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NOTE ON FRACTIONAL DIFFERENCE EQUATIONS WITH PERIODIC AND S-ASYMPTOTICALLY PERIODIC RIGHT-HAND SIDE*

ПРО РІЗНИЦЕВІ РІВНЯННЯ ДРОБОВОГО ПОРЯДКУ З ПЕРІОДИЧНОЮ І S-АСИМПТОТИЧНО ПЕРІОДИЧНОЮ ПРАВОЮ ЧАСТИНОЮ

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Decomposition of a solution of a fractional difference initial value problem with periodic right-hand side is found. For problems with S-asymptotically N-periodic right-hand side, a sufficient condition is proved for the existence of an S-asymptotically N-periodic solution.

Знайдено розклад розв'язку дробово-різницевої початкової задачі з періодичною правою частиною. Для задач з S-асимптотично N-періодичною правою частиною доведено достатню умову існування S-асимптотично N-періодичного розв'язку.

Dedicated to 60th birthday of professor Michal Fečkan

1. Introduction. In this note we consider the fractional difference initial value problem

$$\begin{aligned} \Delta_*^\mu u(k) &= f(k), \quad k \in \mathbb{N}_{1-\mu}, \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

with Caputo like fractional difference operator Δ_*^μ of order $0 < \mu < 1$. Here and after, $\mathbb{N}_a$, $a \in \mathbb{R}$, denotes the shifted set of positive integers, i.e., $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$. We shortly denote $\mathbb{N} := \mathbb{N}_1$. Motivated by [1] we shall improve our recent result from [2], where it was proved using R_a-transform (Laplace transform on the time scale of integers) that despite of having N-periodic right-hand side f, problem (1.1) can not posses an N-periodic solution; instead it has an S-asymptotically N-periodic solution. We shall consider $f: \mathbb{N}_{1-\mu} \to X$, where X is a Banach space equipped with a norm $|\cdot|$, and look for a solution $u: \mathbb{N}_0 \to X$ satisfying fractional difference equation (1.1) along with the corresponding initial condition.

For periodic right-hand side, here we find a decomposition of the solution u to constant, periodic, S-asymptotically periodic and shrinking part. In the case when f has zero arithmetic mean over the period N we obtain a convergence to an N-periodic function. We also derive the rate of this convergence.

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Moreover, for S-asymptotically N-periodic right-hand side we prove sufficient condition under which the solution u to (1.1) is S-asymptotically N-periodic.

Throughout the paper, we assume the property of empty sum and empty product, i.e.,

$$\sum_{k=a}^{b} f(k) = 0, \qquad \prod_{k=a}^{b} f(k) = 1$$

if $a > b$.

2. Preliminaries. First we recall some definitions from the theory of fractional difference calculus. Basic definitions are due to [3, 4]. For properties of fractional difference operator see [5, 6].

Definition 2.1. *Let $\nu \in \mathbb{R}$. Factorial function is defined as*

$$t^{(\nu)} = \begin{cases} 0, & t + 1 - \nu \in \{\dots, -2, -1, 0\}, \\ \dfrac{\Gamma(t+1)}{\Gamma(t+1-\nu)}, & \textit{otherwise,} \end{cases}$$

where Γ is the Euler gamma function.

Lemma 2.1 ([7], Lemma 2.5)**.** *For any $k > 0,\ 0 < \mu < 1$,*

$$\sum_{j=1-\mu}^{k-\mu} (k - \sigma(j))^{(\mu-1)} = \frac{\Gamma(k+\mu)}{\mu\Gamma(k)},$$

where $\sigma(j) = j + 1$.

Definition 2.2. *Let $a \in \mathbb{R},\ \nu > 0$. The νth fractional sum of function f defined on $\mathbb{N}_a$ is given by*

$$\Delta^{-\nu} f(k) = \frac{1}{\Gamma(\nu)} \sum_{j=a}^{k-\nu} (k - \sigma(j))^{(\nu-1)} f(j)$$

for any $k \in \mathbb{N}_{a+\nu}$.

