UDC 517.9

ON THE AVERAGING PRINCIPLE FOR SEMILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN A BANACH SPACE

ПРО ПРИНЦИП УСЕРЕДНЕННЯ НАПІВЛІНІЙНИХ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З НЕСКІНЧЕННИМ ЗАПІЗНЕННЯМ У ПРОСТОРІ БАНАХА

L. Guedda, A. Ouardani

Dep. Math., Ibn Khaldoun Univ.,
P.O. Box 78, Tiaret, 14000, Algeria
e-mail: lahcene_guedda@yahoo.fr
ouardaniabderrahmane@yahoo.com

We establish averaging results for semilinear functional-differential equations with infinite delay on an abstract phase space of Banach space valued functions defined axiomatically, where the unbounded linear part generates a non-compact semigroup and the nonlinear part satisfies a condition with respect to the second argument which is weaker than the usual Lipschitz condition. As a preliminary result, using the technique of the theory of condensing maps, a theorem on the existence and uniqueness of mild solutions is established for such equations.

Встановлено результати щодо усереднення напівлінійних диференціально-функціональних рівнянь із нескінченним запізненням у абстрактному фазовому просторі функцій зі значеннями в банаховому просторі, визначених аксіоматично у випадку, коли необмежена лінійна частина породжує некомпактну півгрупу, а нелінійна частина задовольняє умову відносно другого аргумента більш слабку, ніж звичайна умова Ліпшіца. З використанням техніки ущільнюючих відображень отримано теорему існування та єдиності слабких розв'язків цих рівнянь.

1. Introduction. The averaging principle finds its origins in problems of celestial mechanics (see [1] and the introduction in [2] for the history of the theory of averaging). It consists in replacing the non autonomous right hand side of an equation with small parameter by its average in time. An important step in its history started with the works of Bogoliubov – Mitropolskii [3] and Krylov – Bogoliubov [4], where a rigorous justification of this principle was given for finite dimensional nonlinear systems in standard form. Later this principle was justified for several kinds of finite and infinite dimensional differential equations and functional differential equations with finite delay (see [1, 2, 5 – 13] and the references therein), and it became one of the most efficient methods for the study of systems with a small parameter.

We aim in this paper to give an approach which allows to establish the averaging principle for semilinear functional differential equations with infinite delay on an abstract phase space of a Banach space valued functions defined axiomatically, where the unbounded linear part generates a non-compact semi-group and the nonlinear part satisfies a condition with respect to the second argument which is weaker than the usual Lipschitz condition.

Let us first explain our approach. Let E be a Banach space. Let us consider the following semilinear functional differential equations with infinite delay in E, and with a small positive parameter ε, of the (normal) form

$$\begin{cases} z'(\tau) = \varepsilon \left[A\, z(\tau) + f(\tau, z_\tau)\right], & \tau \geq 0, \\ z_0 = \psi, \end{cases} \tag{1.1}$$

where $A \colon D(A) \subset E \to E$, is a linear operator, $f : [0, +\infty[\times \mathcal{B} \to E$, is a given function and $\psi \in \mathcal{B}$, where $\mathcal{B}$ is a linear topological space of functions mapping $]-\infty, 0]$ into E. For any $z :]-\infty, +\infty[\to E$ and for any $\tau \geq 0$, the function z_τ is defined by, $z_\tau(\theta) = z(\tau + \theta)$, $-\infty < \theta \leq 0$. We are not interested here in specifying the meaning of a solution to (1.1), all that is required:

(1) if $w \colon]-\infty, +\infty[\to E$ is continuous on $[0, +\infty[$ and $w_0 \in \mathcal{B}$, then, for every $\tau \in [0, +\infty[$, we have $w_\tau \in \mathcal{B}$;

(2) If $z \colon]-\infty, +\infty[\to E$ is a solution to (1.1), then $z\,|_{[0,+\infty[}$ is continuous and z satisfies (1.1) in some sens.

Parallel to the problem (1.1), we consider the averaged problem:

$$\begin{cases} z'(\tau) = \varepsilon \big[Az(\tau) + f_0(z_\tau)\big], & \tau \geq 0, \\ z_0 = \psi, \end{cases} \tag{1.2}$$

where $f_0 \colon \mathcal{B} \to E$, is such that for all $u \in \mathcal{B}$

$$f_0(u) = \lim_{t \to +\infty} \frac{1}{t} \int_0^t f(s, u) ds. \tag{1.3}$$

Let $\mathfrak{D}(\psi)$ be a set defined by

$$\mathfrak{D}(\psi) = \{\hat{z} \colon [0, +\infty[\to E, \ \ \hat{z} \text{ is continuous, and } \ \hat{z}(0) = \psi(0)\}.$$

For every $\hat{z} \in \mathfrak{D}(\psi)$, define the function $\hat{z}[\psi] \colon]-\infty, +\infty[\to E$ by

$$\hat{z}[\psi](\tau) = \begin{cases} \psi(\tau), & -\infty < \tau \leq 0, \\ \hat{z}(\tau), & 0 \leq \tau < +\infty. \end{cases} \tag{1.4}$$

We have for every $\tau \geq 0$, $z[\psi]_\tau \in \mathcal{B}$, and for every $\theta \leq 0$,

$$\hat{z}[\psi]_\tau(\theta) = \begin{cases} \psi(\tau + \theta), & \text{if } \ -\infty < \theta \leq -\tau, \\ \hat{z}(\tau + \theta), & \text{if } \ -\tau \leq \theta \leq 0. \end{cases} \tag{1.5}$$

It is clear that, if z is a solution to the problem (1.1), then z has the form, $z = \hat{z}[\psi]$, for some $\hat{z} \in \mathfrak{D}(\psi)$ solution to the problem

$$\begin{cases} \hat{z}'(\tau) = \varepsilon \big[A\hat{z}(\tau) + f\left(\tau, \hat{z}[\psi]_\tau\right)\big], & \tau \geq 0, \\ \hat{z}[\psi]_0 = \psi. \end{cases} \tag{1.6}$$

It results that the problem (1.1) can be written equivalently as (1.6). Notice that, the expression $\hat{z}[\psi]_0 = \psi,$ means that $\hat{z}(0) = \psi(0).$

Now, suppose that the function ψ satisfies

$$\psi\left(\frac{\cdot}{\varepsilon}\right) \in \mathcal{B} \quad \text{for every} \quad \varepsilon > 0. \tag{1.7}$$

In the problem (1.6), let us consider the change of variable

$$\begin{cases} \theta = \dfrac{\theta'}{\varepsilon}, & \tau = \dfrac{t}{\varepsilon}, \quad \theta \le 0, \quad \tau \ge 0, \\ \psi\left(\dfrac{\theta'}{\varepsilon}\right) = \varphi(\theta'), & \hat{z}\left(\dfrac{t}{\varepsilon}\right) = y(t). \end{cases} \tag{1.8}$$

It is clear that for every $\varepsilon > 0,$ the quantity $\theta' = \theta\varepsilon$ varies in $]-\infty, 0]$ when θ varies in $]-\infty, 0].$ Further, by (1.7) and (1.8) the function $\varphi \in \mathcal{B}$ and $\hat{z}(0) = \psi(0) = y(0) = \varphi(0).$ We claim that,

$$\hat{z}[\psi]_\tau(\theta) = y[\varphi]_t(\theta').$$

Indeed, we have

$$\hat{z}[\psi]_\tau(\theta) = \hat{z}[\psi]_{t/\varepsilon}(\theta) =$$

$$= \begin{cases} \psi(\theta + t/\varepsilon), & \text{if} \quad -\infty < \theta \le -t/\varepsilon, \\ \hat{z}(\theta + t/\varepsilon), & \text{if} \quad -t/\varepsilon \le \theta \le 0, \end{cases} =$$

$$= \begin{cases} \psi\left(\dfrac{1}{\varepsilon}(\theta\varepsilon + t)\right), & \text{if} \quad -\infty < \theta\varepsilon \le -t, \\ \hat{z}\left(\dfrac{1}{\varepsilon}(\theta\varepsilon + t)\right), & \text{if} \quad -t \le \theta\varepsilon \le 0, \end{cases} =$$

$$= \begin{cases} \varphi(\theta\varepsilon + t), & \text{if} \quad -\infty < \theta\varepsilon \le -t, \\ y(\theta\varepsilon + t), & \text{if} \quad -t \le \theta\varepsilon \le 0, \end{cases} =$$

$$= \begin{cases} \varphi(\theta' + t), & \text{if} \quad -\infty < \theta' \le -t, \\ y(\theta' + t), & \text{if} \quad -t \le \theta' \le 0, \end{cases} = y[\varphi]_t(\theta').$$

This proves that our claim is true. Thus, under the change of variable given by (1.8), Problem (1.6) (consequently Problem (1.1)) takes the form

$$\begin{cases} y'(t) = A\,y(t) + f\left(\dfrac{t}{\varepsilon}, y[\varphi]_t\right), \\ y[\varphi]_0 = \varphi. \end{cases} \tag{1.9}$$

Now, setting in (1.9), $x(t) = y[\varphi](t),\ t \in]-\infty, +\infty[,$ we deduce that the problem (1.1) can be written equivalently as:

$$\begin{cases} x'(t) = A\,x(t) + f\left(\dfrac{t}{\varepsilon}, x_t\right), \quad t \ge 0, \\ x_0 = \varphi. \end{cases} \tag{1.10}$$

Similarly, under the condition (1.7), the problem (1.2) can be written equivalently as

$$\begin{cases} x'(t) = Ax(t) + f_0(x_t), \quad t \geq 0, \\ x_0 = \varphi. \end{cases}$$

Therefore, we conclude that under the additional condition (1.7) on the phase space $\mathcal{B}$, to establish the averaging principle for the problem (1.1) in the case of finite time interval one has only to show that for any fixed time $T > 0$, the (unique) solution to the problem

$$\begin{cases} x'(t) = A\,x(t) + f\big(\frac{t}{\varepsilon}, x_t\big), \quad t \in [0, T], \\ x_0 = \varphi, \end{cases}$$

is approximated (in some sens) by the (unique) solution of the problem

$$\begin{cases} x'(t) = Ax(t) + f_0(x_t), \quad t \in [0, T], \\ x_0 = \varphi, \end{cases}$$

as $\varepsilon \to 0^+$.

