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ON THE ASYMPTOTIC STABILITY OF SOLUTIONS OF NONLINEAR DELAY DIFFERENTIAL EQUATIONS

ПРО АСИМПТОТИЧНУ СТІЙКІСТЬ РОЗВ'ЯЗКІВ НЕЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ІЗ ЗАПІЗНЕННЯМ

Cemil Tunç, A. Yiğit

Dep. Math., Faculty Sci.
Van Yuzuncu Yil Univ.
65080, Van-Turkey
e-mail: cemtunc@yahoo.com,
a-yigit63@hotmail.com

A nonlinear system of delay differential equations (DDEs) is considered. We obtain some new results of the asymptotic stability of a zero solution of the considered system by using well-known inequalities and Lyapunov–Krasovskii functionals. Two numerical examples illustrate applications of the obtained results. The results of this paper make contributions to the qualitative theory of DDEs and improve some known results in the modern literature.

Розглянуто нелінійну систему диференціальних рівнянь із запізненням. Отримано нові результати про асимптотичну стійкість нульового розв'язку досліджуваної системи за допомогою деяких відомих нерівностей та функціоналів Ляпунова – Красовського. Наведено два числові приклади, які ілюструють застосування здобутих результатів. Результати цієї статті доповнюють якісну теорію диференціальних рівнянь із запізненням, а також покращують відомі в сучасній літературі результати.

1. Introduction. It can be followed from the relevant literature that the problems related to the qualitative analysis of solutions, in particular, stability analysis of solutions of time delay systems of first order are very effective in the qualitative theory of solutions in the literature due to that kind of problems with time delays can be frequently encountered in various engineering systems such as long transmission lines in pneumatic systems, nuclear reactors, rolling mills, hydraulic systems, manufacturing processes, population dynamics, control theory and so on. For instance, we would like to suggest the reader to look at [1 – 23] and references therein).

It is notable that, in 2016, Alla et al. [1] considered the following linear differential system with time-varying delay:

$$\dot{x}(t) = Ax(t) + A_d x(t - d(t)).$$

By means of a Lyapunov – Krasovskii functional, which is appropriately chosen, and the Wirtinger's inequality, the authors derived some new delay dependent asymptotic stability criteria in terms of linear matrix inequalities for the above system.

Later, in 2017, Alla et al. [2] considered the following singular system with time-varying delay:

$$E\dot{x}(t) = Ax(t) + A_d x(t - d(t)).$$

In [2], the authors proposed new delay-dependent stability criteria for this singular system by using Jensen's and Wirtinger's inequalities. The proposed delay-dependent stability criteria have been derived in terms of linear matrix inequalities by use of a common augmented Lyapunov – Krasovskii functional.

In this paper, in particular, motivated by Alla et al. [1, 2] and the works in the references of this paper, we consider the following system of nonlinear DDEs, which include two variable delays:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{2} A_{d_i} x\big(t - d_i(t)\big) + \sum_{i=1}^{2} F_i\big(t, x\big(t - d_i(t)\big)\big),$$
$$x(t) = \phi(t), \qquad t \in [-r, 0], \quad r > 0, \quad r \in \Re, \tag{1}$$

where $t \in [-r, 0)$, r is constant delay, $x \in \Re^n$ is the state vector, $\phi(t)$ is a continuous initial function defined on $[-r, 0], A \in \Re^{n\times n}$ is a negative definite real constant matrix and $A_{d_i} \in \in \Re^{n\times n}$ are real constant matrices and $d_1(t), d_2(t) \in C^1(\Re^+, \Re^+)$ are variable delays, bounded. In addition, $F_i \in C(\Re^+ \times \Re^n, \Re^n)$ with $F_i(t, 0) \equiv 0$ and they satisfy the Lipschitz condition, that is,

$$\big\|F_i(t, x_0) - F_i(t, y_0)\big\| \leq \|U_i(x_0 - y_0)\|, \quad \forall t \in \Re^+, \quad \forall x_0, y_0 \in \Re^n, \tag{2}$$

such that U_i are some known matrices.

Notations. Through this article, $\Re^n$ denotes the n-dimensional Euclidean space, $\Re^{n\times m}$ is the set of all $n \times m$ real matrices. The notation $\begin{bmatrix} K & M \\ * & N \end{bmatrix}$ stands for $\begin{bmatrix} K & M \\ M^T & N \end{bmatrix}$. The notation $P > 0$ $(P \geq 0)$, for $P \in \Re^{n\times n}$, means that P is symmetric and positive definite (positive semi definite) and $P < 0$ $(P \leq 0)$, for $P \in \Re^{n\times n}$, means that P is symmetric and negative definite (negative semi definite).

Lemma 1.1 (Schur complement [6, p. 37]). *For a given symmetric matrix* $S = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}$, *where* $S_{11} \in \Re^{r\times r}$, *the following conditions are equivalent:*

(1°) $S < 0$;
(2°) $S_{11} < 0, S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$;
(3°) $S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

Lemma 1.2 (Jensen inequality [10]). *For any matrix* $Z > 0$ *and a vector function* $x\colon [a, b] \longmapsto \Re^n$ *the following inequality holds:*

$$(b - a) \int_a^b x^T(s) Z x(s) ds \geq \left(\int_a^b x^T(s) ds \right) Z \left(\int_a^b x(s) ds \right)$$

provided that the given integrals are well-defined.

Lemma 1.3 (Wirtinger inequality [11]). *Let* $R \in \Re^{n\times n}$ *be any constant symmetric matrix and* $x\colon [a, b] \longmapsto \Re^n$ *be a continuously differentiable function. Then,the following inequality holds:*

$$\int_a^b \dot{x}^T(s) R \dot{x}(s) ds \geq \frac{1}{b - a} [x(b) - x(a)]^T R [x(b) - x(a)] + \frac{3}{b - a} \Omega^T R \Omega,$$

where

$$\Omega = x(a) + x(b) - \left(\frac{2}{b-a}\right) \int\limits_a^b x(s)ds.$$

2. Stability criteria. Firstly, we present sufficient criteria for the asymptotic stability of the zero solution of the system of DDEs (1).