Definition 2.3. *Let $a \in \mathbb{R},\ \mu > 0,\ m - 1 < \mu < m$ for some $m \in \mathbb{N},\ \nu := m - \mu$ and function f be defined on $\mathbb{N}_a$. The μth fractional Caputo like difference of f is defined as*

$$\Delta_*^{\mu} f(k) = \Delta^{-\nu}(\Delta^m f(k)) = \frac{1}{\Gamma(\nu)} \sum_{j=a}^{k-\nu} (k - \sigma(j))^{(\nu-1)} \big(\Delta^m f\big)(j)$$

for any $k \in \mathbb{N}_{a+\nu}$. Here Δ^m is the mth forward difference operator,

$$\big(\Delta^m f\big)(k) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(k+j).$$

A form of summation by parts formula (discrete analogue to per partes method) is mentioned in the following lemma.

Lemma 2.2. *For any* $a \in \mathbb{R},\ b \in \mathbb{N}_a$,

$$\sum_{j=a}^{b} f(j)\Delta g(j) = \big[f(j)g(j)\big]_{j=a}^{b+1} - \sum_{j=a+1}^{b+1} \Delta f(j-1)g(j).$$

Definition 2.4. *A function* $f\colon \mathbb{N}_a \to X$ *is* N*-periodic if there exists* $N \in \mathbb{N}$ *such that* $f(k+N) - f(k) = 0$ *for each* $k \in \mathbb{N}_a$.

Definition 2.5 [8]**.** *A function* $f\colon \mathbb{N}_a \to X$ *is called* S*-asymptotically* N*-periodic if there exists* $N \in \mathbb{N}$ *such that*

$$\lim_{k\to\infty} \big(f(k+N) - f(k)\big) = 0.$$

In this case, the smallest such N *is called asymptotic period of* f.

Example 2.1. Function $g(k) = \sqrt{k}$ is S-asymptotically 1-periodic, since

$$\lim_{k\to\infty}\left(\sqrt{k+1} - \sqrt{k}\right) = \lim_{k\to\infty} \frac{1}{\sqrt{k+1} + \sqrt{k}} = 0.$$

We shall need the following estimation of a ratio of gamma functions. Another useful estimations can be found in [9].

Lemma 2.3 [10]**.** *For any* $0 < s < 1$ *and* $x > 0$,

$$x^{1-s} \le \frac{\Gamma(x+1)}{\Gamma(x+s)} \le (x+s)^{1-s}.$$

We can apply the latter estimation immediately to prove the next statement.

Lemma 2.4. *For any* $N, J \in \mathbb{N},\ 0 < \mu < 1$,

$$\sum_{j=J}^{\infty} (N+j+\mu-3)^{(\mu-2)} = \frac{\Gamma(N+J+\mu-2)}{(1-\mu)\Gamma(N+J-1)}.$$

Proof. We consider the infinite sum as $\lim_{K\to\infty}\sum_{j=J}^{K}$. Then for any $K \in \mathbb{N}_J$, we evaluate the finite sum by writing it as a telescoping series,

$$\sum_{j=J}^{K} (N+j+\mu-3)^{(\mu-2)} = \sum_{j=J}^{K} \frac{\Gamma(N+j+\mu-2)}{\Gamma(N+j)} =$$

$$= \frac{1}{1-\mu}\sum_{j=J}^{K}\left[\frac{\Gamma(N+j+\mu-2)}{\Gamma(N+j-1)} - \frac{\Gamma(N+j+\mu-1)}{\Gamma(N+j)}\right] =$$

$$= \frac{1}{1-\mu}\left[\frac{\Gamma(N+J+\mu-2)}{\Gamma(N+J-1)} - \frac{\Gamma(N+K+\mu-1)}{\Gamma(N+K)}\right]$$

for any $K \in \mathbb{N}_J$. Using Lemma 2.3 we obtain

$$0 \le \frac{\Gamma(N+K+\mu-1)}{\Gamma(N+K)} \le \frac{1}{(N+K)^{1-\mu}} \xrightarrow{K\to\infty} 0.$$

Lemma 2.4 is proved.