If we consider the problem (1.1) but with finite delay, where the initial condition is written as $z_0 = \psi$, with $\psi \in C([-r, 0], E)$, for some $0 < r < \infty$, then using the same reasoning as above and the change of variable (1.8), we can define equivalently a problem similar to the problem (1.10) but with the initial condition $x(\theta) = \varphi(\theta)$, $\theta \in [-\varepsilon r, 0]$, that is, the effects of the delay is negligible and approaches zero in ε, as a consequence, the function $x(\theta + t)$, $\theta \in [-\varepsilon r, 0]$, can be approximated by $x(t)$ as ε is small enough. This fact was used as a basic idea in many works (see [10] (Section 1) for more details). In our case such approach can not be used, because when θ varies in $]-\infty, 0]$, the quantity $\theta\varepsilon$ varies in $]-\infty, 0]$, for every $\varepsilon > 0$. To guarantee that the problem (1.1) and (1.10) are equivalent we have added a condition given by (1.7). Of course, such a condition is not satisfied for any choice of a phase space $\mathcal{B}$ introduced by Hale and Kato [14], as shows this simple example:

For $\gamma > 0$, take $\mathcal{B} = C_\gamma$, where C_γ is the space of continuous functions $\psi\colon]-\infty, 0] \to \mathbb{R}$ such that $e^{\gamma\theta}\psi(\theta)$ has a limit in $\mathbb{R}$ as $\theta \to -\infty$, endowed with the norm,

$$\|\psi\|_{\mathcal{B}} = \sup\left\{ e^{\gamma\theta}|\psi(\theta)| : \theta \in]-\infty, 0] \right\}.$$

It is clear that the function $\psi(\theta) = e^{-\gamma\theta}$ belongs to $\mathcal{B}$, but for every $0 < \varepsilon < 1$,

$$\left\|\psi\left(\frac{\cdot}{\varepsilon}\right)\right\|_{\mathcal{B}} = \sup_{\theta \in]-\infty, 0]} e^{\gamma\theta} e^{-\gamma(\theta/\varepsilon)} = \sup_{\theta \in]-\infty, 0]} e^{\gamma\theta(1 - 1/\varepsilon)} = +\infty.$$

The paper is organized as follows. In Section 2 we recall some necessary preliminaries. In Section 3 we establish a theorem (Theorem 2) on the existence and uniqueness of mild solutions for semilinear functional differential equations in a Banach space with infinite delay on an abstract phase space $\mathcal{B}$, where the unbounded linear part generates A a non-compact semi-group and the nonlinear part f, satisfies a condition with respect to the second argument which is weaker than the usual Lipschitz condition.

In Section 4, we consider a semi-linear functional differential equation in E with a small positive parameter ε, of the form

$$\begin{cases} x'(t) = A\,x(t) + f\left(\dfrac{t}{\varepsilon}, x_t\right), & t \in [0,T], \\ x_0 = \varphi. \end{cases} \tag{P_ε}$$

Parallel to the problem (P$_\varepsilon$), $\varepsilon > 0$, we consider the averaged problem:

$$\begin{cases} x'(t) = Ax(t) + f_0(x_t), & t \in [0,T], \\ x_0 = \varphi, \end{cases} \tag{P_0}$$

where A generates a strongly continuous semigroup $(S(t))_{t\geq 0}$, and the function $f_0 : \mathcal{B} \to E$, is such that, for all $u \in \mathcal{B}$ and $t_1,\, t_2 \in [0,T]$ with $0 \leq t_1 \leq t_2 \leq t_1 + \Delta_0$, for some $\Delta_0 > 0$,

$$\lim_{\varepsilon \to 0^+} \int\limits_{t_1}^{t_2} S(t_2 - \theta)\left[f\left(\frac{\theta}{\varepsilon}, u\right) - f_0(u)\right] d\theta = 0. \tag{1.11}$$

By strengthening the conditions on f and, adding axioms on the phase space $\mathcal{B}$, we establish averaging result for the problem $(P_\varepsilon)_{\varepsilon>0}$ (Theorem 3). Condition (1.11) is inspired from [15].

(3) Finally, in Section 5 we replace (1.11) by a more natural hypothesis, then we establish the averaging principle in the traditional form (Theorem 4).

The notion of measure of non-compactness is used only in Section 3, where an existence result is established. For the proofs of our averaging results, we make direct estimations by generalizing the approach given in [15] to our case.

2. Preliminaries. Let $\mathcal{E}$ be a Banach space and $(\mathcal{Y}, \leq)$ a partially ordered set. Denote by $\mathcal{P}(\mathcal{E})$ the collection of all nonempty bounded subsets of $\mathcal{E}$.

A map $\Psi : \mathcal{P}(\mathcal{E}) \to \mathcal{Y}$ is called a measure of non-compactness in $\mathcal{E}$ if

$$\Psi(\Omega) = \Psi(\overline{co}\,\Omega)$$

for every $\Omega \in \mathcal{P}(\mathcal{E})$, where $\overline{co}\,\Omega$ denotes the closed convex hull of Ω.

The measure Ψ is called

(i) nonsingular if for every $a \in \mathcal{E},\ \Omega \in \mathcal{P}(\mathcal{E}),\ \Psi(\{a\} \cup \Omega) = \Psi(\Omega)$;

(ii) monotone, if $\Omega_0,\ \Omega_1 \in \mathcal{P}(\mathcal{E})$ and $\Omega_0 \subseteq \Omega_1$ imply $\Psi(\Omega_0) \leq \Psi(\Omega_1)$.

If $\mathcal{Y}$ is a cone in a Banach space, we say that

(iii) Ψ is regular if $\Psi(\Omega) = 0$ is equivalent to the relative compactness of the set Ω.

One of most important example of a measure of non-compactness possessing all these properties is the Hausdorff measure of non-compactness, defined by

$$\chi(\Omega) = \inf\{\epsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net in } \mathcal{E}\}.$$

Recall that, in a metric space $\mathcal{M}$, a finite ϵ-net of a subset Ω is a finite set (x_i) of points of $\mathcal{M}$ such that $\Omega \subset \cup_i B_i$, where B_i denotes the open ball of radius ϵ centered on x_i.

Let Ψ be a measure of non-compactness in $\mathcal{E}$. A multimap $G\colon Z \to \mathcal{E}$, where $Z \subset \mathcal{E}$ is a closed subset, is called Ψ-condensing if for every bounded set $\Omega \subset Z$, the relation $\Psi(G(\Omega)) \geq \geq \Psi(\Omega)$ implies the relative compactness of Ω. For more details, see, e.g., [8, 16, 17].

For the existence result, we will need the following fixed point theorem (see [16] (Theorem 1.5.11) and its generalization in the Subsection 1.5.12).

Theorem 1. *If $\mathcal{G}$ is a closed convex subset of a Banach space $\mathcal{E}$, and $\Gamma\colon \mathcal{G} \to \mathcal{G}$ is a continuous Ψ-condensing map, where Ψ is a nonsingular measure of non-compactness in $\mathcal{E}$, then Γ has at least one fixed point.*

3. Existence results. Let σ be a real number and $T > 0$ be a fixed time. By $C([\sigma, \sigma+T];\, E)$ we denote the space of continuous functions defined on $[\sigma, \sigma+T]$ with values in a Banach space $(E, \|.\|)$, endowed with the uniform convergence norm. For any function $z : \,]-\infty, \sigma+T] \to E$ and for every $t \in [\sigma, \sigma+T]$, z_t represents the function from $]-\infty, 0]$ into E defined by $z_t(\theta) = z(t+\theta)$, $-\infty < \theta \leq 0$. Let $\mathcal{B}$ be a linear topological space of functions mapping $]-\infty, 0]$ into E endowed with a seminorm $\|.\|_{\mathcal{B}}$ and, satisfying the following axioms:

if $z : \,]-\infty, \sigma+T] \to E$ is continuous on $[\sigma, \sigma+T]$ and $z_\sigma \in \mathcal{B}$, then, for every $t \in [\sigma, \sigma+T]$, we have

(B_1) $z_t \in \mathcal{B}$;

(B_2) $\|z_t\|_{\mathcal{B}} \leq K(t-\sigma)\sup_{\sigma \leq s \leq t}\|z(s)\| + N(t-\sigma)\,\|z_\sigma\|_{\mathcal{B}}$, where $K, N : [0, +\infty[\longrightarrow [0, +\infty[$, are independent of z, K is positive and continuous and, N is locally bounded;

(B_3) the function $t \mapsto z_t$ is continuous.