A. Assumptions. (A1) It is assumed that the following inequalities hold:

$$0 \le d_i(t) \le \tau_i, \quad \tau_i > 0, \quad \tau_i \in \Re,$$

$$\dot{d}_i(t) \le \mu_i \le 1, \quad \mu_i > 0, \quad \mu_i \in \Re, \quad i = 1, 2,$$

$$r = \max\{\tau_1, \tau_2\}.$$

(A2) We have positive definite symmetric matrices $P \in \Re^{n\times n}$, $R_i \in \Re^{n\times n}$, $Z \in \Re^{n\times n}$ and some U_i known matrices with appropriate dimensions such that the following matrix inequality holds:

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} & \Xi_{17} \\ * & \Xi_{22} & \Xi_{23} & 0 & 0 & \Xi_{26} & \Xi_{27} \\ * & * & \Xi_{33} & 0 & 0 & \Xi_{36} & \Xi_{37} \\ * & * & * & \Xi_{44} & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55} & 0 & 0 \\ * & * & * & * & * & \Xi_{66} & \Xi_{67} \\ * & * & * & * & * & * & \Xi_{77} \end{bmatrix} < 0,$$

where

$$\Xi_{11} = A^T P + PA + \sum_{i=1}^{2} \tau_i A^T Z A - \sum_{i=1}^{2} \tau_i^{-1} Z + \sum_{i=1}^{2} R_i,$$

$$\Xi_{12} = PA_{d_1} + (\tau_1 + \tau_2) A^T Z A_{d_1}, \qquad \Xi_{13} = PA_{d_2} + (\tau_1 + \tau_2) A^T Z A_{d_2},$$

$$\Xi_{14} = \tau_1^{-1} Z, \Xi_{15} = \tau_2^{-1} Z, \qquad \Xi_{16} = P + (\tau_1 + \tau_2) A^T Z, \qquad \Xi_{17} = P + (\tau_1 + \tau_2) A^T Z,$$

$$\Xi_{22} = (\tau_1 + \tau_2) A_{d_1}^T Z A_{d_1} - (1 - \mu_1) R_1 + \epsilon_1 U_1^T U_1,$$

$$\Xi_{23} = (\tau_1 + \tau_2) A_{d_1}^T Z A_{d_2}, \qquad \Xi_{26} = (\tau_1 + \tau_2) A_{d_1}^T Z,$$

$$\Xi_{27} = (\tau_1 + \tau_2) A_{d_1}^T Z, \qquad \Xi_{33} = (\tau_1 + \tau_2) A_{d_2}^T Z A_{d_2} - (1 - \mu_2) R_2 + \epsilon_2 U_2^T U_2,$$

$$\Xi_{36} = (\tau_1 + \tau_2) A_{d_2}^T Z, \qquad \Xi_{37} = (\tau_1 + \tau_2) A_{d_2}^T Z, \qquad \Xi_{44} = -\tau_1^{-1} Z,$$

$$\Xi_{55} = -\tau_2^{-1} Z, \qquad \Xi_{66} = (\tau_1 + \tau_2) Z - \epsilon_1 I, \qquad \Xi_{67} = (\tau_1 + \tau_2) Z,$$

$$\Xi_{77} = (\tau_1 + \tau_2) Z - \epsilon_2 I,$$

here I is $(n \times n)$-identity matrix.

Theorem 2.1. *The zero solution of the system of DDEs (1) is asymptotically stable if assumptions (A1) and (A2) hold.*

Proof. Let

$$x_t = x(t+\beta), \quad -r \le \beta \le 0.$$

We define a Lyapunov – Krasovskii functional by

$$V(t,x_t) = x^T(t)Px(t) + \sum_{i=1}^{2} \int_{-\tau_i}^{0} \int_{t+\beta}^{t} \dot{x}^T(\alpha) Z \dot{x}(\alpha)\, d\alpha\, d\beta + \sum_{i=1}^{2} \int_{t-d_i(t)}^{t} x^T(\alpha) R_i x(\alpha)\, d\alpha.$$

By the derivative of the functional $V = V(t,x_t)$ along the system of DDEs (1) and by using the Newton – Leibnitz formula and Jensen inequality, that is, Lemma 2, we obtain:

$$\begin{aligned}
\dot{V}(t,x_t) \le x^T(t)\left[A^TP + PA + \sum_{i=1}^{2}\tau_i A^T Z A - \sum_{i=1}^{2}\tau_i^{-1}Z + \sum_{i=1}^{2}R_i\right]x(t) + \\
+ \sum_{i=1}^{2} x^T\big(t-d_i(t)\big)\left[A_{d_i}^T P + \left(\sum_{i=1}^{2}\tau_i\right) A_{d_i}^T Z A\right]x(t) + \\
+ \sum_{i=1}^{2}\tau_i^{-1} x^T(t) Z x(t-\tau_i) + \\
+ \sum_{i=1}^{2} x^T(t)\left[PA_{d_i} + \left(\sum_{i=1}^{2}\tau_i\right) A^T Z A_{d_i}\right]x\big(t-d_i(t)\big) + \\
+ \left(\sum_{i=1}^{2} x^T\big(t-d_i(t)\big)A_{d_i}^T\right)\left[\sum_{i=1}^{2}\tau_i Z\right]\left(\sum_{i=1}^{2} A_{d_i} x\big(t-d_i(t)\big)\right) - \\
- \sum_{i=1}^{2} x^T\big(t-d_i(t)\big)(1-\mu_i)R_i x\big(t-d_i(t)\big) + \\
+ \sum_{i=1}^{2} x^T(t-\tau_i)\tau_i^{-1} Z x(t) - \sum_{i=1}^{2} x^T(t-\tau_i)\tau_i^{-1} Z x(t-\tau_i) + \\
+ \sum_{i=1}^{2} x^T(t)\left[P + \sum_{i=1}^{2}\tau_i A^T Z\right] F_i\big(t, x\big(t-d_i(t)\big)\big) + \\
+ \left(\sum_{i=1}^{2} x^T\big(t-d_i(t)\big)A_{d_i}^T\right)\left[\sum_{i=1}^{2}\tau_i Z\right]\left(\sum_{i=1}^{2} F_i\big(t, x\big(t-d_i(t)\big)\big)\right) + \\
+ \sum_{i=1}^{2} F_i^T\big(t, x\big(t-d_i(t)\big)\big)\left[P + \left(\sum_{i=1}^{2}\tau_i\right) Z A\right]x(t) +
\end{aligned}$$