3. Periodic right-hand side. From [7] (Lemma 2.4) we know that function u is a solution to initial value problem (1.1) if and only if it satisfies

$$u(k) = u_0 + \frac{1}{\Gamma(\mu)} \sum_{j=1-\mu}^{k-\mu} (k - \sigma(j))^{(\mu-1)} f(j) \tag{3.1}$$

for each $k \in \mathbb{N}_0$. Assume that $f : \mathbb{N}_{1-\mu} \to X$ is N-periodic for some $N \in \mathbb{N}_2$ and decompose

$$f(k) = \bar{f} + \tilde{f}(k), \quad k \in \mathbb{N}_0,$$

with

$$\bar{f} := \frac{1}{N} \sum_{k=1-\mu}^{N-\mu} f(k).$$

Consequently,

$$\sum_{j=1-\mu}^{N-\mu} \tilde{f}(j) = \sum_{j=1-\mu}^{N-\mu} f(j) - \sum_{j=1-\mu}^{N-\mu} \bar{f} = 0.$$

Using Lemma 2.1, we obtain

$$u(k) = u_0 + \frac{\bar{f}}{\Gamma(\mu)} \sum_{j=1-\mu}^{k-\mu} (k - \sigma(j))^{(\mu-1)} + \frac{1}{\Gamma(\mu)} \sum_{j=1-\mu}^{k-\mu} (k - \sigma(j))^{(\mu-1)} \tilde{f}(j) =$$

$$= u_0 + \frac{\bar{f}\Gamma(k+\mu)}{\Gamma(\mu+1)\Gamma(k)} + \frac{1}{\Gamma(\mu)} \sum_{j=1-\mu}^{k-\mu} (k - \sigma(j))^{(\mu-1)} \tilde{f}(j). \tag{3.2}$$

Now we have the following observations.

Lemma 3.1. *Function* $k \mapsto \dfrac{\Gamma(k+\mu)}{\Gamma(k)}$ *is* S*-asymptotically* 1*-periodic. Consequently, it is also* S*-asymptotically* N*-periodic.*

Proof. Lemma 2.3 with $x = k + \mu,\ s = 1 - \mu$ implies

$$(k+\mu)^{\mu} \le \frac{\Gamma(k+1+\mu)}{\Gamma(k+1)} \le (k+1)^{\mu}, \tag{3.3}$$

and with $x = k + \mu - 1,\ s = 1 - \mu$ we get

$$(k+\mu-1)^{\mu} \le \frac{\Gamma(k+\mu)}{\Gamma(k)} \le k^{\mu}. \tag{3.4}$$

Therefrom, using mean value theorem we derive

$$\frac{\Gamma(k+1+\mu)}{\Gamma(k+1)} - \frac{\Gamma(k+\mu)}{\Gamma(k)} \le (k+1)^{\mu} - (k+\mu-1)^{\mu} =$$

$$= \frac{d}{dx} \left[x^{\mu}\right]_{x=\theta_k} (k+1-(k+\mu-1)) = \frac{\mu(2-\mu)}{\theta_k^{1-\mu}}$$

for some $\theta_k \in (k+\mu-1, k+1)$. Note that $\theta_k \to \infty$ as $k \to \infty$. Similarly, from (3.3) and (3.4) we obtain

$$\frac{\Gamma(k+1+\mu)}{\Gamma(k+1)} - \frac{\Gamma(k+\mu)}{\Gamma(k)} \geq (k+\mu)^\mu - k^\mu = \frac{d}{dx}\left[x^\mu\right]_{x=\vartheta_k}(k+\mu-k) = \frac{\mu^2}{\vartheta_k^{1-\mu}}$$

for some $\vartheta_k \in (k, k+\mu)$. Again, $\vartheta_k \to \infty$ as $k \to \infty$. Summarizing, we get

$$\lim_{k\to\infty}\left(\frac{\Gamma(k+1+\mu)}{\Gamma(k+1)} - \frac{\Gamma(k+\mu)}{\Gamma(k)}\right) = 0.$$

Lemma 3.1 is proved.