Such a space $\mathcal{B}$ was introduced by Hale and Kato [14] and has been considered as a phase space in the theory of functional differential equations with infinite delay.

Let us denote by the symbol $\mathcal{C}(]-\infty, \sigma+T]; E)$ the linear topological space consisting of functions $z\colon]-\infty, \sigma+T] \to E$ such that $z_\sigma \in \mathcal{B}$ and the restriction $z\,|_{[\sigma,\, \sigma+T]}$ is continuous, endowed with a seminorm

$$\|z\|_{\mathcal{C}} = \|z_\sigma\|_{\mathcal{B}} + \left\|z\,|_{[\sigma,\,\sigma+T]}\right\|_{C([\sigma,\sigma+T];E)}.$$

Let us consider the following semilinear functional differential equation in E of the form

$$\begin{cases} x'(t) = Ax(t) + f(t, x_t), & t \in [\sigma, \sigma+T], \\ x_\sigma = \varphi, \end{cases} \tag{3.1}$$

where

(A) A is the generator of a $\mathcal{C}_0$ semigroup $(S(t))_{t\geq 0}$ on E.

The function f is acting from $[\sigma, \sigma+T] \times \mathcal{B}$ to E and satisfies

(f_1) for all $u \in \mathcal{B}$, the mapping $t \to f(t, u)$ is measurable;

(f_2) for all $(u, v) \in \mathcal{B} \times \mathcal{B}$,

$$\|f(t,u) - f(t,v)\| \leq L(t, \|u - v\|_{\mathcal{B}}), \quad \text{a.e.} \quad t \in [\sigma, \sigma+T],$$

where $L\colon [\sigma, \sigma+T] \times [0, +\infty[\to [0, +\infty[$ is a given mapping such that

(i) $L(\cdot, \theta)$ is locally integrable for each $\theta \in [0, +\infty[$ and for a.e. $t \in [\sigma, \sigma+T]$, $L(t, \cdot)$ is continuous, monotone nondecreasing and, $L(t, 0) = 0$;

(ii) for every nonnegative continuous function $h\colon [\sigma,\sigma+T]\to[0,+\infty[$ and for every constant $\zeta,$ the following implication holds true:

$$\left[\forall t\in[\sigma,\sigma+T],\ h(t)\le\zeta\int\limits_{\sigma}^{t}L\Big(s,h(s)\Big)\,ds\right]\Rightarrow h\equiv 0; \tag{3.2}$$

(f_3) there exists a function $\alpha\in L^1\big([\sigma,\sigma+T],\mathbb{R}^+\big)$ such that, for all $u\in\mathcal{B},$

$$\|f(t,u)\|\le\alpha(t)(1+\|u\|_{\mathcal{B}})\quad\text{a.e.}\quad t\in[\sigma,\sigma+T].$$

Remark 1. Concrete examples of such a function L can be found, e.g., [18–20].

Remark 2. Let $L\colon [a,b]\times[0,+\infty[\to[0,+\infty[$ be a mapping satisfying Hypotheses (i) and (ii) of (f_2). According to [19] (Lemma 2.2), if a nonnegative monotone nondecreasing function $h\colon [a,b]\to[0,+\infty[$ such that $h(a)=0$ satisfies (3.2) for some constant $\zeta>0,$ then $h(t)=0$ for all $t\in[a,b]$.

Definition 1. *We say that* $z\in\mathcal{C}(\,]-\infty,\sigma+T];E)$ *is a* mild solution *to the problem (3.1) if* $z_\sigma=\varphi$ *and,*

$$z(t)=S(t-\sigma)\varphi(0)+\int\limits_{\sigma}^{t}S(t-s)f(s,z_s)\,ds\quad\textit{for}\quad\sigma\le t\le\sigma+T.$$

We can now state our preliminary result.

Theorem 2. *Assume that the phase space* $\mathcal{B}$ *satisfies the axioms* (B_1)–(B_3)*. Under Hypotheses (A) and* (f_1)–(f_3) *the problem (3.1) has a unique mild solution in* $\mathcal{C}(\,]-\infty,\sigma+T];E)$.

Remark 3. Theorem 2 is a deterministic version of [21] (Theorem 3.2), its proof may be deduced from the latter's proof, except some modifications. As this is not the same context (the conditions on the phase space $\mathcal{B}$ are not exactly the same) and for easier reading, we prefer to give the proof. Instead of observing the solution as a limit of a sequence of approximating solutions constructed via Tonelli's scheme (the approach that was used in [21]), here, for the proof of Theorem 2, the mild solutions are observed as a fixed points of an integral operator which is condensing with respect to a measure of non-compactness with good properties. Then, to end the proof, a fixed point theorem for such class of operators (Theorem 1) is invoked.

Another reason for which we prefer to give the proof of Theorem 2 is due to the fact that in the literature, we did not find existence results for the problem (3.1) with a general non Lipschitz condition given by (f_2).

Before giving the proof of Theorem 2, we prove some auxiliary results. We will use the same notation as in [22].

Let $t\in[\sigma,\sigma+T]$. In the space $C([\sigma,t],E)$, let us define the set

$$\mathfrak{D}(\varphi,t)=\Big\{x\in C([\sigma,t];E),\ x(\sigma)=\varphi(0)\Big\}. \tag{3.3}$$

It is clear that $\mathfrak{D}(\varphi,t)$ is a closed convex subset in $C([\sigma,t];E)$.

For every $x \in \mathfrak{D}(\varphi, \sigma + T)$, let us define the function $x[\varphi] \in \mathcal{C}(\,]-\infty, \sigma + T]; E)$ by

$$x[\varphi](t) = \begin{cases} \varphi(t - \sigma), & -\infty < t < \sigma, \\ x(t), & \sigma \le t \le \sigma + T. \end{cases} \tag{3.4}$$

Then

$$x[\varphi]_t(\theta) = \begin{cases} \varphi(t - \sigma + \theta), & \text{if} \quad -\infty < \theta < \sigma - t, \\ x(t + \theta), & \text{if} \quad \sigma - t \le \theta \le 0. \end{cases} \tag{3.5}$$

The function $x[\varphi]\,|_{[\sigma,\sigma+T]} = x(\cdot)$ is continuous and $x[\varphi]_\sigma = \varphi$, hence by Axiom (B$_1$), $x[\varphi]_t \in \mathcal{B}$ for all $t \in [\sigma, \sigma + T]$.

Let us denote by Φ the mapping which, with every element $x \in \mathfrak{D}(\varphi, \sigma + T)$ associates the element $\Phi(x) \in \mathfrak{D}(\varphi, \sigma + T)$, given by

$$S(t - \sigma)\varphi(0) + \int_\sigma^t S(t - s) f(s, x[\varphi]_s)\, ds, \quad \sigma \le t \le \sigma + T.$$

Remark 4. It is clear that each mild solution to (3.1) has the form $x[\varphi](.)$, for some a fixed point $x \in \mathfrak{D}(\varphi, \sigma + T)$ of Φ. Thus, for the proof of Theorem 2, it is enough to prove that the mapping Φ has a unique fixed point in $\mathfrak{D}(\varphi, \sigma + T)$.

Remark 5. Since $(S(t))_{t \ge 0}$ is a $\mathcal{C}_0$-semigroup, there exists, for each $t \in [0, T]$, a constant $M_t > 0$ such that

$$\sup_{0 \le s \le t} \|S(s)\| \le M_t. \tag{3.6}$$

See, e.g., [23] (Theorem 2.2).

Lemma 1. *Let $L \colon [\sigma, \sigma + T] \times [0, +\infty[\, \to [0, +\infty[$ be a mapping satisfying Hypotheses (i)–(ii) of (f$_2$). Let $(v_n)_n$ be a sequence of positive number such that $v_n \to 0$ as $n \to \infty$. Then for any $\mu \ge 0$, we have*

$$\int_\sigma^T L(s, \mu\, v_n)\, ds \to 0, \quad \textit{as} \quad n \to \infty.$$

The result follows immediately from the Lebesgue dominated convergence theorem.

Lemma 2. *The map $\Phi \colon \mathfrak{D}(\varphi, \sigma + T) \to \mathfrak{D}(\varphi, \sigma + T)$ is continuous.*

Proof. Let

$$\lambda_K = \sup_{0 \le s \le T} K(s), \tag{3.7}$$

where the function K is introduced in Axiom (B$_2$). It is clear that $\lambda_k > 0$.

Let $(x_n)_n$ be a sequence of $\mathfrak{D}(\varphi, \sigma + T)$, such that $x_n \to x$ as $n \to \infty$. Using Axiom (B$_2$) and the fact that the function $L(s, \cdot)$ is monotone nondecreasing a.e $s \in [\sigma, \sigma + T]$, we get

$$\|\Phi(x)(t) - \Phi(x_n)(t)\| \le M_t \int_\sigma^t \|f(s, x[\varphi]_s) - f(s, x_n[\varphi]_s)\|\, ds \le$$

$$\leq M_t \int_\sigma^t L\Big(s, K(s-\sigma) \sup_{\sigma\leq\tau\leq s} \|x(\tau) - x_n(\tau)\|\Big) ds \leq$$

$$\leq M_T \int_\sigma^T L\big(s, \lambda_K \|x - x_n\|_{C([\sigma,\sigma+T],E)}\big) ds,$$

where M_T is from (3.6). It remains to apply Lemma 1.