$$+\left(\sum_{i=1}^{2} F_i^T\big(t, x\big(t-d_i(t)\big)\big)\right)\left[\sum_{i=1}^{2}\tau_i Z\right]\left(\sum_{i=1}^{2} A_{d_i} x\big(t-d_i(t)\big)\right)+$$

$$+\left(\sum_{i=1}^{2} F_i^T\big(t, x\big(t-d_i(t)\big)\big)\right)\left[\sum_{i=1}^{2}\tau_i Z\right]\left(\sum_{i=1}^{2} F_i\big(t, x\big(t-d_i(t)\big)\big)\right). \qquad (3)$$

For nonlinear functions $F_i(.)$ endowed with $\epsilon_i > 0,\ i = 1, 2,$ we can derive

$$0 \le -\epsilon_i F_i^T\big(t, x\big(t-d_i(t)\big)\big) F_i\big(t, x\big(t-d_i(t)\big)\big) + \epsilon_i x^T\big(t-d_i(t)\big) U_i^T U_i x\big(t-d_i(t)\big). \qquad (4)$$

Next, by the inequalities (3) and (4), it follows that

$$\dot V \le \Big[x^T(t) \quad x^T(t-d_1(t)) \quad x^T(t-d_2(t)) \quad x^T(t-\tau_1) \quad x^T(t-\tau_2)$$

$$F_1^T\big(t, x(t-d_1(t))\big) \quad F_2^T\big(t, x(t-d_2(t))\big)\Big]\times$$

$$\times\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} & \Xi_{17} \\ * & \Xi_{22} & \Xi_{23} & 0 & 0 & \Xi_{26} & \Xi_{27} \\ * & * & \Xi_{33} & 0 & 0 & \Xi_{36} & \Xi_{37} \\ * & * & * & \Xi_{44} & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55} & 0 & 0 \\ * & * & * & * & * & \Xi_{66} & \Xi_{67} \\ * & * & * & * & * & * & \Xi_{77} \end{bmatrix}\begin{bmatrix} x(t) \\ x(t-d_1(t)) \\ x(t-d_2(t)) \\ x(t-\tau_1) \\ x(t-\tau_2) \\ F_1\big(t, x(t-d_1(t))\big) \\ F_2\big(t, x(t-d_2(t))\big) \end{bmatrix},$$

where

$$\Xi_{11} = A^T P + PA + \sum_{i=1}^{2}\tau_i A^T Z A - \sum_{i=1}^{2}\tau_i^{-1} Z + \sum_{i=1}^{2} R_i,$$

$$\Xi_{12} = PA_{d_1} + (\tau_1+\tau_2)A^T Z A_{d_1}, \qquad \Xi_{13} = PA_{d_2} + (\tau_1+\tau_2)A^T Z A_{d_2},$$

$$\Xi_{14} = \tau_1^{-1} Z, \qquad \Xi_{15} = \tau_2^{-1} Z, \qquad \Xi_{16} = P + (\tau_1+\tau_2)A^T Z, \qquad \Xi_{17} = P + (\tau_1+\tau_2)A^T Z,$$

$$\Xi_{22} = (\tau_1+\tau_2)A_{d_1}^T Z A_{d_1} - (1-\mu_1)R_1 + \epsilon_1 U_1^T U_1,$$

$$\Xi_{23} = (\tau_1+\tau_2)A_{d_1}^T Z A_{d_2}, \qquad \Xi_{26} = (\tau_1+\tau_2)A_{d_1}^T Z,$$

$$\Xi_{27} = (\tau_1+\tau_2)A_{d_1}^T Z, \qquad \Xi_{33} = (\tau_1+\tau_2)A_{d_2}^T Z A_{d_2} - (1-\mu_2)R_2 + \epsilon_2 U_2^T U_2,$$

$$\Xi_{36} = (\tau_1+\tau_2)A_{d_2}^T Z, \qquad \Xi_{37} = (\tau_1+\tau_2)A_{d_2}^T Z,$$

$$\Xi_{44} = -\tau_1^{-1} Z, \Xi_{55} = -\tau_2^{-1} Z, \qquad \Xi_{66} = (\tau_1+\tau_2)Z - \epsilon_1 I, \qquad \Xi_{67} = (\tau_1+\tau_2)Z,$$

$$\Xi_{77} = (\tau_1+\tau_2)Z - \epsilon_2 I.$$

Hence, we can easily obtain the following inequality:

$$\dot{V}(t, x_t) \leq \xi^T(t)\Xi\xi(t),$$

where

$$\xi^T(t) = \Big[x^T(t) \quad x^T(t - d_1(t)) \quad x^T(t - d_2(t)) \quad x^T(t - \tau_1) \quad x^T(t - \tau_2)$$

$$F_1^T\big(t, x(t - d_1(t))\big) \quad F_2^T\big(t, x(t - d_2(t))\big)\Big],$$

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} & \Xi_{17} \\ * & \Xi_{22} & \Xi_{23} & 0 & 0 & \Xi_{26} & \Xi_{27} \\ * & * & \Xi_{33} & 0 & 0 & \Xi_{36} & \Xi_{37} \\ * & * & * & \Xi_{44} & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55} & 0 & 0 \\ * & * & * & * & * & \Xi_{66} & \Xi_{67} \\ * & * & * & * & * & * & \Xi_{77} \end{bmatrix}.$$

Applying the Schur complement [6], that is, Lemma 1.1, we can show that $\dot{V}(t, x_t) < 0$. In this case, we can conclude the zero solution of the system of DDEs (1) is asymptotically stable provide that $\Xi < 0$.