Lemma 3.2. *Function* $\widetilde{F}(k) = \sum_{j=1-\mu}^{k-\mu} \tilde{f}(j)$ *is* N*-periodic on* $\mathbb{N}$.

Proof. For any $k \in \mathbb{N}$ we have

$$F(k+N) - F(k) = \sum_{j=1-\mu}^{k+N-\mu} \tilde{f}(j) - \sum_{j=1-\mu}^{k-\mu} \tilde{f}(j) = \sum_{j=k-\mu+1}^{k+N-\mu} \tilde{f}(j) = \sum_{j=k-\mu+1}^{k+N-\mu} \big(f(j) - \bar{f}\big) =$$

$$= \sum_{j=k-\mu+1}^{k+N-\mu} f(j) - N\bar{f} = \sum_{j=1-\mu}^{N-\mu} f(j) - \sum_{j=1-\mu}^{N-\mu} f(j) = 0.$$

Lemma 3.2 is proved.

Due to the empty sum property one can see that $\widetilde{F}(k) = 0$ whenever $k < 1$. Later we will need $\widetilde{F}$ to be defined and periodic on $\mathbb{Z}$ — the set of all integers. Thus to simplify the notation from now on we will assume that $\widetilde{F}$ denotes an N-periodic extension of the function

$$\{1, 2, \ldots, N\} \ni k \mapsto \sum_{j=1-\mu}^{k-\mu} \tilde{f}(j)$$

to $\mathbb{Z}$ or, in other words,

$$\widetilde{F}(k) = \sum_{j=1-\mu}^{\Phi(k)-\mu} \tilde{f}(j), \quad \Phi(k) = k - N\left\lfloor \frac{k}{N} \right\rfloor \tag{3.5}$$

for $k \in \mathbb{Z}$, where $\lfloor\cdot\rfloor$ is the floor function defined as $\lfloor x \rfloor = \max\{y \in \mathbb{Z} \mid y \leq x\}$ for any $x \in \mathbb{R}$. Obviously, if $k \in \mathbb{N}$, the value of $\widetilde{F}(k)$ remains unchanged and $\widetilde{F}(k)$ can be understood as in Lemma 3.2.

Since we are interested in asymptotic properties of u, we can assume that $k > N$. Let us further split the last term in (3.2) to get

$$u(k) = u_0 + \frac{\bar{f}\Gamma(k+\mu)}{\Gamma(\mu+1)\Gamma(k)} + \frac{1}{\Gamma(\mu)} \sum_{j=1-\mu}^{k-N-\mu} (k-\sigma(j))^{(\mu-1)} \tilde{f}(j) +$$

$$+ \frac{1}{\Gamma(\mu)} \sum_{j=k-N+1-\mu}^{k-\mu} (k-\sigma(j))^{(\mu-1)} \tilde{f}(j). \tag{3.6}$$

After substituting $l = j + \mu$ we can apply discrete per partes from Lemma 2.2 to work with the sum in the third term:

$$\sum_{j=1-\mu}^{k-N-\mu}(k-j-1)^{(\mu-1)}\tilde{f}(j) = \sum_{l=1}^{k-N}(k-l+\mu-1)^{(\mu-1)}\tilde{f}(l-\mu) =$$

$$= (k+\mu-2)^{(\mu-1)}\tilde{f}(1-\mu) + \sum_{l=2}^{k-N}(k-l+\mu-1)^{(\mu-1)}\tilde{f}(l-\mu) =$$