Let Λ be a subset of $C([\sigma, \sigma+T]; E)$. For any $t \in [\sigma, \sigma+T]$, by $\Lambda_{|t}$ we denote the restrictions to $[\sigma, t]$ of elements of Λ. It is clear that $\Lambda_{|t}$ is a subset of $C([\sigma, t]; E)$.

Let $\mathcal{T}[\sigma, \sigma+T]$ denote the partially ordered linear space of all real monotone nondecreasing functions defined on $[\sigma, \sigma+T]$. Let us consider a measure of non-compactness $\Psi_{C([\sigma,\sigma+T];E)}$ defined on bounded subsets of $C([\sigma, \sigma+T]; E)$ and with values in $\mathcal{T}[\sigma, \sigma+T]$, given by

$$\Psi_{C([\sigma,\sigma+T];E)}(\Lambda)(t) = \inf\big\{\epsilon > 0;\ \Lambda_{|t} \text{ has a finite } \epsilon\text{-net in } C([\sigma, t]; E)\big\}.$$

Now on bounded subsets of $\mathfrak{D}(\varphi, \sigma+T)$, let us define a new measure of non-compactness $\Psi_{\mathfrak{D}}$ with values in $\mathcal{T}[\sigma, \sigma+T]$, given by:

$$\Psi_{\mathfrak{D}}(\Lambda)(t) = \inf\big\{\epsilon > 0;\ \Lambda_{|t} \text{ has a finite } \epsilon\text{- net in } \mathfrak{D}(\varphi, t)\big\}.$$

Remark 6. Note that for each $t \in [\sigma, \sigma+T]$, the set $\mathfrak{D}(\varphi, t)$ endowed with the metric $d_{\mathfrak{D}(\varphi, t)}$, given by,

$$d_{\mathfrak{D}(\varphi, t)}(x, y) = \|x - y\|_{C([\sigma,t];E)},$$

is a (complete) metric space.

Remark 7. It is obvious that a subset Λ of $C([\sigma, \sigma+T]; E)$ is relatively compact if and only if

$$\Psi_{C([\sigma,\sigma+T];E)}(\Lambda)(T) = 0.$$

Remark 8. One can easily show that the restriction of $\Psi_{C([\sigma,\sigma+T];E)}$ on bounded subsets of $\mathfrak{D}(\varphi, \sigma+T)$ satisfies for each $t \in [\sigma, \sigma+T]$ and for any bounded set $\Lambda \subset \mathfrak{D}(\varphi, \sigma+T)$, the relation

$$\Psi_{C([\sigma,\sigma+T];E)}(\Lambda)(t) \leq \Psi_{\mathfrak{D}}(\Lambda)(t) \leq 2\Psi_{C([\sigma,\sigma+T];E)}(\Lambda)(t). \tag{3.8}$$

Lemma 3. *Let Λ be a bounded subset of $\mathfrak{D}(\varphi, \sigma+T)$. Assume that Hypotheses (A), (f_1) – (f_3) and (B_1) – (B_3) are satisfied. Then for every $t \in [\sigma, \sigma+T]$, we have*

$$\Psi_{\mathfrak{D}}(\Phi \circ \Lambda)(t) \leq M_T \int_\sigma^t L(s, K(s)\, \Psi_{\mathfrak{D}}(\Lambda)(s)) ds,$$

where M_T is from (3.6) and $K(\cdot)$ from Axiom (B_2).

Proof. Lemma 3 is a deterministic version of [21] (Lemma 3.4) (see Remark 3).

Let Λ be a bounded subset of $\mathfrak{D}(\varphi, \sigma+T)$.

Let $\epsilon > 0$. As the function $t \mapsto \Psi_{\mathfrak{D}}(\Lambda)(t)$ is nondecreasing and bounded, it admits at most a finite number of points $\sigma \leq t_1 \leq \ldots \leq t_n \leq \sigma + T$ for which

$$\left|\Psi_{\mathfrak{D}}(\Lambda)(t_i + 0) - \Psi_{\mathfrak{D}}(\Lambda)(t_i - 0)\right| \geq \epsilon, \quad i = 1, \ldots, n.$$

Remove these points with their disjoint δ_1-neighborhoods from the segment $[\sigma, \sigma + T]$. If $t_1 \neq \sigma$, remove also the segment $[\sigma, \sigma + \delta_1[$ from the segment $[\sigma, \sigma + T]$. Using points $\beta_j,\ j = 1, \ldots, m$, divide the remaining part into intervals on which the increment of the function $\Psi_{\mathfrak{D}}(\Lambda)(\cdot)$ is smaller than ϵ, i.e.,

$$\sup_{s,t \in [\beta_{j-1}, \beta_j]} \left|\Psi_{\mathfrak{D}}(\Lambda)(s) - \Psi_{\mathfrak{D}}(\Lambda)(t)\right| < \epsilon, \quad j = 2, \ldots, m. \tag{3.9}$$

Now, in order to be able to construct a net of continuous functions, surround the points β_j, $j = 1, \ldots, m$, by δ_2-neighborhoods and consider the family $\mathcal{R}$ in $\mathfrak{D}(\varphi, \sigma + T)$ obtained by taking all continuous functions which coincide on each $[\beta_{j-1} + \delta_2, \beta_j - \delta_2],\ 2 \leq j \leq m$, with some element of a finite $(\Psi_{\mathfrak{D}}(\Lambda)(\beta_j) + \epsilon)$-net of Λ in $\mathfrak{D}(\varphi, \beta_j)$ and which have affine trajectories on the complementary segments. By construction, the elements of $\mathcal{R}$ are affine on $[\sigma, \sigma + \delta_1]$, to stay in $\mathfrak{D}(\varphi, \sigma + T)$, we choose them equal to $\varphi(0)$ at $t = \sigma$.

The set $\mathcal{R}$ in $\mathfrak{D}(\varphi, \sigma + T)$ is finite. Denote by $z_k,\ k = 1, \ldots, p$ its elements. For each $j = 1, \ldots, m$, let $\left(z_l^j\right)_{1 \leq l \leq q_j}$ be a $(\Psi_{\mathfrak{D}}(\Lambda)(\beta_j) + \epsilon)$-net of $\Lambda\,|_{\beta_j}$ in $\mathfrak{D}(\varphi, \beta_j)$. Consider a fixed $x \in \Lambda$. Then, one can find an element z_l^j such that,

$$d_{\mathfrak{D}(\varphi,\beta_j)}\big(x, z_l^j\big) := \left\|x - z_l^j\right\|_{C([\sigma,\beta_j];\,E)} \leq \Psi_{\mathfrak{D}}(\Lambda)(\beta_j) + \epsilon. \tag{3.10}$$

By construction, for each $j \in \{1, \ldots, m\}$ and fixed $l \in \{1, \ldots, q_j\}$, one can find an element z_k of $\mathcal{R}$, such that $z_l^j\,|_{[\beta_{j-1}+\delta_2, \beta_j - \delta_2]} = z_k\,|_{[\beta_{j-1}+\delta_2, \beta_j - \delta_2]}$. From (3.10) and (3.9), it results that for every $t \in [\beta_{j-1} + \delta_2, \beta_j - \delta_2]$

$$\|x(t) - z_k(t)\| = \left\|x(t) - z_l^j(t)\right\| \leq d_{\mathfrak{D}(\varphi,\beta_j)}\big(x, z_l^j\big) \leq$$

$$\leq \Psi_{\mathfrak{D}}(\Lambda)(\beta_j) + \epsilon \leq \Psi_{\mathfrak{D}}(\Lambda)(t) + 2\epsilon. \tag{3.11}$$

Because the function $\Psi_{\mathfrak{D}}(\Lambda)(\cdot)$ is nondecreasing, for every $t \in [\beta_{j-1} + \delta_2, \beta_j - \delta_2]$, we get

$$\sup_{\beta_{j-1}+\delta_2 \leq s \leq t} \big\|x(s) - z_k(s)\big\| \leq \Psi_{\mathfrak{D}}(\Lambda)(t) + 2\epsilon. \tag{3.12}$$

