is a closed and convex subspace of X. Let $\Phi : S \to S$ be such that

$$(\Phi y)(t) = \begin{cases} (\Phi y)(t_3+\rho), & t \in [t_3, t_3+\rho], \\ -p(t)y(t-\tau) + \dfrac{9+a}{10} + F(t) \\ \qquad -\displaystyle\int_t^\infty \frac{1}{r(\eta)}\Big[\int_\eta^\infty q(\zeta)G\big(y(\zeta-\sigma)\big)\,d\zeta + \\ \qquad + \sum\nolimits_{k=1}^\infty h(\tau_k)G\big(y(\tau_k-\sigma)\big)\Big]d\eta, & t \geq t_3+\rho. \end{cases}$$

For every $y \in X,\ (\Phi y)(t) \leq F(t) + \dfrac{9+a}{10} \leq 1$ and

$$(\Phi y)(t) \geq -p(t)y(t-\tau) - G(1)\int_t^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta + \sum_{k=1}^\infty h(\tau_k)\right]d\eta + F(t) + \frac{9+a}{10} \geq$$

$$\geq -a - \frac{1-a}{5} - \frac{1-a}{10} + \frac{9+a}{10} = \frac{3}{5}(1-a)$$

implies that $(\Phi y) \in S$. Now for y_1 and $y_2 \in S$, we have

$$\big|(\Phi y_1)(t) - (\Phi y_2)(t)\big| \leq a|y_1(t-\tau) - y_2(t-\tau)| +$$

$$+ \int_t^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)\big|G(y_1(\zeta-\sigma)) - G(y_2(\zeta-\sigma))\big|d\zeta + \right.$$

$$\left. + \sum_{k=1}^\infty h(\tau_k)\big|G(y_1(\tau_k-\sigma)) - G(y_2(\tau_k-\sigma))\big|\right]d\eta,$$

that is,

$$\big|(\Phi y_1)(t) - (\Phi y_2)(t)\big| \leq$$

$$\leq a\|y_1 - y_2\| + \|y_1 - y_2\|K_1\int_t^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta + \sum_{k=1}^\infty h(\tau_k)\right]d\eta \leq$$

$$\leq \left(a + \frac{1-a}{5}\right)\|y_1 - y_2\| = \frac{4a+1}{5}\|y_1 - y_2\|.$$

Therefore, $\big\|(\Phi y_1) - (\Phi y_2)\big\| \leq \dfrac{4a+1}{5}\|y_1 - y_2\|$ implies that Φ is a contraction. By using Banach's fixed point theorem, it follows that Φ has a unique fixed point $y(t)$ in $\left[\dfrac{3}{5}(1-a), 1\right]$. Hence, $(\Phi y) = y$ and the proof of the theorem is complete.

Remark 3.1. We can not apply Lebesgue's dominated convergence theorem for other ranges of $p(t)$, except $-1 \leq p(t) \leq 0$ due to the technical difficulties arising in the method. However, we can apply Banach's fixed point theorem to other ranges of $p(t)$ similar to Theorem 3.2.

4. Discussion and example. In this work, we have undertaken the problem and established the sufficient conditions for oscillation and nonoscillation of solutions of the impulsive system (E). However, we failed to establish the necessary and sufficient conditions for oscillation of all solutions of the system (E). It seems that some other method may require to establish the necessary and sufficient conditions for oscillation.

Open Problem: In this work, we have seen that $(A_6)-(A_{13})$ and $(A_{15})-(A_{23})$ are the sufficient conditions for oscillation of all solutions of (E) in which we are depending explicitly on the forcing function. However, we are locked with the forcing function in (A_{23}) to establish the existence of nonoscillatory solutions of (E). In contrast to (A_{23}), can we find the sufficient condition for oscillation of solutions of (E) in the format of

$$(A_{24})\quad \int_0^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta + \sum_{k=1}^\infty h(\tau_k)\right]d\eta = \infty.$$

Indeed, this format could be more feasible not only for (E), but also for its homogeneous counterpart. We conclude this section with the following example to illustrate our main result:

Example 4.1. Consider the impulsive system

$$\begin{cases} \big(y(t)+y(t-\pi)\big)'' + y\Big(t-\dfrac{\pi}{4}\Big) = \cos\Big(t-\dfrac{\pi}{4}\Big), \quad t>\dfrac{\pi}{4}, \\ \Delta\big(y(\tau_k)+y(\tau_k-\pi)\big)' + h(\tau_k)y\Big(\tau_k-\dfrac{\pi}{4}\Big) = 2\sin(h)\cos\Big(k-\dfrac{\pi}{4}\Big), \end{cases} \tag{E_5}$$

where

$$h(\tau_k) = \frac{2}{1+\cot(h)}, \quad \tau_k = k, \quad k\in\mathbb{N}, \qquad G(u)=u, \qquad f(t) = \cos\Big(t-\frac{\pi}{4}\Big).$$

