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UNIQUE SOLVABILITY OF THE BOUNDARY-VALUE PROBLEMS FOR NONLINEAR FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS*

ОДНОЗНАЧНА РОЗВ'ЯЗНІСТЬ КРАЙОВИХ ЗАДАЧ ДЛЯ НЕЛІНІЙНИХ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРОБОВОГО ПОРЯДКУ

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By using the Krasnoselskii theorem, we obtain general conditions for the unique solvability of boundary-value problems for (non)linear fractional functional-differential equations.

Із застосуванням теореми Красносельського отримано загальні умови однозначної розв'язності крайових задач для (не)лінійних функціонально-диференціальних рівнянь дробового порядку.

Paper dedicated to Professor Michal Fečkan
on the occasion of his 60^{th} anniversary

The perspectives of a wide application of the fractional functional-differential equations (FFDE) made them interesting for numerous researchers. In analogue to the ordinary differential equation with derivatives of any natural order, conditions on the unique solvability of the boundary-value problem for FFDE is a basic stage that should be solved for further application.

1. Introduction. The fractional differential equations (FDE), which are currently a hot topic, are represented by the numerous papers, here is referred a few of them only [1 – 7].

The application scale of mentioned equations is quite wide. We would like to highlight the [5], where authors made a complex overview of possible applications of FDE. As the first of all, it was mentioned follow topics: the theories of differential, integral, and integro-differential equations, and special functions of mathematical physics, as well as their extensions and generalizations for one and more variables. Then, there are some present-day applications of fractional calculus, include fluid flow, rheology, dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, electrical networks, probability and statistics, control theory of dynamical systems, viscoelasticity, electrochemistry of corrosion, chemical physics, optics, and signal processing, and so on.

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Conditions on the unique solvability of the boundary value problem for functional differential equations is a fundamental and non-trivial part of the study, and, it is not surprising that many publications are focused to them, for example, [8 – 11].

In the present paper is utilized our previous experience for functional-differential equations with derivatives of natural order [12 – 15]. Besides, our motivation were the work [1], where authors studied conditions on the unique solvability of the ordinary differential equations of the fractional order and the work [7], where author study conditions on the unique solvability of the initial-value problem for functional differential equations of the fractional order.

The main focus of the present investigation is the conditions lookup of the unique solvability of the boundary value problem for the FFDE. There is a perspective way to solve the specified task is using the Krasnoselski Theorem on the unique solvability in the suitable cone. Thereby, here is arisen the problem to construct that cone, that was successfully done in the present work.

2. Problem formulation. Here is considered the FFDE

$$ {}^{c}D_a^q x(t) = (lx)(t) + r(t), \quad t \in [a,b], \tag{1}$$

with boundary-value condition

$$ x(a) = \phi(x), \tag{2}$$

where ${}^{c}D_a^q$ is the Caputo fractional derivative of order $q \in (0,1)$ with the lower limit zero and $l \in \mathcal{W}\big([a,b],\mathbb{R}^n\big) \to L_1\big([a,b],\mathbb{R}^n\big)$ is generally speaking nonlinear operator, $\phi \in \mathcal{W}\big([a,b],\mathbb{R}^n\big) \to$ $\to \mathbb{R}^n$ is nonlinear functional defined in the space $\in \mathcal{W}\big([a,b],\mathbb{R}^n\big)$ of vector functions with absolutely continuous components of $u,$ function $r \in L_1\big([a,b],R^n\big).$

The main goal of this investigation is to establish the conditions on the unique solvability of the boundary value problem for FFDE (1), (2).

The paper is constructed in a next way. In the Section 3 we give the necessary notation and definitions, in Section 4 one can find the auxiliary Theorems and Lemmas, next in Section 5 is the main result on the unique solvability of the nonlinear FFDE with boundary-value conditions. Then, in Section 6 we give the proof of the main result Theorem 2. We give the conditions on the unique solvability of the boundary value problem for the linear FFDE in Section 7. Finally, in Section 8 is some discussion.

3. Notations and definitions.

(a) $q \in (0,1)$ is an order of the Caputo fractional derivative ${}^{c}D_a^q$.