By (B_2), we have for every $t \in [\sigma, \sigma + T]$,

$$\sup_{\sigma \leq s \leq t} \big\|\Phi(x)(s) - \Phi(z_k)(s)\big\| \leq$$

$$\leq \sup_{\sigma \leq s \leq t} \left\| \int_{\sigma}^{s} (s - \tau)\left(f(\tau, x[\varphi]_\tau) - f(\tau, z_k[\varphi]_\tau)\right) d\tau \right\| \leq$$

$$\leq M_t \int_{\sigma}^{t} \big\|f(\tau, x[\varphi]_\tau) - f(\tau, z_k[\varphi]_\tau)\big\| d\tau \leq$$

$$\leq M_t \int\limits_{\sigma}^{t} L\Bigg(\tau, K(\tau-\sigma) \sup_{\sigma\leq\theta\leq\tau} \big\|x(\theta) - z_k(\theta)\big\|\Bigg) d\tau$$

where M_t is from (3.6). Let us denote by

$$J(t) = [\sigma, t] \cap \Big([\sigma, \sigma+\delta_1[\cup (\cup_{1\leq i\leq n}]t_i - \delta_1, t_i + \delta_1[) \cup (\cup_{1\leq j\leq m}]\beta_j - \delta_2, \beta_j + \delta_2[)\Big),$$

$$I(t) = [\sigma, t] \setminus J(t).$$

Because the sets Λ and $\mathcal{R}$ are bounded in $\mathfrak{D}(\varphi, \sigma+T)$, using the last estimation and taking δ_1 and δ_2 small enough, one can ensure that

$$d_{\mathfrak{D}(\varphi,\, t)}\left(\Phi(x), \Phi(z)\right) \leq M_T \left(\int\limits_{I(t)} L(s, K(s) \sup_{\sigma\leq\tau\leq s} \|x(\tau) - z(\tau)\|)\, ds + \epsilon \right).$$

Now, using (3.12), we deduce

$$d_{\mathfrak{D}(\varphi,\, t)}\left(\Phi(x), \Phi(z)\right) \leq M_T \left(\int\limits_{\sigma}^{t} L\left(s, K(s)\left(\Psi(\Lambda)(s) + 2\epsilon\right)\right) ds + \epsilon \right). \tag{3.13}$$

The result follows from the arbitrariness in the choice of ϵ and x.

Corollary 1. *The map Φ is $\Psi_{C([\sigma,\sigma+T];E)}$-condensing on bounded subsets of $\mathfrak{D}(\varphi, \sigma+T)$.*

Indeed, Let Λ be a bounded subset of $\mathfrak{D}(\varphi, \sigma+T)$ such that

$$\Psi_{C([\sigma,\sigma+T];E)}(\Phi\circ\Lambda) \geq \Psi_{C([\sigma,\sigma+T];E)}(\Lambda). \tag{3.14}$$

By Lemma 3, (3.8), (3.14), and the fact that $L(t, \cdot)$ monotone nondecreasing, we have for every $t \in [\sigma, \sigma+T]$,

$$\frac{1}{2}\,\Psi_{\mathfrak{D}}(\Lambda)(t) \leq \Psi_{C([\sigma,\sigma+T];E)}(\Lambda)(t) \leq$$

$$\leq \Psi_{C([\sigma,\sigma+T];E)}(\Phi\circ\Lambda)(t) \leq \Psi_{\mathfrak{D}}(\Phi\circ\Lambda)(t) \leq$$

$$\leq M_T \int\limits_{\sigma}^{t} L\Big(s, K(s)\,\Psi_{\mathfrak{D}}(\Lambda)(s)\Big)\, ds.$$

Hence,

$$\lambda_K \Psi_{\mathfrak{D}}(\Lambda)(t) \leq 2\lambda_K M_T \int\limits_{\sigma}^{t} L(s, \lambda_K \Psi_{\mathfrak{D}}(\Lambda)(s))\, ds,$$

where λ_K is from (3.7). Because $\lambda_K > 0$, from Remark 2, it results that for every $t \in [\sigma, \sigma+T]$, $\Psi_{\mathfrak{D}}\,(\Lambda)(t) = 0$. As a consequence, for every $t \in [\sigma, \sigma+T]$, $\Psi_{C([\sigma,\sigma+T];E)}(\Lambda)(t) = 0$. Thus, Λ is relatively compact.

Proof of Theorem 2. Let us begin with the existence problem. According to Remark 4, is it enough to prove that Φ has a fixed point. In the space $C([\sigma,\sigma+T];E)$, let us consider an equivalent norm $\|\cdot\|_*$ given by the formula

$$\|y\|_* = \max_{t\in[\sigma,\sigma+T]} e^{-\mathcal{L}(t-\sigma)}\|y(t)\|,$$

where $\mathcal{L}>0$ is chosen so that

$$\max_{t\in[\sigma,\sigma+T]} M_T\lambda_K \int\limits_\sigma^t e^{-\mathcal{L}(t-s)}\alpha(s)ds \le q < 1, \tag{3.15}$$

where $\alpha(\cdot)$ is from (f_3). With this norm, let us consider the closed ball $\bar{B}_{\mathfrak{D}}(\bar{\varphi},r)\subset\mathfrak{D}(\varphi,\sigma+T)$ of the radius $r>0$ centered at the function $\bar{\varphi}\in\mathfrak{D}(\varphi,\sigma+T)$, where $\bar{\varphi}(t)=\varphi(0)$ for all $t\in[\sigma,\sigma+T]$, i.e.,

$$\bar{B}_{\mathfrak{D}}(\bar{\varphi},r) = \left\{x\in\mathfrak{D}(\varphi,\sigma+T)\colon \max_{t\in[\sigma,\sigma+T]} e^{-\mathcal{L}(t-\sigma)}\|x(t)-\varphi(0)\|\le r\right\},$$

Choose $r>0$ such that

$$r \ge \big[(M_T+2)\|\varphi(0)\|_E + M_T\|\alpha\|_{L^1}(1+\lambda_N\|\alpha\|_{\mathcal{B}})\big](1-q)^{-1}$$

where q is from (3.15) and,

$$\lambda_N = \sup_{0\le s\le T} N(s). \tag{3.16}$$

The last inequality implies

$$(M_T+2)\|\varphi(0)\|_E + M_T\|\alpha\|_{L^1}(1+\lambda_N\|\alpha\|_{\mathcal{B}}) + qr \le r, \tag{3.17}$$

Let us chow that Φ maps $\bar{B}_{\mathfrak{D}}(\bar{\varphi},r)$ into itself. Let $x\in\bar{B}_{\mathfrak{D}}(\bar{\varphi},r)$. For all $t\in[\sigma,\sigma+T]$, we have

$$e^{-\mathcal{L}(t-\sigma)}\|\Phi(x)(t)-\bar{\varphi}(t)\| = e^{-\mathcal{L}(t-\sigma)}\|\Phi(x)(t)-\varphi(0)\| \le$$

$$\le \|S(t-\sigma)\varphi(0)-\varphi(0)\| + e^{-\mathcal{L}(t-\sigma)}M_T\int\limits_\sigma^t \alpha(s)\,(1+\|x[\varphi]_s\|)ds \le$$

$$\le (M_T+1)\|\varphi(0)\|_E +$$

$$+\, e^{-\mathcal{L}(t-\sigma)}M_T\int\limits_\sigma^t \alpha(s)\left(1+\lambda_K\sup_{\theta\in[\sigma,s]}\|x(\theta)\| + \lambda_N\,\|\varphi\|_{\mathcal{B}}\right)ds \le$$

$$\le (M_T+1)\|\varphi(0)\|_E + M_T\|\alpha\|_{L^1}(1+\lambda_N\,\|\varphi\|_{\mathcal{B}}) +$$

$$+ M_T \lambda_K \int\limits_{\sigma}^{t} e^{-\mathcal{L}(t-\sigma)} e^{\mathcal{L}(s-\sigma)} \alpha(s) e^{-\mathcal{L}(s-\sigma)} \sup_{\theta\in[\sigma,s]} \|x(\theta)\| \, ds \le$$

$$\le (M_T+1)\|\varphi(0)\| + M_T \|\alpha\|_{L^1}(1+\lambda_N\|\varphi\|_{\mathcal{B}}) +$$

$$+ M_T \lambda_K \int\limits_{\sigma}^{t} e^{-\mathcal{L}(t-s)} \alpha(s) \sup_{\theta\in[\sigma,s]} e^{-\mathcal{L}(\theta-\sigma)} \left[\|x(\theta)-\varphi(0)\| + \|\varphi(0)\|_E\right] ds \le$$

$$\le (M_T+1)\|\varphi(0)\|_E + M_T\|\alpha\|_{L^1}(1+\lambda_N\|\varphi\|_{\mathcal{B}}) + qr + q\|\varphi(0)\|_E \le$$

$$\le (M_T+2)\|\varphi(0)\|_E + M_T\|\alpha\|_{L^1}(1+\lambda_N\|\varphi\|_{\mathcal{B}}) + qr \le r.$$

It results that the operator Φ maps $\bar{B}_{\mathfrak{D}}(\bar{\varphi}, r)$ into itself, moreover, according to Corollary 1, $\Phi\colon \bar{B}_{\mathfrak{D}}(\bar{\varphi}, r) \to \bar{B}_{\mathfrak{D}}(\bar{\varphi}, r)$ is $\Psi_{C([\sigma,\sigma+T];E)}$-condensing. From Theorem 1, it follows that Φ admits a fixed point.