$$= (k+\mu-2)^{(\mu-1)}\tilde{f}(1-\mu) + \sum_{l=2}^{k-N}(k-l+\mu-1)^{(\mu-1)}\Delta\widetilde{F}(l-1) =$$

$$= (k+\mu-2)^{(\mu-1)}\tilde{f}(1-\mu) + \Big[(k-l+\mu-1)^{(\mu-1)}\widetilde{F}(l-1)\Big]_{l=2}^{k-N+1} -$$

$$- \sum_{l=3}^{k-N+1}\Delta_l(k-l+\mu)^{(\mu-1)}\widetilde{F}(l-1) =$$

$$= (k+\mu-2)^{(\mu-1)}\tilde{f}(1-\mu) + (N+\mu-2)^{(\mu-1)}\widetilde{F}(k-N) -$$

$$- (k+\mu-3)^{(\mu-1)}\widetilde{F}(1) - \sum_{l=2}^{k-N}\Delta_l(k-l+\mu-1)^{(\mu-1)}\widetilde{F}(l).$$

Here the lower index l in Δ_l means that the difference should be taken with respect to l. Now, note that

$$(k+\mu-2)^{(\mu-1)}\tilde{f}(1-\mu) - (k+\mu-3)^{(\mu-1)}\widetilde{F}(1) = -\Big[\Delta_l(k-l+\mu-1)^{(\mu-1)}\widetilde{F}(l)\Big]_{l=1}.$$

Hence,

$$\sum_{j=1-\mu}^{k-N-\mu}(k-j-1)^{(\mu-1)}\tilde{f}(j) = (N+\mu-2)^{(\mu-1)}\widetilde{F}(k-N) -$$

$$- \sum_{l=1}^{k-N}\Delta_l(k-l+\mu-1)^{(\mu-1)}\widetilde{F}(l).$$

Rewriting the factorial function using gamma functions we can simplify the Δ_l-term as follows

$$\Delta_l(k-l+\mu-1)^{(\mu-1)} = (k-l+\mu-2)^{(\mu-1)} - (k-l+\mu-1)^{(\mu-1)} =$$

$$= \frac{\Gamma(k-l+\mu-1)}{\Gamma(k-l)} - \frac{\Gamma(k-l+\mu)}{\Gamma(k-l+1)} =$$

$$= \frac{\Gamma(k-l+\mu-1)}{\Gamma(k-l+1)}\left[k-l-(k-l+\mu-1)\right] =$$

$$= \frac{(1-\mu)\Gamma(k-l+\mu-1)}{\Gamma(k-l+1)} = (1-\mu)(k-l+\mu-2)^{(\mu-2)}.$$

So we can write

$$\sum_{j=1-\mu}^{k-N-\mu} (k-j-1)^{(\mu-1)} \tilde{f}(j) = (N+\mu-2)^{(\mu-1)} \widetilde{F}(k) -$$

$$- (1-\mu) \sum_{l=1}^{k-N} (k-l+\mu-2)^{(\mu-2)} \widetilde{F}(l) \tag{3.7}$$

using the N-periodicity of $\widetilde{F}$.

The last term in (3.6) is an N-periodic function. Indeed, taking the substitution $l = j-k+N$ in the sum leads to

$$\sum_{j=k-N+1-\mu}^{k-\mu} (k-\sigma(j))^{(\mu-1)} \tilde{f}(j) = \sum_{l=1-\mu}^{N-\mu} (N-\sigma(l))^{(\mu-1)} \tilde{f}(l+k-N) \tag{3.8}$$

where the N-periodicity of the right-hand side follows from the N-periodicity of function $\tilde{f}$.