(b) The interval $I_a = [a,b]$.

(c) $\mathbb{R} := (-\infty,\infty),$ $\|x\| := \max_{1\le i\le n}|x_i|$ for $x = (x_i)_{i=1}^n \in \mathbb{R}^n$.

(d) $L_1\big(I_a,\mathbb{R}^n\big)$ is the Banach space of all the summable vector-functions $u\colon [a,b] \to \mathbb{R}^n$ with the standard norm

$$ L_1\big(I_a,\mathbb{R}^n\big) \ni u \longmapsto \int\limits_a^b \|u(s)\| ds. $$

(e) $C\big(I_a,\mathbb{R}^n\big)$ is the Banach space of continuous functions u from I_a to $\mathbb{R}^n$ with the norm

$$ C\big(I_a,\mathbb{R}^n\big) \ni u \longmapsto \|u\|_C := \max|u(t)|, \quad t \in I_a. $$

(f) $\mathcal{W}\big(I_a, \mathbb{R}^n\big)$ is the Banach space of absolutely continuous functions u from I_a to $\mathbb{R}^n$ with the norm

$$\mathcal{W}\big(I_a, \mathbb{R}^n\big) \ni u \longmapsto \|u\|_{\mathcal{W}} := \int\limits_a^b \|\, {}^cD^u(s)\| \; ds + \|u(a)\|. \tag{3}$$

(g) The set $\mathcal{W}^+\big(I_a, \mathbb{R}^n\big)$ is defined by the formula

$$\mathcal{W}^+\big(I_a, \mathbb{R}^n\big) := \Big\{ u = (u_i)_{i=1}^n \in \mathcal{W}\big(I_a, \mathbb{R}^n\big) \colon \min_{t \in I_a} u_i(t) \ge 0, \; i = 1, 2, \ldots, n \Big\}. \tag{4}$$

(h) The set $\mathcal{W}^{++}\big(I_a, \mathbb{R}^n\big)$ is defined by the formula

$$\mathcal{W}^{++}\big(I_a, \mathbb{R}^n\big) := \Big\{ u = (u_i)_{i=1}^n(t) \in \mathcal{W}^+\big(I_a, \mathbb{R}^n\big) \colon \\ \operatorname*{ess\,sup}_{t \in I_a} {}^cD_a^q u_i(t) \ge 0, \; u_i(a) \ge 0, \; i = 1, 2, \ldots, n \Big\}. \tag{5}$$

In what follows, the symbols $\mathcal{W}\big(I_a, \mathbb{R}^n\big)$, $\mathcal{W}^+\big(I_a, \mathbb{R}^n\big)$, $\mathcal{W}^{++}\big(I_a, \mathbb{R}^n\big)$ corresponding to the fixed a, b, and n will usually appear simply as $\mathcal{W}$, $\mathcal{W}^+$, $\mathcal{W}^{++}$.

Definition 1 [16]. *By a solution of the problem (1), (2) we mean a vector-function $x \in \mathcal{W}$ with property (2) and satisfying (1) for a.e. $t \in I_a$.*

Definition 2 [2]. *For a function f given on the interval I_a, the Caputo derivative of fractional order q is defined by*

$${}^cD_a^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int\limits_a^t (t-s)^{-q} f(s) ds,$$

where $\Gamma(q) \colon [0, \infty) \to \mathbb{R}$ is Gamma-function and

$$\Gamma(q) := \int\limits_0^\infty t^{q-1} e^{-t} dt.$$

The q-th Riemann-Liowille fractional order derivative of f, is defined by

$${}^LD_a^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \left(\int\limits_a^t (t-s)^{-q} f(s) ds + (t-a)^{-q} f(a) \right).$$

Definition 3 [5]. *The Caputo derivative of order q for a function $f \colon I_a \to \mathbb{R}$ can be written as*

$${}^cD_a^q f(t) = {}^LD_a^q \left(f(t) - (t-a)^{-q} f(a) \right).$$

It is known [5] that the Riemann – Liowille fractional derivative depends on initial conditions.