Example 2.1. For the particular case of the system of DDEs (1), when $n = 2$, let us consider the following delay differential system:

$$\frac{d}{dt}\left(\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\right) = \begin{bmatrix} -6 & 0 \\ 0 & -5 \end{bmatrix} \times \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} x_1\left(t - 20^{-1}(1 + \sin t)\right) \\ x_2\left(t - 20^{-1}(1 + \sin t)\right) \end{bmatrix} +$$

$$+ \begin{bmatrix} x_1\left(t - 20^{-1}(1 + \sin t)\right) e^{-x_1^2\left(t - 20^{-1}(1+\sin t)\right)} \\ x_2\left(t - 20^{-1}(1 + \sin t)\right) e^{-x_2^2\left(t - 20^{-1}(1+\sin t)\right)} \end{bmatrix}, \quad t \geq \frac{1}{10}. \tag{5}$$

When we compare the system of DDEs (5) with the system of DDEs (1), it is derived the following relations:

$$A = \begin{bmatrix} -6 & 0 \\ 0 & -5 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$F_1(t, x(t - d(t))) = \begin{bmatrix} x_1\left(t - \dfrac{1 + \sin t}{20}\right) e^{-x_1^2(t - \frac{1+\sin t}{20})} \\ x_2\left(t - \dfrac{1 + \sin t}{20}\right) e^{-x_2^2(t - \frac{1+\sin t}{20})} \end{bmatrix}, \quad t \geq \frac{1}{10},$$

$$\epsilon = 1.15, \qquad 0 \leq d(t) = d_1(t) = \frac{1 + \sin t}{20} \leq 0.1 = \tau_1,$$

$$\frac{d}{dt}\, d(t) = \dot{d}_1(t) = \frac{\cos t}{20} \leq 0.05 = \mu = \mu_1 < 1, \quad r = 0.05.$$

Firstly, it is clear that assumption (A1) of Theorem 2.1 is satisfied. Next, we choose the matrices $P,\ R,\ Z$, and U as the following:

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \qquad R = R_1 = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad U = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For this particular case, the Lyapunov – Krasovskii functional, which is given in Theorem 2.1, takes the following form:

$$V(t, x_t) = x^T(t)Px(t) + \int\limits_{-\tau_1}^{0}\int\limits_{t+\beta}^{t} \dot{x}^T(\alpha)Z\dot{x}(\alpha)d\alpha\, d\beta + \int\limits_{t-d_1(t)}^{t} x^T(\alpha)R_1x(\alpha)\, d\alpha =$$

$$= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^T \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \int\limits_{-\tau_1}^{0}\int\limits_{t+\beta}^{t} \begin{bmatrix} \dot{x}_1(\alpha) \\ \dot{x}_2(\alpha) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \dot{x}_1(\alpha) \\ \dot{x}_2(\alpha) \end{bmatrix} d\alpha\, d\beta +$$

$$+ \int\limits_{t-d_1(t)}^{0} \begin{bmatrix} x_1(\alpha) \\ x_2(\alpha) \end{bmatrix}^T \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1(\alpha) \\ x_2(\alpha) \end{bmatrix} d\alpha.$$

If we calculate the time derivative of this functional along the system of DDEs (5) and follow the way of Theorem 2.1, we can easily arrive the following inequality:

$$\dot{V}(t, x_t) \le \xi^T(t)\Xi_1\xi(t),$$

where

$$\Xi_1 = \begin{bmatrix} -27.4 & 0 & -1.4 & 0 & 10 & 0 & 1.4 & 0 \\ 0 & -41 & 0 & 0 & 0 & 20 & 0 & 2 \\ -1.4 & 0 & -1.6 & 0 & 0 & 0 & -0.1 & 0 \\ 0 & 0 & 0 & -2.65 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & -10 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 & 0 & -20 & 0 & 0 \\ 1.4 & 0 & -0.1 & 0 & 0 & 0 & -1.05 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & -0.95 \end{bmatrix}.$$

Hence, we can verify that the matrix Ξ_1 is symmetric and negative definite. In addition, the eigenvalues of the matrix Ξ_1, can be derived as $\lambda_1 = -53.1449,\ \lambda_2 = -32.0602,\ \lambda_3 = -8.0605,\ \lambda_4 = -5.5957,\ \lambda_5 = -2.6500,\ \lambda_6 = -1.5501,\ \lambda_7 = -0.8441,\ \lambda_8 = -0.7447$. Thus, secondly, it is clear that assumption (A2) of Theorem 2.1 is satisfied. Then, all assumptions of Theorem 2.1 are hold. From this point, we can conclude that the zero solution of the system of DDEs (5) is asymptotically stable (see also Fig. 1).

We now present an additional assumption for the next theorem. Coming theorem also includes new stability criteria for the system of DDEs (1). Here, the proof of the next theorem is given by using the Wirtinger inequality.