At this moment we can use (3.7), (3.8) to rewrite $u(k)$ of (3.6) as

$$u(k) = u_0 + \frac{\bar{f}\Gamma(k+\mu)}{\Gamma(\mu+1)\Gamma(k)} +$$

$$+ \left[\frac{(N+\mu-2)^{(\mu-1)} \widetilde{F}(k)}{\Gamma(\mu)} + \frac{1}{\Gamma(\mu)} \sum_{j=1-\mu}^{N-\mu} (N-\sigma(j))^{(\mu-1)} \tilde{f}(k+j) \right] -$$

$$- \frac{1-\mu}{\Gamma(\mu)} \sum_{j=1}^{k-N} (k-j+\mu-2)^{(\mu-2)} \widetilde{F}(j) \tag{3.9}$$

for $k \in \mathbb{N}$, where there is an N-periodic function in the bracket (we omitted N in the argument of $\tilde{f}$ due to its N-periodicity).

Now we focus on the last sum. We substitute $l = k-N+1-j$ to get

$$\sum_{j=1}^{k-N} (k-j+\mu-2)^{(\mu-2)} \widetilde{F}(j) = \sum_{l=1}^{k-N} (l+N+\mu-3)^{(\mu-2)} \widetilde{F}(k-N+1-l). \tag{3.10}$$

We want to get rid of k in the upper bound of the sum, so we would get another N-periodic term. We can do it by extending the sum to infinity and subtracting what we added. Here we need the periodicity of $\widetilde{F}$ on the whole $\mathbb{Z}$. Without extending to non-positive integers empty sum property would cause the sum to contain a finite number (although changing with k) of nonzero terms whereas the other sum would be empty. Fortunately (due to periodic extension in (3.5)), $\widetilde{F}$ is periodic on $\mathbb{Z}$ and it only needs to be verified that the sums converge. Here we apply Lemma 2.4 with $J = 1$ to get the estimation

$$\left\| \sum_{l=1}^{\infty} (l+N+\mu-3)^{(\mu-2)} \widetilde{F}(k-N+1-l) \right\| \leq \|\widetilde{F}\|_{\infty} \frac{\Gamma(N+\mu-1)}{(1-\mu)\Gamma(N)}$$

with
$$\|\widetilde{F}\|_\infty = \sup_{k\in\mathbb{Z}} |\widetilde{F}(k)| = \max_{k=1,\ldots,N} |\widetilde{F}(k)|,$$
and analogously
$$\left\| \sum_{l=k-N+1}^{\infty} (l+N+\mu-3)^{(\mu-2)} \widetilde{F}(k-N+1-l) \right\| \le \|\widetilde{F}\|_\infty \frac{\Gamma(k+\mu-1)}{(1-\mu)\Gamma(k)} \tag{3.11}$$
by applying Lemma 2.4 with $J = k-N+1$.

We summarize this section into the next result.

Theorem 3.1. *Let $N \in \mathbb{N}_2$. If f is N-periodic on $\mathbb{N}_{1-\mu}$, then a solution u to (1.1) is S-asymptotically N-periodic on $\mathbb{N}_0$. More precisely, it can be written as*
$$u(k) = u_0 + v_1(k) + v_2(k) + v_3(k)$$
for $k \in \mathbb{N}_{N+1}$, where
$$v_1(k) = \frac{\bar{f}\Gamma(k+\mu)}{\Gamma(\mu+1)\Gamma(k)}$$
is S-asymptotically N-periodic,
$$v_2(k) = \frac{(N+\mu-2)^{(\mu-1)}\widetilde{F}(k)}{\Gamma(\mu)} + \frac{1}{\Gamma(\mu)} \sum_{j=1-\mu}^{N-\mu} (N-\sigma(j))^{(\mu-1)} \tilde{f}(k+j) -$$
$$- \frac{1-\mu}{\Gamma(\mu)} \sum_{j=1}^{\infty} (j+N+\mu-3)^{(\mu-2)} \widetilde{F}(k+1-j)$$
is an N-periodic function and
$$v_3(k) = \frac{1-\mu}{\Gamma(\mu)} \sum_{j=k-N+1}^{\infty} (j+N+\mu-3)^{(\mu-2)} \widetilde{F}(k+1-j) \xrightarrow{k\to\infty} 0.$$

Proof. It only remains to prove the convergence of $v_3(k)$. But this follows by applying Lemma 2.3 on the right-hand side of (3.11), i.e.,
$$\|\widetilde{F}\|_\infty \frac{\Gamma(k+\mu-1)}{(1-\mu)\Gamma(k)} \le \frac{\|\widetilde{F}\|_\infty}{(1-\mu)k^{1-\mu}} \to 0 \quad \text{as} \quad k \to \infty.$$
Moreover, we omitted N in the arguments of $\widetilde{F}$ due to its N-periodicity.