It remain to show the uniqueness. Again, according to Remark 4 it is enough to prove that Φ has a unique fixed point. Suppose that Φ has two fixed points in $\mathfrak{D}(\varphi, \sigma+T)$, say x and y, then following the same line of calculations as above, we get for any $t \in [\sigma, \sigma+T]$,

$$\sup_{\sigma\le s\le t} \|x(s)-y(s)\| = \sup_{\sigma\le s\le t} \|\Phi(x)(s)-\Phi(y)(s)\| \le$$

$$\le M_T \int\limits_{\sigma}^{t} \|f(\tau, x[\varphi]_\tau) - f(\tau, y[\varphi]_\tau)\| \, d\tau \le$$

$$\le M_T \int\limits_{\sigma}^{t} L\left(\tau, K(\tau-\sigma) \sup_{\sigma\le\theta\le\tau} \|x(\theta)-y(\theta)\|\right) d\tau.$$

Consequently

$$\lambda_K \sup_{\sigma\le s\le t} \|x(s)-y(s)\| \le \lambda_K M_T \int\limits_{\sigma}^{t} L\left(\tau, \lambda_K \left(\sup_{\sigma\le\theta\le\tau} \|x(\theta)-y(\theta)\|\right)\right) d\tau.$$

where M_T is from (3.6) and, λ_K is from (3.7). Since $\lambda_K > 0$, by (f_2), it results that

$$\sup_{\sigma\le s\le t} \|x(s)-y(s)\| = 0, \quad \text{for any} \quad t \in [\sigma, \sigma+T].$$

Thus, the problem (3.1) has a unique mild solution.

4. Averaging result. Throughout this section, the phase space $\mathcal{B}$ is considered as a normed space satisfying Axioms (B_1) – (B_3).

As an example of such phase space $\mathcal{B}$, one can take the Banach space C_∞ [24], constituted of continuous functions $\psi\colon]-\infty, 0] \to E$, such that $\lim\limits_{\theta\to-\infty} \psi(\theta)$ exists, endowed with the sup-norm,

$$\|\psi\|_{C_\infty} = \sup\big\{\|\psi(\theta)\| \colon \theta \in]-\infty, 0]\big\}.$$

Let us consider a semilinear functional differential equation in E with a small positive parameter ε, of the form

$$\begin{cases} z'(t) = A\,z(t) + f\left(\dfrac{t}{\varepsilon}, z_t\right), \quad t \in [0,T], \\ z_0 = \varphi, \end{cases} \tag{P$_\varepsilon$}$$

Now parallel to the problem (P$_\varepsilon$), $\varepsilon > 0$, we consider the averaged problem

$$\begin{cases} z'(t) = A\,z(t) + f_0(z_t), \\ z_0 = \varphi. \end{cases} \tag{P$_0$}$$

Remark 9. It is clear that any $\psi \in C_\infty$, satisfies (1.7). Then, for the choice $\mathcal{B} = C_\infty$, the problems in the normal form

$$\begin{cases} w'(\tau) = \varepsilon\left[A\,w(\tau) + f(\tau, w_\tau)\right], \quad \tau \in \left[0, \dfrac{T}{\varepsilon}\right], \\ w_0 = \psi \end{cases} \tag{4.1}$$

and

$$\begin{cases} w'(t) = \varepsilon\big[A\,w(t) + f_0(w_t)\big], \\ w_0 = \psi \end{cases} \tag{4.2}$$

can be written equivalently as (P$_\varepsilon$) and (P$_0$) respectively (see Section 1).

Hypotheses. Suppose that the unbounded linear operator A satisfies the condition (A) and the function f is acting from $\mathbb{R}_+ \times \mathcal{B}$ to E.

Let us consider the following hypotheses:

($\mathtt{f}_1$) for all $u \in \mathcal{B}$, the mapping $t \to f(t, u)$ is measurable;

($\mathtt{f}_2$) for all $(u, v) \in \mathcal{B} \times \mathcal{B}$,

$$\|f(t,u) - f(t,v)\| \le \mathbb{L}(\|u - v\|_{\mathcal{B}}) \quad \text{a.e.} \quad t \in \mathbb{R}^+,$$

where $\mathbb{L}\colon [0, +\infty[\, \to [0, +\infty[$ is a given mapping such that:

(i) $\mathbb{L}(\cdot)$ is continuous, monotone nondecreasing and, $\mathbb{L}(0) = 0$;

(ii) for every nonnegative continuous mapping $h\colon [0,T] \to [0, +\infty[$ and for every constant ζ, the following implication holds true:

$$\left[(\forall t \in [0,T]) \quad h(t) \le \zeta \int\limits_0^t \mathbb{L}(h(s))\,ds\right] \Rightarrow h \equiv 0; \tag{4.3}$$

($\mathtt{f}_3$) there exists a constant $C > 0$, such that for all $u \in \mathcal{B}$

$$\|f(t,u)\| \le C(1 + \|u\|_{\mathcal{B}}) \quad \text{a.e.} \quad t \in \mathbb{R}^+.$$

$(\mathtt{Af})$ There exist $\Delta_0 > 0$ and a function $f_0 \colon \mathcal{B} \to E$ satisfying conditions similar to $(\mathtt{f}_2)$ and $(\mathtt{f}_3)$, and for all $u \in \mathcal{B}$ and $t_1, t_2 \in [0, T]$ with $0 \le t_1 \le t_2 \le t_1 + \Delta_0 \le T$,

$$\lim_{\varepsilon \to 0^+} \int_{t_1}^{t_2} S(t_2 - \theta)\left[f\left(\frac{\theta}{\varepsilon}, u\right) - f_0(u)\right] d\theta = 0. \tag{4.4}$$

Theorem 3. *Assume that the hypotheses (A),* $(\mathtt{f}_1) - (\mathtt{f}_3)$ *and* $(\mathtt{Af})$ *are satisfied. Then, the sequence* $\left(z^{\varepsilon}\right)_{\varepsilon > 0}$*, where* z^{ε} *is the mild solution to the problem (*P_{ε}*), converges to the mild solution* z^{∞} *of the problem (*P_0*) as* $\varepsilon \to 0^+$*.*

Remark 10. In Hypothesis $(\mathtt{Af})$ the expression f_0 satisfies a condition similar to $(\mathtt{f}_2)$ and $(\mathtt{f}_3)$, means

(1) for all $(u, v) \in \mathcal{B} \times \mathcal{B}$,

$$\|f_0(u) - f_0(v)\| \le \mathbb{L}(\|u - v\|_{\mathcal{B}});$$

(2) there exists a constant $C' > 0$, such that for all $u \in \mathcal{B}$

$$\|f(u)\| \le C'(1 + \|u\|_{\mathcal{B}}).$$

It is clear that f_0 is a continuous function.

Remark 11. Hypothesis $(\mathtt{Af})$ is inspired from [15].

Proof of Theorem 3. According to Theorem 2, for each $\varepsilon > 0$ there exists a unique mild solution z^{ε} to the problem (P$_{\varepsilon}$), and there exists a unique mild solution z^{∞} to the problem (P$_0$). Moreover, for each $\varepsilon > 0,\ z^{\varepsilon} = x^{\varepsilon}[\varphi]$, where x^{ε} is the unique fixed point of the operator, $\Phi_{\varepsilon}: \mathfrak{D}(\varphi, T) \to \mathfrak{D}(\varphi, T)$, defined by

$$\Phi_{\varepsilon}(x)(t) = S(t)\varphi(0) + \int_0^t S(t - s) f\left(\frac{s}{\varepsilon}, x[\varphi]_s\right) ds, \quad 0 \le t \le T,$$

and $z^{\infty} = x^{\infty}[\varphi]$, where x^{∞} is the unique fixed point of the operator, $\Phi_{\infty}: \mathfrak{D}(\varphi, T) \to \mathfrak{D}(\varphi, T)$, defined by

$$\Phi_{\infty}(x)(t) = S(t)\varphi(0) + \int_0^t S(t - s) f_0(x[\varphi]_s)\, ds, \quad 0 \le t \le T. \tag{4.5}$$

Note that, if $x^{\varepsilon} \to x^{\infty}$ in $C([0, T]; E)$, then $x^{\varepsilon}[\varphi] \to x^{\infty}[\varphi]$ in $\mathcal{C}(]-\infty, T]; E)$. Therefore, for the proof of Theorem 3, we have only to prove that

$$\lim_{\varepsilon \to 0^+} \sup_{t \in [0, T]} \|x^{\varepsilon}(t) - x^{\infty}(t)\| = 0.$$

First, let us prove that the sequence $\left(x^{\varepsilon}\right)_{\varepsilon > 0}$ is bounded. Let $\varepsilon > 0$. By using $(\mathtt{f}_3)$ and Axiom (B$_2$), for any $t \in [0, T]$, we have

$$\|x^{\varepsilon}(t)\| = \|\Phi_{\varepsilon}(x^{\varepsilon})(t)\| \le M_T \|\varphi(0)\| + M_T \int_0^t \left\| f\left(\frac{\tau}{\varepsilon}, x^{\varepsilon}[\varphi]_{\tau}\right)\right\| d\tau \le$$

$$\leq M_T\,\|\varphi(0)\| + M_T \int_0^t C\left(1+\lambda_K \sup_{0\leq\theta\leq\tau}\|x^\varepsilon(\theta)\| + \lambda_N\,\|\varphi\|_{\mathcal{B}}\right)d\tau \leq$$

$$\leq M_T\Big[\|\varphi(0)\| + T\,C\Big(1+\lambda_N\,\|\varphi\|_{\mathcal{B}}\Big)\Big] + M_T\,C\,\lambda_K \int_0^t \sup_{0\leq\theta\leq\tau}\|x^\varepsilon(\theta)\|\;d\tau.$$

where M_T is from (3.6), λ_K is from (3.7), λ_N is from (3.16), and C from Hypothesis $(\mathtt{f}_3)$. Since the last expression does not decrease, setting

$$\omega = M_T\big[\,\|\varphi(0)\| + T\,C\,(1+\lambda_N\,\|\varphi\|_{\mathcal{B}})\big] + M_T\,C\,\lambda_K,$$

we get

$$\sup_{0\leq\tau\leq t}\|x^\varepsilon(\tau)\| \leq \omega + \omega\int_0^t \sup_{0\leq\theta\leq\tau}\|x^\varepsilon(\theta)\|\;d\tau.$$

Hence, by Gronwall Lemma, we obtain, for any $t\in[0,T]$,

$$\sup_{0\leq\tau\leq t}\|x^\varepsilon(\tau)\| \leq \omega e^{\omega t},$$

which proves the boundedness of the sequence $\left(x^\varepsilon\right)_{\varepsilon>0}$.