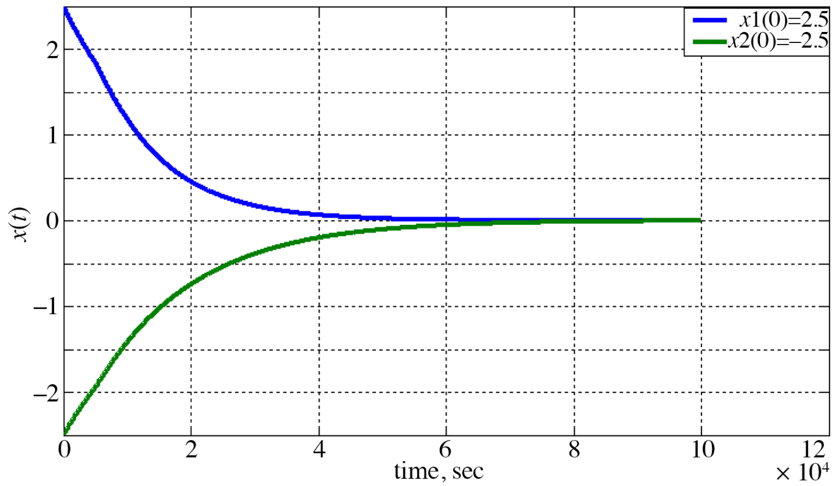

Fig. 1. Trajectories of the solution $x(t)$ of the system of DDEs (5) when $d(t) = 20^{-1}(1 + \sin t)$.

(A3) We have symmetric positive definite matrices $P \in \Re^{n\times n},\ R_i \in \Re^{n\times n},\ Z \in \Re^{n\times n}$ and some U_i known matrices with appropriate dimensions such that the following matrix inequality holds:

$$\overline{\Xi} = \begin{pmatrix} \overline{\Xi}_{11} & \overline{\Xi}_{12} & \overline{\Xi}_{13} & \overline{\Xi}_{14} & \overline{\Xi}_{15} & \overline{\Xi}_{16} & \overline{\Xi}_{17} & \overline{\Xi}_{18} & \overline{\Xi}_{19} \\ * & \overline{\Xi}_{22} & \overline{\Xi}_{23} & 0 & 0 & \overline{\Xi}_{26} & \overline{\Xi}_{27} & 0 & 0 \\ * & * & \overline{\Xi}_{33} & 0 & 0 & \overline{\Xi}_{36} & \overline{\Xi}_{37} & 0 & 0 \\ * & * & * & \overline{\Xi}_{44} & 0 & 0 & 0 & \overline{\Xi}_{48} & 0 \\ * & * & * & * & \overline{\Xi}_{55} & 0 & 0 & 0 & \overline{\Xi}_{59} \\ * & * & * & * & * & \overline{\Xi}_{66} & \overline{\Xi}_{67} & 0 & 0 \\ * & * & * & * & * & * & \overline{\Xi}_{77} & 0 & 0 \\ * & * & * & * & * & * & * & \overline{\Xi}_{88} & 0 \\ * & * & * & * & * & * & * & * & \overline{\Xi}_{99} \end{pmatrix} < 0,$$

where

$$\overline{\Xi}_{11} = A^T P + PA + \sum_{i=1}^{2} \tau_i A^T Z A - \sum_{i=1}^{2} 4\tau_i^{-1} Z + \sum_{i=1}^{2} R_i,$$

$$\overline{\Xi}_{12} = PA_{d_1} + (\tau_1 + \tau_2) A^T Z A_{d_1},$$

$$\overline{\Xi}_{13} = PA_{d_2} + (\tau_1 + \tau_2) A^T Z A_{d_2},$$

$$\overline{\Xi}_{14} = -2\tau_1^{-1} Z, \qquad \overline{\Xi}_{15} = -2\tau_2^{-1} Z,$$

$$\overline{\Xi}_{16} = P + \sum_{i=1}^{2} \tau_i A^T Z, \qquad \overline{\Xi}_{17} = P + \sum_{i=1}^{2} \tau_i A^T Z,$$

$$\overline{\Xi}_{18} = \frac{6Z}{\tau_1^2}, \qquad \overline{\Xi}_{19} = \frac{6Z}{\tau_2^2},$$

$$\overline{\Xi}_{22} = (\tau_1 + \tau_2)A_{d_1}^T Z A_{d_1} - (1 - \mu_1)R_1 + \epsilon_1 U_1^T U_1,$$

$$\overline{\Xi}_{23} = (\tau_1 + \tau_2)A_{d_1}^T Z A_{d_2}, \qquad \overline{\Xi}_{26} = (\tau_1 + \tau_2)A_{d_1}^T Z,$$

$$\overline{\Xi}_{27} = (\tau_1 + \tau_2)A_{d_1}^T Z,$$

$$\overline{\Xi}_{33} = (\tau_1 + \tau_2)A_{d_2}^T Z A_{d_2} - (1 - \mu_2)R_2 + \epsilon_2 U_2^T U_2,$$

$$\overline{\Xi}_{36} = (\tau_1 + \tau_2)A_{d_2}^T Z, \qquad \overline{\Xi}_{37} = (\tau_1 + \tau_2)A_{d_2}^T Z,$$

$$\overline{\Xi}_{44} = -\frac{4Z}{\tau_1}, \qquad \overline{\Xi}_{48} = \frac{6Z}{\tau_1^2},$$

$$\overline{\Xi}_{55} = -\frac{4Z}{\tau_2}, \qquad \overline{\Xi}_{59} = \frac{6Z}{\tau_2^2},$$

$$\overline{\Xi}_{66} = (\tau_1 + \tau_2)Z - \epsilon_1 I, \qquad \overline{\Xi}_{67} = (\tau_1 + \tau_2)Z,$$

$$\overline{\Xi}_{77} = (\tau_1 + \tau_2)Z - \epsilon_2 I, \qquad \overline{\Xi}_{88} = -\frac{12Z}{\tau_1^3}, \qquad \overline{\Xi}_{99} = -\frac{12Z}{\tau_2^3}.$$