Definition 4. *Let $h = (h_k)_{k=1}^n : \mathcal{W} \to \mathbb{R}^n$ be a linear mapping. We say that a linear operator $p = (p_k)_{k=1}^n : \mathcal{W} \to L_1$ belongs to the set $S_{a,h}$ if the boundary-value problem*

$$^cD_a^q u = (pu)(t) + \alpha(t), \quad t \in I_a, \tag{6}$$

$$u(a) = h(u) + c, \tag{7}$$

has a unique solution $u = (u_k)_{k=1}^n$ for any $\alpha = (\alpha_k)_{k=1}^n \subset L_1$ and $c = (c_k)_{k=1}^n \subset \mathbb{R}^n$ and, moreover, the solution of (6), (7) possesses the property

$$\min_{t \in I_a} u_k(t) \geq 0, \quad k = 1, 2, \ldots, n,$$

whenever the functions $\alpha = (\alpha_k)_{k=1}^n$, and the constants $c = (c_k)_{k=1}^n$, appearing in (6) and (7) are non-negative.

4. Auxiliary propositions. For further investigation we will need the next lemmas from [5].

Lemma 1 ([5], Lemma 2.21). *Let $0 < q < 1$ and let $x(t) \in L_1$, $x(t) \in C$ or $x(t) \in \mathcal{W}$, then*

$$^cD_a^q I_a^q x(t) = x(t)$$

and

$$^cD_b^q I_b^q x(t) = x(t),$$

where

$$I_a^q x(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} x(s) ds, \quad x > a, \tag{8}$$

and

$$I_b^q x(t) = \frac{1}{\Gamma(q)} \int_t^b (t-s)^{q-1} x(s) ds, \quad x < b. \tag{9}$$

Lemma 2 ([5], Lemma 2.22). *Let $0 < q < 1$. If $x(t) \in \mathcal{W}$ or $x(t) \in C$, then*

$$I_a^q\, {}^cD_a^q x(t) = x(t) - x(a)$$

and

$$I_b^q\, {}^cD_b^q x(t) = x(t) - x(b),$$

where $I_a^q x(t)$ and I_b^q are defined by (8) and (9) correspondingly.

In view of Definition 2 and Lemma 1 and Lemma 2 the next obvious Lemma is true.

Lemma 3. *The problem (1), (2) is equivalent to the equation*

$$x(t) = \phi(x) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} (lx)(s) ds + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} r(s) ds, \quad t \in I_a,$$

where $x(t) \in \mathcal{W}$.

To prove our main result, we use the following statement on the unique solvability of an equation with a Lipschitz type non-linearity established in [17].

Let us consider the abstract operator-equation

$$Fx = z, \tag{10}$$

where $F\colon E_1 \to E_2$ is a mapping between a normed space $\langle E_1, \|\cdot\|_{E_1}\rangle$ and a Banach space $\langle E_2, \|\cdot\|_{E_2}\rangle$ over the field $\mathbb{R}$, and z is an arbitrary element from E_2.

Let $K_i \subset E_i,\ i = 1,2,$ be cones [18]. The cones $K_i,\ i = 1,2,$ induce natural partial orderings of the respective spaces. Thus, for each $i = 1,2,$ we write $x \leqq_{K_i} y$ and $y \geqq_{K_i} x$ if and only if $\{x, y\} \subset E_i$ and $y - x \in K_i$.

Theorem 1 ([17], Theorem 49.4). *Let the cone K_2 be normal and generating. Furthermore, let $\Psi_k\colon E_1 \to E_2,\ k = 1,2,$ be additive and homogeneous operators such that Ψ_1^{-1} and $(\Psi_1 + \Psi_2)^{-1}$ exist and possess the properties*

$$\Psi_1^{-1}(K_2) \subset K_1, \tag{11}$$

$$(\Psi_1 + \Psi_2)^{-1}(K_2) \subset K_1 \tag{12}$$

and, furthermore, let the order relation

$$\Psi_1(x - y) \leqq_{K_2} Fx - Fy \leqq_{K_2} \Psi_2(x - y) \tag{13}$$

be satisfied for any pair $(x, y) \in E_1^2$ such that $x \geqq_{K_1} y$.