By Axiom (B_3), the function, $x^\infty[\varphi]_{(.)}: t\to x^\infty[\varphi]_t$ is continuous on $[0,T]$. Let $\delta>0$. There exists a partition $0=t_0<t_1<\ldots<t_q=T$ of $[0,T]$, such that

$$\max_{1\leq i\leq q}(t_i-t_{i-1}) \leq \min\{\delta,\Delta_0\}, \tag{4.6}$$

$$\|x^\infty[\varphi]_t - x^\infty[\varphi]_{t_i}\| \leq \delta \quad \forall t\in[t_{i-1},t_i], \quad i=1,\ldots,q, \tag{4.7}$$

where Δ_0 is from $(\mathtt{Af})$. Define the function $\overline{x}^\infty[\varphi]_{(.)}$ by

$$\text{for}\quad t\in[t_{i-1},t_i[,\quad \overline{x}^\infty[\varphi]_t = x^\infty[\varphi]_{t_{i-1}},\quad i=1,\ldots,q. \tag{4.8}$$

Setting $\tau(t)=\max\{i, t_i\leq t\}$ and, $\vartheta(t)=t_{\tau(t)}$, for every $t\in[0,T]$ we have

$$x^\varepsilon[\varphi](t) - x^\infty[\varphi](t) = \Phi_\varepsilon\,(x^\varepsilon)(t) - \Phi_\infty\,(x^\infty)(t) =$$

$$= \int_{\vartheta(t)}^t S(t-\theta)\left[f\left(\frac{\theta}{\varepsilon}, x^\varepsilon[\varphi]_\theta\right) - f_0\,(x^\infty[\varphi]_\theta)\right]d\theta +$$

$$+ \int_0^{\vartheta(t)} S(t-\theta)\left[f\left(\frac{\theta}{\varepsilon}, x^\varepsilon[\varphi]_\theta\right) - f\left(\frac{\theta}{\varepsilon}, x^\infty[\varphi]_\theta\right)\right]d\theta +$$

$$+ \int_0^{\vartheta(t)} S(t-\theta)\left[f\left(\frac{\theta}{\varepsilon}, x^\infty[\varphi]_\theta\right) - f\left(\frac{\theta}{\varepsilon}, \overline{x}^\infty[\varphi]_\theta\right)\right]d\theta +$$

$$+\int_0^{\vartheta(t)} S(t-\theta)\left[f\left(\frac{\theta}{\varepsilon},\overline{x}^{\infty}[\varphi]_\theta\right)-f_0\left(\overline{x}^{\infty}[\varphi]_\theta\right)\right]d\theta+$$

$$+\int_0^{\vartheta(t)} S(t-\theta)\big[f_0\left(\overline{x}^{\infty}[\varphi]_\theta\right)-f_0\left(x^{\infty}[\varphi]_\theta\right)\big]d\theta=$$

$$=I_1+\ldots+I_5.$$

Since the sequence $\left(x^{\varepsilon}\right)_{\varepsilon>0}$ is bounded, by Axiom (B$_2$), the sequence $\left(x^{\varepsilon}[\varphi]_{(.)}\right)_{\varepsilon>0}$ is bounded too in $C([0,T];\mathcal{B})$. Then, setting

$$\varpi=M_T\left[C\left(1+\sup_{\varepsilon>0}\left\|x^{\varepsilon}[\varphi]_{(.)}\right\|_{C([0,T];\mathcal{B})}\right)+C'\left(1+\max_{t\in[0,T]}\|x^{\infty}[\varphi]_t\|\right)\right],$$

where C is from $(\mathtt{f}_3)$ and C' is from Remark 10, by (4.6) we get

$$\|I_1\|\le(t-\vartheta(t))\,\varpi\le\min\{\delta,\,\Delta_0\}\,\varpi\le\,\varpi\,\delta.$$

By using (3.6), Axiom (B$_2$), and Hypothesis $(\mathtt{f}_2)$, we get

$$\|I_2\|\le\int_0^t S(t-\theta)\left[f\left(\frac{\theta}{\varepsilon},x^{\varepsilon}[\varphi]_\theta\right)-f\left(\frac{\theta}{\varepsilon},x^{\infty}[\varphi]_\theta\right)\right]d\theta\le$$

$$\le M_T\int_0^t\mathbb{L}\left(K(\theta)\sup_{0\le s\le\theta}\|x^{\varepsilon}[\varphi](s)-x^{\infty}[\varphi](s)\|\right)d\theta.$$

Now, by using (3.6), $(\mathtt{f}_2)$, (4.7), and (4.8), we have

$$\|I_3\|\le M_T\sum_{i=1}^{q}\int_{t_{i-1}}^{t_i}\left\|f\left(\frac{\theta}{\varepsilon},x^{\infty}[\varphi]_\theta\right)-f\left(\frac{\theta}{\varepsilon},\overline{x}^{\infty}[\varphi]_\theta\right)\right\|d\theta\le$$

$$\le M_T\sum_{i=1}^{q}\int_{t_{i-1}}^{t_i}\left\|f\left(\frac{\theta}{\varepsilon},x^{\infty}[\varphi]_\theta\right)-f\left(\frac{\theta}{\varepsilon},x^{\infty}[\varphi]_{t_{i-1}}\right)\right\|d\theta\le$$

$$\le M_T\,T\,\mathbb{L}(\delta).$$

Since the function f_0 satisfies a condition similar to $(\mathtt{f}_2)$, the term I_5 can be estimated as I_3,

$$\|I_5\|\le M_T\,T\,\mathbb{L}(\delta).$$

It remains to estimate the term I_4. We have

$$\|I_4\|=\left\|\int_0^{\vartheta(t)} S(t-\theta)\left[f\left(\frac{\theta}{\varepsilon},\overline{x}^{\infty}[\varphi]_\theta\right)-f_0\left(\overline{x}^{\infty}[\varphi]_\theta\right)\right]d\theta\right\|\le$$

$$\leq M_T \sum_{i=1}^{\tau(t)} \left\| \int_{t_{i-1}}^{t_i} S(t_i-\theta) \left[f\left(\frac{\theta}{\varepsilon}, x^{\infty}[\varphi]_{t_{i-1}}\right) - f_0\left(x^{\infty}[\varphi]_{t_{i-1}}\right)\right] d\theta \right\|.$$

Bearing in mind (4.6), from Hypothesis $(\mathtt{Af1})$ we have

$$\max_{1\leq i\leq q} \lim_{\varepsilon\to 0^+} \int_{t_{i-1}}^{t_i} S(t_i-\theta) \left[f\left(\frac{\theta}{\varepsilon}, x^{\infty}[\varphi]_{t_{i-1}}\right) - f_0\left(x^{\infty}[\varphi]_{t_{i-1}}\right)\right] d\theta = 0.$$

Thus

$$\max_{1\leq i\leq q} \left\| \int_{t_{i-1}}^{t_i} S(t_i-\theta) \left[f\left(\frac{\theta}{\varepsilon}, x^{\infty}[\varphi]_{t_{i-1}}\right) - f_0\left(x^{\infty}[\varphi]_{t_{i-1}}\right)\right] d\theta \right\| \leq \frac{\delta}{q} \quad \text{as} \quad \varepsilon\to 0^+.$$

Thus

$$\|I_4\| \leq M_T \, \frac{\tau(t)}{q} \, \delta \leq M_T \, \delta \quad \text{as} \quad \varepsilon \to 0^+.$$