Theorem 2.2. *The zero solution of the system of DDEs (1) is asymptotically stable if assumptions (A1) and (A3) hold.*

Proof. We now define a Lyapunov – Krasovskii functional by

$$V_1(t, x_t) = x^T(t)Px(t) + \sum_{i=1}^{2} \int\limits_{t-d_i(t)}^{t} x^T(\alpha)R_i x(\alpha)\, d\alpha + \sum_{i=1}^{2} \int\limits_{-\tau_i}^{0} \int\limits_{t+\beta}^{t} \dot{x}^T(\alpha) Z \dot{x}(\alpha)\, d\alpha\, d\beta.$$

Calculating the derivative of $V_1(t, x_t)$ along the system of DDEs (1) and using the Wirtinger inequality, that is, Lemma 1.3, we get the following inequality:

$$\dot{V}_1(t, x_t) \le x^T(t)A^T Px(t) + x^T(t)PAx(t) + \sum_{i=1}^{2} x^T(t)PA_{d_i} x\big(t - d_i(t)\big) +$$

$$+ \sum_{i=1}^{2} x^T\big(t - d_i(t)\big)A_{d_i}^T Px(t) + \sum_{i=1}^{2} x^T(t)PF_i\big(t, x\big(t - d_i(t)\big)\big) +$$

$$+ \sum_{i=1}^{2} F_i^T\big(t, x\big(t - d_i(t)\big)\big)Px(t) + \sum_{i=1}^{2} x^T(t)R_i x(t) -$$

$$- \sum_{i=1}^{2} x^T\big(t - d_i(t)\big)R_i x\big(t - d_i(t)\big)(1 - \mu_i) +$$

$$+ \sum_{i=1}^{2} \tau_i x^T(t)A^T Z A x(t) + \sum_{i=1}^{2} \tau_i x^T(t)A^T Z \sum_{i=1}^{2} A_{d_i} x\big(t - d_i(t)\big) +$$

$$+\sum_{i=1}^{2}\tau_i x^T(t)A^T Z\left(\sum_{i=1}^{2}F_i\big(t,x\big(t-d_i(t)\big)\big)\right)+$$

$$+\sum_{i=1}^{2}\tau_i\left(\sum_{i=1}^{2}x^T\big(t-d_i(t)\big)A_{d_i}^T\right)ZAx(t)+$$

$$+\sum_{i=1}^{2}\tau_i\left(\sum_{i=1}^{2}x^T\big(t-d_i(t)\big)A_{d_i}^T\right)Z\left(\sum_{i=1}^{2}A_{d_i}x\big(t-d_i(t)\big)\right)+$$

$$+\sum_{i=1}^{2}\tau_i\left(\sum_{i=1}^{2}x^T\big(t-d_i(t)\big)A_{d_i}^T\right)Z\left(\sum_{i=1}^{2}F_i\big(t,x\big(t-d_i(t)\big)\big)\right)+$$

$$+\sum_{i=1}^{2}\tau_i\left(\sum_{i=1}^{2}F_i^T\big(t,x\big(t-d_i(t)\big)\big)\right)ZAx(t)+$$

$$+\sum_{i=1}^{2}\tau_i\left(\sum_{i=1}^{2}F_i^T\big(t,x\big(t-d_i(t)\big)\big)\right)Z\left(\sum_{i=1}^{2}A_{d_i}x\big(t-d_i(t)\big)\right)+$$

$$+\sum_{i=1}^{2}\tau_i\left(\sum_{i=1}^{2}F_i^T\big(t,x\big(t-d_i(t)\big)\big)\right)Z\left(\sum_{i=1}^{2}F_i\big(t,x\big(t-d_i(t)\big)\big)\right)-$$

$$-\sum_{i=1}^{2}\Bigg\{x^T(t)\,\frac{4Z}{\tau_i}x(t)+x^T(t)\,\frac{2Z}{\tau_i}\,x(t-\tau_i)+$$

$$+x^T(t-\tau_i)\frac{2Z}{\tau_i}\,x(t)+x^T(t-\tau_i)\frac{4Z}{\tau_i}x(t-\tau_i)-$$

$$-x^T(t)\,\frac{6Z}{\tau_i^2}\left(\int\limits_{t-\tau_i}^{t}x(s)ds\right)-x^T(t-\tau_i)\frac{6Z}{\tau_i^2}\left(\int\limits_{t-\tau_i}^{t}x(s)ds\right)-$$

$$-\left(\int\limits_{t-\tau_i}^{t}x(s)ds\right)^T\frac{6Z}{\tau_i^2}\,x(t)-\left(\int\limits_{t-\tau_i}^{t}x(s)ds\right)^T\frac{6Z}{\tau_i^2}\,x(t-\tau_i)+$$

$$+\left(\int\limits_{t-\tau_i}^{t}x(s)ds\right)^T\frac{12Z}{\tau_i^3}\left(\int\limits_{t-\tau_i}^{t}x(s)ds\right)\Bigg\}. \qquad (6)$$

For nonlinear functions $F_i(.)$ endowed with $\epsilon_i > 0,\ i = 1, 2,$ we can obtain

$$0 \le -\epsilon_i F_i^T\big(t,x\big(t-d_i(t)\big)\big)F_i\big(t,x\big(t-d_i(t)\big)\big)+\epsilon_i x^T\big(t-d_i(t)\big)U_i^T U_i x\big(t-d_i(t)\big). \qquad (7)$$