Then equation (10) has a unique solution for an arbitrary z from E_2.

Let us recall two definitions (see, e.g., [17, 18]).

Definition 5. *A cone $K_2 \subset E_2$ is called normal if there exists a constant $\gamma \in (0, +\infty)$ such that $\|x\|_{E_2} \le \gamma \|y\|_{E_2}$ for arbitrary $\{x, y\} \subset E_2$ with the property $0 \leqq_{K_2} x \leqq_{K_2} y$.*

Definition 6. *A cone K_1 is called generating in E_1 if every element $u \in E_1$ can be represented in the form $u = u_1 - u_2$, where $\{u_1, u_2\} \subset K_1$.*

4.1. Lemmas. We need some technical lemmas.

Lemma 4. *The following propositions on the space $\mathcal{W}$ are true:*

(a) the set $\mathcal{W}^+$ is a cone in the space $\mathcal{W}$;

(b) the set $\mathcal{W}^{++}$ is a normal and reproducing cone in the space $\mathcal{W}$.

Proof. Let us proof assertion (a). If $\{u_1, u_2\} \subset \mathcal{W}^{++}$ and $\{\lambda_1, \lambda_2\} \subset [0, +\infty)$, then, obviously, $\lambda_1 u_1 + \lambda_2 u_2$ lies in $\mathcal{W}^{++}$ as well. Suppose that $u \in \mathcal{W}^{++}$ and $-u \in \mathcal{W}^{++}$ simultaneously. Taking into account the definition of $\mathcal{W}^{++}$, we have ${}^cD_a^q u \equiv 0$ and, moreover, $u(a) = 0$, whence it is obvious that $u \equiv 0$. Thus, $\mathcal{W}^{++}$ is a cone in $\mathcal{W}$.

Let us proof assertion (b). In order to check that the cone $\mathcal{W}^{++}$ is normal, it is sufficient to show that every set of the form

$$\Big\{x \in \mathcal{W}:\ \{x - u, v - x\} \subset \mathcal{W}^{++}\},\ \ u, v \in \mathcal{W},\ \ \max\{\|u\|_{\mathcal{W}}, \|v\|_{\mathcal{W}}\} \le 1\Big\}, \tag{14}$$

is bounded with respect to the norm $\|\cdot\|_{\mathcal{W}}$ (see (3)). Indeed, if an arbitrary x belongs to set (14), then for a.e. $t \in I_a$

$$ {}^cD_a^q u(t) \le {}^cD_a^q x(t) \le {}^cD_a^q v(t), \quad 0 \le u(a) \le x(a) \le v(a),$$

componentwise. Therefore,

$$\|x\|_{\mathcal{W}} = \int_a^b \|{}^cD_a^q x(s)\|\, ds + \|x(a)\| \le \|u\|_{\mathcal{W}} + \|v\|_{\mathcal{W}} \le 2,$$

which, in view of the arbitrariness of $x,$ implies that set (14) is bounded.

Finally, let us check, that the cone $\mathcal{W}^{++}$ is generating cone in the space $\mathcal{W}$. To proof that, it is sufficient to show that every element x of $\mathcal{W}$ admits a majorant in $\mathcal{W}^{++}$. Let $x \in \mathcal{W}$ be arbitrary. Then, in view of Lemma 3,

$$x(t) = \frac{1}{\Gamma(q)} \frac{d}{dt} \int_a^t (t-s)^{q-1} X(s) ds + x(a), \quad t \in I_a, \tag{15}$$

where $X \in L_1,\ X = {}^cD_a^q x$. Equality (15) implies that, componentwise,

$${}^cD_a^q x(t) \le {}^cD_a^q u(t), \quad t \in I_a,$$

where

$$u(t) = \frac{1}{\Gamma(q)} \frac{d}{dt} \int_a^t (t-s)^{q-1} |X(s)| ds + |x(a)|, \quad t \in I_a. \tag{16}$$

It is obvious from (16) that $u(a) \ge 0$ and ${}^cD_a^q u$ are non-negative and, therefore, u is an element of $\mathcal{W}^{++}$. This, due to the arbitrariness of $x,$ proves that $\mathcal{W}^{++}$ is generating.