From the estimations of the terms $I_1,\ldots,I_5$, for every $t\in[0,T]$, we obtain

$$\|x^{\varepsilon}[\varphi](t) - x^{\infty}[\varphi](t)\| \leq (\varpi + M_T)\,\delta + 2M_T\,T\,\mathbb{L}(\delta) +$$
$$+ M_T \int_0^t \mathbb{L}\left(K(\theta) \sup_{0\leq s\leq\theta} \|x^{\varepsilon}[\varphi](s) - x^{\infty}[\varphi](s)\| \right) d\theta,$$

as $\varepsilon\to 0^+$. Since the last expression does not decrease, setting

$$\rho(\delta) = (\varpi + M_T)\delta + 2\,M_T\,T\,\mathbb{L}(\delta),$$

we get for every $t\in[0,T]$

$$K(t) \sup_{0\leq s\leq t} \|x^{\varepsilon}[\varphi](s) - x^{\infty}[\varphi](s)\| \leq$$
$$\leq \lambda_K \rho(\delta) + \lambda_K M_T \int_0^t \mathbb{L}\left(K(\theta) \sup_{0\leq s\leq\theta} \|x^{\varepsilon}[\varphi](s) - x^{\infty}[\varphi](s)\| \right) d\theta,$$

as $\varepsilon\to 0^+$, where λ_K is from (3.7) (recall that $K(\cdot)$ is with positive values). From $(\mathtt{f}_2)$, the function $\delta\to\lambda_K\,\rho(\delta)$, is continuous on $[0,+\infty[$ and $\lambda_K\,\rho(0)=0$. Then, from the arbitrariness of δ, we have necessarily that,

$$K(t) \sup_{0\leq s\leq t} \|x^{\varepsilon}[\varphi](s) - x^{\infty}[\varphi](s)\| \leq$$
$$\leq \lambda_K M_T \int_0^t \mathbb{L}\left(K(\theta) \sup_{0\leq s\leq\theta} \|x^{\varepsilon}[\varphi](s) - x^{\infty}[\varphi](s)\| \right) d\theta \quad \text{as} \quad \varepsilon\to 0^+.$$

By using $(\mathtt{f}_2)$ again, we have, for every $t \in [0,T]$,

$$K(t) \sup_{0\le s\le t} \|x^\varepsilon[\varphi](s) - x^\infty[\varphi](s)\| = 0 \quad \text{as} \quad \varepsilon \to 0^+.$$

Since $K(\cdot)$ is a positive function, we deduce that for every $t \in [0,T]$,

$$\sup_{0\le s\le t} \|x^\varepsilon[\varphi](s) - x^\infty[\varphi](s)\| = 0 \quad \text{as} \quad \varepsilon \to 0^+.$$

The proof is complete.

5. Averaging principle in the traditional form. Now we aim to establish the averaging principle in traditional form for the problem (P$_\varepsilon$). Recall that the phase space $\mathcal{B}$ is considered as a normed space stisfying Axioms (B$_1$) – (B$_3$).

Assume that the hypotheses $(\mathtt{f}_1)$ – $(\mathtt{f}_3)$ are satisfied. Let us replace the hypothesis $(\mathtt{Af})$ by a more natural one:

$(\mathtt{Avf})$ for every $u \in \mathcal{B}$,

$$f_0(u) = \lim_{t\to+\infty} \frac{1}{t} \int\limits_0^t f(s,u)\, ds. \tag{5.1}$$

We have immediately the next lemma.

Lemma 4. *The function f_0 satisfies hypothesis similar to $(\mathtt{f}_2)$ and $(\mathtt{f}_3)$.*

The proof of Lemma 4 is easy.

Now, invoking [15] (Lemma 3), we deduce the next lemma.

Lemma 5. *Assume that hypotheses $(\mathtt{f}_1)$ – $(\mathtt{f}_3)$ are satisfied. Moreover, assume that E is reflexive and $(S(t))_t$ is a analytic semigroup. Then, under the hypothesis $(\mathtt{Avf})$, for every $u \in \mathcal{B}$ and $t_1,\ t_2$ such that $0 \le t_1 \le t_2 \le T$,*

$$\lim_{\varepsilon\to0^+} \int\limits_{t_1}^{t_2} S(t_2-\theta)\left[f\left(\frac{\theta}{\varepsilon}, u\right) - f_0(u)\right] d\theta = 0.$$

Since E is reflexive, the proof of Lemma 5 is exactly the same as in [15] (Lemma 3) except some trivial modifications (for analytic simigroups see, e.g., [25]).

From Lemma 4 and Lemma 5, we deduce

Corollary 2. *Assume that the hypotheses $(\mathtt{f}_1)$ – $(\mathtt{f}_3)$ are satisfied. Moreover, assume that E is reflexive, and the semigroup $(S(t))_t$ is analytic. Then, $(\mathtt{Avf}) \Rightarrow (\mathtt{Af})$.*

Applying Lemma 4, Corollary 2 and Theorem 3, we have immediately.

Theorem 4. *Assume that Hypotheses (A), $(\mathtt{f}_1)$ – $(\mathtt{f}_3)$ and $(\mathtt{Avf})$ are satisfied. Moreover, assume that E is reflexive and, the semigroup $(S(t))_t$ is analytic. Then*

$$\lim_{\varepsilon\to0} \|z^\varepsilon - z^\infty\|_{C(]-\infty,T];E)} = 0,$$

where z^ε is the unique mild solution to the problem (P$_\varepsilon$), $\varepsilon > 0$, and z^∞ is the unique mild solution to the problem (P$_0$).

References

1. J. A. Sanders, F. Verhulst, J. Murdock, *Averaging methods in nonlinear dynamical systems*, Applied Mathematical Sciences, Vol. 59, Springer, New York (2007).
2. A. M. Samoilenko, R. Petryshyn, *Multifrequency oscillations of nonlinear systems*, Mathematics and its Applications, Vol. 567, Kluwer Acad. Publ. Group, Dordrecht (2004).
3. N. N. Bogoliubov, Y. A. Mitropolsky, *Asymptotic methods in the theory of non-linear oscillations*, International Monographs on Advanced Mathematics and Physics, Hindustan Publ. Corp., Delhi; Gordon and Breach Sci. Publ., New York (1961).
4. N. Kryloff, N. Bogoliuboff, *Introduction to non-linear mechanics*, Annals of Mathematics Studies, № 11, Princeton Univ. Press, Princeton, N. J.(1943).
5. V. Burd, *Method of averaging for differential equations on an infinite interval*, Lecture Notes in Pure and Applied Mathematics, Vol. 255, Chapman & Hall/CRC, Boca Raton, FL (2007).
6. J. K. Hale, *Averaging methods for differential equations with retarded arguments and a small parameter*, J. Differential Equations, № 2, 57 – 73 (1966).
7. J. K. Hale, S. M. Verduyn Lunel, *Averaging in infinite dimensions*, J. Integral Equations Appl., **2**, 463 – 494 (1990).
8. M. Kamenskii, V. Obukhovskii, P. Zecca, *Condensing multivalued maps and semilinear differential inclusions in Banach spaces*, De Gruyter Series in Nonlinear Analysis and Applications, Vol. 7, Walter de Gruyter & Co., Berlin (2001).
9. M. Lakrib, T. Sari, *Results on the averaging of functional differential equations*, Sib. Mat. Zh., **45**, 375 – 386 (2004).
10. B. Lehman, V. Strygin, *Partial and generalized averaging of functional differential equations*, Funct. Differ. Equ., № 9, 165 – 200 (2002).
11. B. Lehman, S. P. Weibel, *Fundamental theorems of averaging for functional-differential equations*, J. Differential Equations, **152**, 160 – 190 (1999).
12. I. B. Simonenko, *A justification of the averaging method for abstract parabolic equations*, Mathematics, **10**, № 1, 51 – 59 (1970).
13. V. V. Strygin, *The averaging principle for equations with heredity*, Ukr. Mat. Zh., **22**, № 4, 503 – 513 (1970).
14. J. K. Hale, J. Kato, *Phase space for retarded equations with infinite delay*, Funkcial. Ekvac., № 21, 11 – 41 (1978).
15. J. Seidler, I. Vrkoč, *An averaging principle for stochastic evolution equations*, Časopis Pěst. Mat., № 115, 240 – 263(1990).
16. R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina, B. N. Sadovskiĭ, *Measures of noncompactness and condensing operators*, Operator Theory: Advances and Applications, Vol. 55, Birkhäuser Verlag, Basel (1992).
17. J. Banaś, K. Goebel, *Measures of noncompactness in Banach spaces*, Lecture Notes in Pure and Applied Mathematics, Vol. 60, Marcel Dekker, Inc., New York (1980).
18. P. R. Agarwal, V. Lakshmikantham, *Uniqueness and nonuniqueness criteria for ordinary differential equations*, Series in Real Analysis, Vol. 6, World Sci. Publ. Co., Inc., River Edge, NJ (1993).
19. D. Barbu, *Local and global existence for mild solutions of stochastic differential equations*, Portugal. Math., **55**, № 4, 411 – 424 (1998).
20. Ph. Hartman, *Ordinary differential equations*, Classics in Applied Mathematics, Vol. 38, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, Vol. 38 (2002).
21. L. Guedda, P. Raynaud de Fitte, *Existence and dependence results for semilinear functional stochastic differential equations with infinite delay in a Hilbert space*, Mediterr. J. Math., **13**, № 6, 4153 – 4174 (2016).

22. C. Gori, V. Obukhovskii, M. Ragni, P. Rubbioni, *Existence and continuous dependence results for semilinear functional differential inclusions with infinite delay*, Nonlinear Anal., **51**, 765 – 782 (2002).

23. A. Paz, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, Springer-Verlag, New York, Vol. 44 (1983).

24. Y. Hino, S. Murakami, T. Naito, *Functional-differential equations with infinite delay*, Lecture Notes in Mathematics, Vol. 1473, Springer-Verlag, Berlin (1991).

25. T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Vol. 3, Springer-Verlag, Berlin (1995).

26. A. Halanay, *On the method of averaging for differential equations with retarded argument*, J. Math. Anal. Appl., № 14, 70 – 76 (1966).

Received 29.11.16,
after revision — 28.12.20