Indeed, if we choose $F(t) = -\cos\Big(t-\dfrac{\pi}{4}\Big)$, then $\big(r(t)F'(t)\big)' = F''(t) = f(t)$ and

$$\begin{aligned} \Delta\big(r(\tau_k)F'(\tau_k)\big) &= F'(\tau_k+h) - F'(\tau_k-h) = \\ &= F'(k+h) - F'(k-h) = \\ &= \sqrt{2}\sin(h)\big(\sin(k)+\cos(k)\big) = g(\tau_k), \quad k\in\mathbb{N}. \end{aligned}$$

Clearly,

$$F^+(t) = \begin{cases} -\cos\Big(t-\dfrac{\pi}{4}\Big), & 2n\pi+\dfrac{3\pi}{4} \le t \le 2n\pi+\dfrac{7\pi}{4}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$F^-(t) = \begin{cases} \cos\Big(t-\dfrac{\pi}{4}\Big), & 2n\pi+\dfrac{7\pi}{4} \le t \le 2n\pi+\dfrac{11\pi}{4}, \\ 0, & \text{otherwise}, \end{cases}$$

implies that

$$F^+\Big(t-\frac{\pi}{4}\Big) = \begin{cases} -\sin(t), & 2n\pi+\pi \le t \le 2n\pi+2\pi, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$F^{-}\left(t-\frac{\pi}{2}\right)=\begin{cases}\sin(t), & 2n\pi+2\pi\leq t\leq 2n\pi+3\pi,\\ 0, & \text{otherwise.}\end{cases}$$

Since

$$\int\limits_{\frac{\pi}{4}}^{\infty}F^{+}\left(\eta-\frac{\pi}{4}\right)d\eta=\sum_{n=0}^{\infty}\int\limits_{2n\pi+\pi}^{2n\pi+2\pi}[-\sin(\eta)]d\eta=\infty,$$

then for $n=0,1,2,\ldots,$ we get

$$\int\limits_{\frac{\pi}{4}}^{\infty}F^{+}\left(\eta-\frac{\pi}{4}\right)d\eta+\sum_{k=1}^{\infty}\left(\frac{2}{1+\cot(h)}\right)F^{+}\left(k-\frac{\pi}{4}\right)=\infty.$$

Clearly, $(A_1)-(A_7)$ are satisfied. Hence, by Theorem 2.1 every solution of (E_5) is oscillatory.

References

1. D. D. Bainov, P. S. Simeonov, *Systems with impulse effect: stability, theory and applications*, Ellis Horwood, Chichester (1989).
2. D. D. Bainov, P. S. Simeonov, *Theory of impulsive differential equations: asymptotic properties of the solutions and applications*, World Sci. Publ., Singapore (1995).
3. I. Gyori, G. Ladas, *Oscillation theory of delay differential equations with applications*, Clarendon Press, Oxford (1991).
4. G. S. Ladde, V. Lakshmikantham, B. G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Pure Appl. Math., **110**, Marcel Dekker (1987).
5. V. Lakshmikantham, D. D. Bainov, P. S. Simieonov, *Oscillation theory of impulsive differential equations*, World Sci., Singapore (1989).
6. S. G. Pandit, S. G. Deo, *Differential systems involving impulses*, Lecture Notes in Math., **954**, Springer-Verlag, New York (1982).
7. A. K. Tripathy, *Oscillation criteria for a class of first order neutral impulsive differential-difference equations*, J. Appl. Anal. Comput., **4**, 89 – 101 (2014).
8. A. K. Tripathy, S. S. Santra, *Pulsatile constant and characterisation of first order neutral impulsive differential equations*, Comm. Appl. Anal., **20**, 65 – 76 (2016).
9. A. Tripathy, S. Santra, *Oscillation properties of a class of second order impulsive differential systems of neutral type*, Funct. Differ. Equ., **23**, № 1-2, 57 – 71 (2016).
10. D. D. Bainov, M. B. Dimitrova, *Oscillation of sub and super linear impulsive differential equations with constant delay*, Appl. Anal., **64**, 57 – 67 (1997).
11. E. M. Bonotto, L. P. Gimenes, M. Federson, *Oscillation for a second order neutral differential equation with impulses*, Appl. Math. Comput., **215**, 1 – 15 (2009).
12. A. Domoshnitsky and M. Drakhlin, *On boundary value problems for first order impulse functional differential equations*, Boundary Value Problems for Functional Differential Equations, Ed. J. Henderson, World Sci., Singapore; New Jersey; London; Hong Kong (1995), pp. 107 – 117.
13. A. Domoshnitsky, M. Drakhlin, *Nonoscillation of first order impulse differential equations with delay*, J. Math. Anal. Appl., **206**, 254 – 269 (1997).
14. A. Domoshnitsky, *Wronskian of fundamental system of delay differential equations*, Funct. Differ. Equ., **7**, 445 – 467 (2002).