Lemma 4 is proved.

Taking into account Definition 2 and Lemma 3, let us define a linear operator $\Theta_{p,h} : \mathcal{W} \to \mathcal{W}$ by putting

$$(\Theta_{p,h} u)(t) := u(t) - \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} (pu)(s) ds - u(a) \tag{17}$$

for all $u \in \mathcal{W}$.

Lemma 5. *Function x from the space $\mathcal{W}$ is a solution of the equation*

$$(\Theta_{p,h} x)(t) = \frac{1}{\Gamma(q)} \int_a^t (t-\xi)^{q-1} r(\xi) d\xi + c, \quad t \in I_a,$$

where $r \in L_1,\ c \in \mathbb{R}^n,$ if and only if it is a solution of the non-local boundary-value problem (1), (2).

The next lemma states the relation between the property described in Definition 4 and the positive invertibility of operator (17).

Lemma 6. *Let* $p = (p_k)_{k=1}^n : \mathcal{W} \to L_1$ *is linear operator such that*

$$p \in \mathcal{S}_{a,h}, \tag{18}$$

then the linear operator $\Theta_{p,h} : \mathcal{W} \to \mathcal{W}$ *given by formula (17) is invertible and, moreover, its inverse* $\Theta_{p,h}^{-1}$ *is satisfies the inclusion*

$$\Theta_{p,h}^{-1}(\mathcal{W}^{++}) \subset \mathcal{W}^{+}. \tag{19}$$

Proof. Suppose that mapping l belongs to the set $\mathcal{S}_{a,h}$. Given an arbitrary function $y = (y_k)_{k=1}^n \in \mathcal{W}$, consider the equation

$$\Theta_{p,h} u = y. \tag{20}$$

Since $y \in \mathcal{W}$, then ${}^cD_a^q y \in L_1$ and

$$y(t) - y(a) = \frac{1}{\Gamma(q)} \frac{d}{dt} \int_a^t (t-s)^{q-1}\, {}^cD_a^q y(s) ds.$$

According to (18), there exists a unique function $u \in \mathcal{W}$ such that

$$ {}^cD_a^q u(t) = (pu)(t) + {}^cD_a^q y(t), \quad t \in I_a,$$
$$u(a) = h(u) + y(a).$$

By Lemma 5, it follows that u is a unique solution of equation (20). Due to the arbitrariness of $y \in \mathcal{W}$, it follows that $\Theta_{p,h}^{-1}$ exists and, hence, $u = \Theta_{p,h}^{-1} y$.

Inclusion (18) also guarantees that if the functions $y_k,\ k = 1, 2, \ldots, n,$ are such that

$$ {}^cD_a^q y_k(t) \geq 0, \quad y(a) \geq 0, \tag{21}$$

then the components of u are non-negative and, therefore, $\Theta_{p,h}^{-1} y \in \mathcal{W}^{+}$. However, relations (21) mean that $y \in \mathcal{W}^{++}$ (see Notation (h)). Since y is arbitrary, we thus arrive at the required inclusion (19).

From the relation (17) follows the next obvious lemma.

Lemma 7. *The identity*

$$\Theta_{p,h} + \Theta_{\xi,\gamma} = 2\Theta_{\frac{1}{2}(p+\xi), \frac{1}{2}(h+\gamma)}. \tag{22}$$

holds for arbitrary linear operators $\{p, \xi\} : \mathcal{W} \to L_1,\ i = 1, 2.$