By the inequalities (6) and (7), it follows that

$$\dot V_1(t,x_t) \le \overline{\xi}^T(t)\,\overline{\Xi\xi}(t), \qquad (8)$$

where

$$\overline{\Xi}=\begin{bmatrix}\overline{\Xi}_{11} & \overline{\Xi}_{12} & \overline{\Xi}_{13} & \overline{\Xi}_{14} & \overline{\Xi}_{15} & \overline{\Xi}_{16} & \overline{\Xi}_{17} & \overline{\Xi}_{18} & \overline{\Xi}_{19}\\ * & \overline{\Xi}_{22} & \overline{\Xi}_{23} & 0 & 0 & \overline{\Xi}_{26} & \overline{\Xi}_{27} & 0 & 0\\ * & * & \overline{\Xi}_{33} & 0 & 0 & \overline{\Xi}_{36} & \overline{\Xi}_{37} & 0 & 0\\ * & * & * & \overline{\Xi}_{44} & 0 & 0 & 0 & \overline{\Xi}_{48} & 0\\ * & * & * & * & \overline{\Xi}_{55} & 0 & 0 & 0 & \overline{\Xi}_{59}\\ * & * & * & * & * & \overline{\Xi}_{66} & \overline{\Xi}_{67} & 0 & 0\\ * & * & * & * & * & * & \overline{\Xi}_{77} & 0 & 0\\ * & * & * & * & * & * & * & \overline{\Xi}_{88} & 0\\ * & * & * & * & * & * & * & * & \overline{\Xi}_{99}\end{bmatrix},$$

$$\bar{\xi}^T(t)=\Big[\; x^T(t)\quad x^T(t-d_1(t))\quad x^T(t-d_2(t))\quad x^T(t-\tau_1)\quad x^T(t-\tau_2)$$

$$F_1^T\big(t,x(t-d_1(t))\big)\quad F_2^T\big(t,x(t-d_2(t))\big)\quad \left(\int_{t-\tau_1}^{t}x(s)ds\right)^T\quad \left(\int_{t-\tau_2}^{t}x(s)ds\right)^T\;\Big],$$

$$\overline{\Xi}_{11}=A^TP+PA+\sum_{i=1}^{2}\tau_iA^TZA-\sum_{i=1}^{2}4\tau_i^{-1}Z+\sum_{i=1}^{2}R_i,$$

$$\overline{\Xi}_{12}=PA_{d_1}+(\tau_1+\tau_2)A^TZA_{d_1},$$

$$\overline{\Xi}_{13}=PA_{d_2}+(\tau_1+\tau_2)A^TZA_{d_2},$$

$$\overline{\Xi}_{14}=-2\tau_1^{-1}Z,\qquad \overline{\Xi}_{15}=-2\tau_2^{-1}Z,$$

$$\overline{\Xi}_{16}=P+\sum_{i=1}^{2}\tau_iA^TZ,\qquad \overline{\Xi}_{17}=P+\sum_{i=1}^{2}\tau_iA^TZ,$$

$$\overline{\Xi}_{18}=\frac{6Z}{\tau_1^2},\qquad \overline{\Xi}_{19}=\frac{6Z}{\tau_2^2},$$

$$\overline{\Xi}_{22}=(\tau_1+\tau_2)A_{d_1}^TZA_{d_1}-(1-\mu_1)R_1+\epsilon_1U_1^TU_1,$$

$$\overline{\Xi}_{23}=(\tau_1+\tau_2)A_{d_1}^TZA_{d_2},$$

$$\overline{\Xi}_{26}=(\tau_1+\tau_2)A_{d_1}^TZ,\qquad \overline{\Xi}_{27}=(\tau_1+\tau_2)A_{d_1}^TZ,$$

$$\overline{\Xi}_{33}=(\tau_1+\tau_2)A_{d_2}^TZA_{d_2}-(1-\mu_2)R_2+\epsilon_2U_2^TU_2,$$

$$\overline{\Xi}_{36}=(\tau_1+\tau_2)A_{d_2}^TZ,\qquad \overline{\Xi}_{37}=(\tau_1+\tau_2)A_{d_2}^TZ,$$

$$\overline{\Xi}_{44}=-\frac{4Z}{\tau_1},\qquad \overline{\Xi}_{48}=\frac{6Z}{\tau_1^2},$$

$$\overline{\Xi}_{55} = -\frac{4Z}{\tau_2}, \qquad \overline{\Xi}_{59} = \frac{6Z}{\tau_2^2},$$

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$$\overline{\Xi}_{77} = (\tau_1 + \tau_2)Z - \epsilon_2 I, \qquad \overline{\Xi}_{88} = -\frac{12Z}{\tau_1^3}, \qquad \overline{\Xi}_{99} = -\frac{12Z}{\tau_2^3}.$$

By using the Schur complement, that is, Lemma 1.1, we have $\dot{V}_1(t, x_t) < 0$. It is now notable that the inequality (8) is considered as a quadratic form. Here, the matrix $\overline{\Xi} < 0$ is symmetric and negative definite. Then, it can be written that

$$\dot{V}_1(t, x_t) \le \overline{\xi}^T(t)\,\overline{\Xi}\overline{\xi}(t) < 0, \quad \overline{\xi}(t) \ne 0.$$

Thus, we can conclude that the zero solution of the system of DDEs (1) is asymptotically stable. This fact completes the proof of Theorem 2.2.