15. A. Domoshnitsky, I. Volinsky, R. Shklyar, *About Green's functions for impulsive differential equations*, Funct. Differ. Equ., **20**, 55 – 81 (2013).

16. A. Domoshnitsky, G. Landsman, S. Yanetz, *About sign-constancy of Green's functions for impulsive second order delay equations*, Opuscula Math., **34**, № 2, 339 – 362 (2014).

17. A. Domoshnitsky and I. Volinsky, *About positivity of Green's functions for nonlocal boundary value problems with impulsive delay equations*, Sci. World J., Article ID 978519 (2014).

18. A. Domoshnitsky, I. Volinsky, *About differential inequalities for nonlocal boundary value problems with impulsive delay equations*, Math. Bohem., **140**, № 2, 121 – 128 (2015).

19. A. Domoshnitsky, Iu. Mizgireva, *Sign-constancy of Green's functions for impulsive nonlocal boundary value problems*, Bound. Value Probl., **175**, № 1 (2019).

20. L. P. Gimenes, M. Federson, *Oscillation by impulses for a second order delay differential equations*, Cadernos de Matemática, **6**, 181 – 191 (2005).

21. J. R. Graef, M. K. Grammatikopoulos, *On the behaviour of a first order nonlinear neutral delay differential equations*, Appl. Anal., **40**, 111 – 121 (1991).

22. J. R. Graef, J. H. Shen, I. P. Stavroulakis, *Oscillation of impulsive neutral delay differential equations*, J. Math. Anal. Appl., **268**, 310 – 333 (2002).

23. Z. M. He, W. Ge, *Oscillation in second order linear delay differential equation with nonlinear impulses*, Math. Slovaca, **52**, 331 – 341 (2002).

24. G. R. Jiang, Q. S. Lu, *Impulsive state feedback control of a prey-predator model*, J. Comput. Appl. Math., **200**, 193 – 207 (2007).

25. X. Liu, G. Ballinger, *Existence and continuability of solutions for differential equations with delays and state-dependent impulses*, Nonlinear Anal., **51**, 633 – 647 (2002).

26. W. Luo, J. Luo, L. Debnath, *Oscillation of second order quasilinear delay differential equations with impulses*, J. Appl. Math. Comput., **13**, 165 – 182 (2003).

27. Z. Luo, J. Shen, *Oscillation of second order linear differential equation with impulses*, Appl. Math. Lett., **20**, 75 – 81 (2007).

28. L. Qian, Q. Lu, Q. Meng, Z. Feng, *Dynamical behaviours of a prey-predator system with impulsive control*, J. Math. Anal. Appl., **363**, 345 – 356 (2010).

29. J. Shen, Y. Liu, *Asymptotic behaviour of solutions for nonlinear delay differential equation with impulses*, Appl. Math. Comput., **213**, 449 – 454 (2009).

30. E. Thandapani, R. Sakthivel, E. Chandrasekaran, *Oscillation of second order nonlinear impulsive differential equations with deviating arguments*, Differ. Equ. Appl., **4**, 571 – 580 (2012).

31. Wu Xiu-li, Chen Si-Yang, Tang Hong-Ji, *Oscillation of a class of second-order delay differential equation with impulses*, Appl. Math. Comput., **145**, 561 – 567 (2003).

32. Wu Xiu-li, Chen Si-Yang, Tang Hong-Ji, *Oscillation of a class of second-order nonlinear ODE with impulses*, Appl. Math. Comput., **138**, 181 – 188 (2003).

33. J. Yan, *Oscillation properties of a second-order impulsive delay differential equations*, Comput. Math. Appl., **47** (2004), 253 – 258.

Received 25.09.16,
after revision — 19.05.20