5. Main result. The main general result of this paper is the next theorem.

Theorem 2. *Assume that there exist some linear operators* $p = (p_k)_{k=1}^n : \mathcal{W} \to L_1,\ \xi = (\xi_k)_{k=1}^n : \mathcal{W} \to L_1,$ *and linear functionals* $h = (h_i)_{i=1}^n : \mathcal{W} \to \mathbb{R}^n,\ \gamma = (\gamma_i)_{i=1}^n : \mathcal{W} \to \mathbb{R}^n,$ *which satisfy inclusions*

$$p \in \mathcal{S}_{a,h}, \quad \frac{1}{2}(p+\xi) \in \mathcal{S}_{a, \frac{1}{2}(h+\gamma)}, \tag{23}$$

and such that for arbitrary functions $u = (u_k)_{k=1}^n : I_a \to \mathbb{R}^n,\ v = (v_k)_{k=1}^n : I_a \to \mathbb{R}^n$ *from* $\mathcal{W}$ *with the properties*

$$u_k(t) \geq v_k(t), \quad t \in I_a, \quad k = 1, 2, \ldots, n, \tag{24}$$

the inequalities

$$\xi_k(u - v)(t) \leq (l_k u)(t) - (l_k v)(t) \leq p_k(u - v)(t), \quad t \in I_a, \quad k = 1, 2, \ldots, n, \tag{25}$$

and

$$\gamma_k(u - v)(t) \leq \phi_k(u) - \phi_k(v) \leq h_k(u - v), \quad k = 1, 2, \ldots, n,$$

hold.

Then the non-local nonlinear boundary-value problem (2) for nonlinear FFDE (1) has a unique solution for an arbitrary function $r \in L_1$.

6. Proof of Theorem 2. Let us take $E_1 = E_2 = \mathcal{W}$ and define a mapping $F : \mathcal{W} \to \mathcal{W}$ by setting

$$(Fu)(t) := (\Theta_{l,\phi} u)(t), \quad t \in [a, b], \tag{26}$$

for any u from $\mathcal{W}$, where $V_{l,\phi}$ is given by (17). Then equation (26) takes form (10) with

$$z(t) := \frac{1}{\Gamma(q)} \int_a^t (t - \xi)^{q-1} r(\xi) d\xi, \quad t \in I_a.$$

Consider problem (1), (2). It is clear from Lemma 5 that an absolutely continuous vector function $u = (u_k)_{k=1}^n : I_a \to \mathbb{R}^n$ is a solution of (1), (2) if, and only if it satisfies the equation

$$\Theta_{l,\phi} u = z.$$

Assumption (25) means that the estimate

$$-p_k(u - v)(t) \leq -(l_k u)(t) + (l_k v)(t) \leq -\xi_k(u - v)(t), \quad t \in I_a,$$

is true for any u and v with property (24). The relation

$$\begin{aligned} &{}^c D_a^q u_k(t) - {}^c D_a^q v_k(t) - p_k(u - v)(t) \leq \\ &\leq {}^c D_a^q u_k(t) - {}^c D_a^q v_k(t) - (l_k u)(t) - (l_k v)(t) \leq \\ &\leq {}^c D_a^q u_k(t) - {}^c D_a^q v_k(t) - \xi_k(u - v)(t), \end{aligned} \tag{27}$$

hold for almost all t from I_a.

Let us specify the linear mappings $\Psi_{ik} : \mathcal{W} \to \mathcal{W},\ i = 1, 2,\ k = 1, 2, \ldots, n,$ by the next way

$$(\Psi_{1k} u)(t) := \Theta_{p,h}, \quad t \in I_a, \tag{28}$$

$$(\Psi_{2k} u)(t) := \Theta_{\xi,\gamma}, \quad t \in I_a, \tag{29}$$

where $\{u, v\} \in \mathcal{W}$ have the properties (24). Then integrating (27) and taking (28), (29) into account, we have

$$\Psi_{1k}(u-v)(t) \leq u(t) - \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1}(l_k u)(s)ds - \phi(u) -$$

$$- \left(v(t) - \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1}(l_k v)(s)ds - \phi(v)\right) \leq$$

$$\leq \Psi_{2k}(u-v)(t), \quad t \in I_a, \tag{30}$$

for any $u = (u_k)_{k=1}^n$ and $v = (v_k)_{k=1}^n$ with properties (24).