Example 2.2. Let us consider the system of DDEs (1) for the particular case given below:

$$\frac{d}{dt}\left(\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\right) = \begin{bmatrix} -6 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1\left(t - \dfrac{1+\sin t}{20}\right) \\ x_2\left(t - \dfrac{1+\sin t}{20}\right) \end{bmatrix} +$$

$$+\begin{bmatrix} x_1\left(t - \dfrac{1+\sin t}{20}\right) e^{-x_1^2\left(t - \frac{1+\sin t}{20}\right)} \\ x_2\left(t - \dfrac{1+\sin t}{20}\right) e^{-x_2^2\left(t - \frac{1+\sin t}{20}\right)} \end{bmatrix}, \quad t \ge \frac{1}{10}. \tag{9}$$

When we compare the system of DDEs (9) with the system of DDEs (1), it follows that

$$A = \begin{bmatrix} -6 & 0 \\ 0 & -5 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$F_1(t, x(t - d(t))) = \begin{bmatrix} x_1\left(t - \dfrac{1+\sin t}{20}\right) e^{-x_1^2\left(t - \frac{1+\sin t}{20}\right)} \\ x_2\left(t - \dfrac{1+\sin t}{20}\right) e^{-x_2^2\left(t - \frac{1+\sin t}{20}\right)} \end{bmatrix}, \quad t \ge \frac{1}{10},$$

$$\epsilon = 0.60, \qquad 0 \le d(t) = d_1(t) = \frac{1+\sin t}{20} \le 0.1 = \tau_1,$$

$$\frac{d}{dt}d(t) = \dot{d}_1(t) = \frac{\cos t}{20} \le 0.05 = \mu = \mu_1 < 1, \quad r = 0.05.$$

As before, we see that assumption (A1) of Theorem 2.2 is satisfied. From this point, for the next step, we choose the matrices $P,\ R,\ Z,$ and U as the following:

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \qquad R_1 = R = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad Z = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \qquad U = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For the above choices, the Lyapunov – Krasovskii functional given in Theorem 2.2 takes the form

$$V_1(t,x_t) = x^T(t)Px(t) + \int\limits_{t-d_1(t)}^{t} x^T(\alpha)R_1x(\alpha)\,d\alpha + \int\limits_{-\tau_1}^{0}\int\limits_{t+\beta}^{t} \dot{x}^T(\alpha)Z\dot{x}(\alpha)\,d\alpha\,d\beta =$$

$$= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^T \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \int\limits_{t-d_1(t)}^{0} \begin{bmatrix} x_1(\alpha) \\ x_2(\alpha) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(\alpha) \\ x_2(\alpha) \end{bmatrix} d\alpha +$$

$$+ \int\limits_{-\tau_1}^{0}\int\limits_{t+\beta}^{t} \begin{bmatrix} \dot{x}_1(\alpha) \\ \dot{x}_2(\alpha) \end{bmatrix}^T \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} \begin{bmatrix} \dot{x}_1(\alpha) \\ \dot{x}_2(\alpha) \end{bmatrix} d\alpha\,d\beta.$$

If we calculate the time derivative of this Lyapunov – Krasovskii functional along the system of DDEs (9) and follow the way of Theorem 2.2, we can easily derive the following inequality:

$$\dot{V}_1(t,x_t) \leq \overline{\xi}^T(t)\,\overline{\Xi}_1\overline{\xi}(t),$$

where

$$\overline{\Xi}_1 = \begin{bmatrix} -23.364 & 0 & -1.994 & 0 & -0.2 & 0 & 1.994 & 0 & 6 & 0 \\ 0 & -28.375 & 0 & 0 & 0 & -0.2 & 0 & 2.995 & 0 & 6 \\ -1.994 & 0 & -0.349 & 0 & 0 & 0 & -0.001 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & -0.4 & 0 & 0 & 0 & 6 & 0 \\ 0 & -0.2 & 0 & 0 & 0 & -0.4 & 0 & 0 & 0 & 6 \\ 1.994 & 0 & -0.001 & 0 & 0 & 0 & -0.599 & 0 & 0 & 0 \\ 0 & 2.995 & 0 & 0 & 0 & 0 & 0 & -0.599 & 0 & 0 \\ 6 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & -120 & 0 \\ 0 & 6 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & -120 \end{bmatrix} < 0.$$

Then, the eigenvalues of the matrix $\overline{\Xi}_1$ can be calculated as

$$\lambda_1 = -120.6909, \qquad \lambda_2 = -120.6708, \qquad \lambda_3 = -28.3074,$$

$$\lambda_4 = -23.3406, \qquad \lambda_5 = -1.3000, \qquad \lambda_6 = -0.5147,$$

$$\lambda_7 = -0.2770, \qquad \lambda_8 = -0.1062, \qquad \lambda_9 = -0.0988, \qquad \lambda_{10} = -0.0798.$$

From this point, we see that assumption (A3) of Theorem 2.2 is held. Then, all assumptions of Theorem 2.2 hold. Thus, for the considered particular case, we can conclude that the zero solution of the system of DDEs (9) is asymptotically stable (see also Fig. 2).

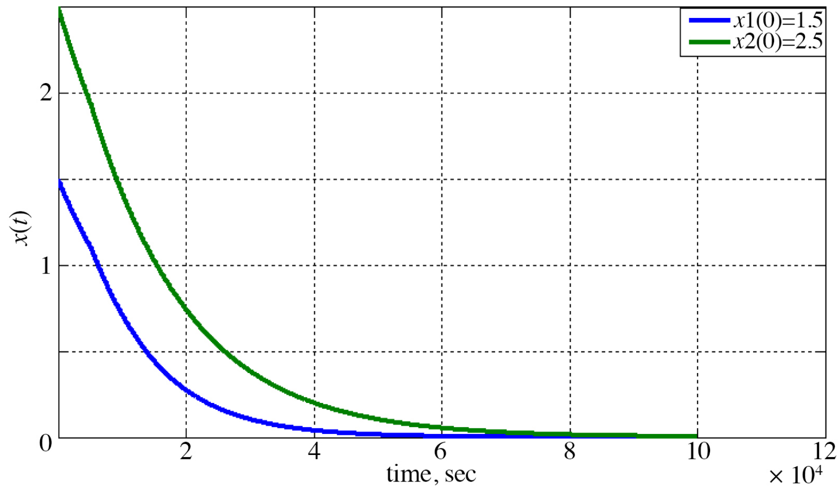

Fig. 2. Trajectories of the solution $x(t)$ of the system of DDEs (9) when $d(t) = 20^{-1}(1 + \sin t)$.

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