In view of the mapping $\Theta_{l,\phi}$ (see formulae (17)) and the sets $\mathcal{W}^+$ and $\mathcal{W}^{++}$ (see (4) and (5)) we see that estimates (27) and (30) ensure the validity of the inclusion

$$\Psi_1(u-v) \leq_{\mathcal{W}^{++}} \Theta_{l,\phi} u - \Theta_{l,\phi} v \leq_{\mathcal{W}^{++}} \Psi_2(u-v)$$

for any function u and v with properties (24) from $\mathcal{W}$.

Now we determine K_1 and K_2 by the formulae

$$K_1 := \mathcal{W}^+, \qquad K_2 := \mathcal{W}^{++}. \tag{31}$$

By Lemma 4, the set K_1 forms a cone in the normed space $\mathcal{W}$, whereas K_2 is a normal and generating cone in the Banach space $\mathcal{W}$.

From Lemma 7 follows, that identity (22) is fulfilled and, therefore,

$$\Psi_1 + \Psi_2 = 2\Theta_{\frac{1}{2}(p+\xi), \frac{1}{2}(h+\gamma)}. \tag{32}$$

Taking into account (23), Lemma 6 guarantees the invertibility of the operators

$$\Theta_{p,h} \qquad \text{and} \qquad \Theta_{\frac{1}{2}(p+\xi), \frac{1}{2}(h+\gamma)}.$$

So, we have that $\Psi_1^{-1} = \Theta_{p,h}^{-1}$ and by (32), the relation

$$(\Psi_1 + \Psi_2)^{-1} = \frac{1}{2}\, \Theta^{-1}_{\frac{1}{2}(p+\xi), \frac{1}{2}(h+\gamma)}$$

is true.

Lemma 6 also ensures the positivity of the inverse operators in the sense that

$$\Theta_{p,h}^{-1}(\mathcal{W}^{++}) \subset \mathcal{W}^+,$$

$$\Theta^{-1}_{\frac{1}{2}(p+\xi), \frac{1}{2}(h+\gamma)}(\mathcal{W}^{++}) \subset \mathcal{W}^+$$

and, hence, inclusions (11), (12) hold.

Finally, in view of assumption (25), we see that relation (13) holds with F, Ψ_1, and Ψ_2 given by (26), (28), (29) with respect to the cones K_1 and K_2 defined by (31).

Applying Theorem 1, we establish the unique solvability of the boundary value problem (1), (2) for arbitrary $r \in L_1$.

We complete the proof of Theorem 2.

7. Unique solvability conditions for the linear FFDE.

Theorem 3. *Assume that there exist some linear operators* $p = (p_k)_{k=1}^n : \mathcal{W} \to L_1,\ \xi = (\xi_k)_{k=1}^n : \mathcal{W} \to L_1$ *which satisfy inclusions*

$$p \in \mathcal{S}_{a,\phi}, \quad \frac{1}{2}(p+\xi) \in \mathcal{S}_{a,\phi},$$

and such that for arbitrary function $z = (z_k)_{k=1}^n : I_a \to \mathbb{R}^n$ *from* $\mathcal{W}^+$ *the inequalities*

$$(\xi_k z)(t) \le (l_k z)(t) \le (p_k z)(t), \quad t \in I_a, \quad k = 1, 2, \ldots, n, \tag{33}$$

hold.

Then the non-local linear boundary-value problem (2) for FFDE (1) has a unique solution for an arbitrary function $r \in L_1$.

Proof. It is easy to see, that inequality (33) is simplest case of (25) with linear operator $l : \mathcal{W} \to L_1$, absolutely continuous function $u - v = z \in \mathcal{W}^+$. Then we can apply Theorem 2 with linear functionals $\gamma = h = \phi$.

Theorem 3 is proved.

8. Conclusions. Here we can conclude that using the Krasnoselskii Theorem (see Theorem 2) is a perspective method for obtaining the general conditions on the unique solvability of the (non)linear FFDE. It was shown that the set defined by (5) is normed and reproducing cone in the space absolutely continuous functions (see Lemma 4). Never the less, the question of proper establishing the exact conditions on the unique solvability of the (non)linear FFDE is still open.

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