Theorem 3.1 is proved.

Remark 3.1. Since by Lemma 2.3,
$$\frac{\Gamma(k+\mu)}{\Gamma(k)} \ge (k+\mu-1)^{\mu} \to \infty \quad \text{as} \quad k \to \infty,$$
$v_1(k)$ is bounded if and only if $\bar{f} = 0$. Consequently, the boundedness of u_0, $v_2(k)$ and $v_3(k)$ implies that $u(k)$ is bounded on $\mathbb{N}_0$ if and only if $\bar{f} = 0$.

Remark 3.2. Without splitting the last term in (3.9) it would be possible to prove (by using (3.10), N-periodicity of $\widetilde{F}$, a lemma analogous to Lemmas 2.1 and 2.3) that this term is S-asymptotically N-periodic directly by showing that it satisfies Definition 2.5. Nevertheless, by writing as a difference of two infinite sums, we showed that the term is asymptotic to a periodic function, which is more precise, since not every S-asymptotically periodic function has this property (see, e.g., Example 2.1).

4. S-asymptotically periodic right-hand side. In this section we assume that the right-hand side f of (1.1) is S-asymptotically N-periodic and we investigate the solution u having the form of (3.1). We look for a condition under which u is S-asymptotically N-periodic. First we substitute $l = k - j - \mu + 1$ to remove the dependence on k in the factorial function,

$$u(k) = u_0 + \frac{1}{\Gamma(\mu)} \sum_{l=1}^{k} (l + \mu - 2)^{(\mu-1)} f(k - l - \mu + 1).$$

Next we calculate the difference

$$u(k+N) - u(k) = \frac{1}{\Gamma(\mu)} \sum_{l=1}^{k+N} (l + \mu - 2)^{(\mu-1)} f(k + N - l - \mu + 1) -$$

$$- \frac{1}{\Gamma(\mu)} \sum_{l=1}^{k} (l + \mu - 2)^{(\mu-1)} f(k - l - \mu + 1) =$$

$$= \frac{G_1(k) + G_2(k)}{\Gamma(\mu)} \tag{4.1}$$

where

$$G_1(k) = \sum_{l=1}^{k} (l + \mu - 2)^{(\mu-1)} \big(f(k + N - l - \mu + 1) - f(k - l - \mu + 1)\big),$$

$$G_2(k) = \sum_{l=k+1}^{k+N} (l + \mu - 2)^{(\mu-1)} f(k + N - l - \mu + 1).$$

First we take a look at G_2. If $l \in \{k+1, k+2, \ldots, k+N\}$ then

$$k + N - l - \mu + 1 \in \{1 - \mu, 2 - \mu, \ldots, N - \mu\}.$$

Thus by Lemma 2.3 we get the estimation

$$|G_2(k)| \le \sum_{l=k+1}^{k+N} \frac{\Gamma(l + \mu - 1)}{\Gamma(l)} |f(k + N - l - \mu + 1)| \le$$

$$\le \max_{j=1-\mu,\ldots,N-\mu} |f(j)| \sum_{l=k+1}^{k+N} l^{\mu-1} \le \frac{N \max_{j=1-\mu,\ldots,N-\mu} |f(j)|}{(k+1)^{1-\mu}}. \tag{4.2}$$

Note that the right-hand side tends to 0 as $k \to \infty$.

Now, we shall study G_1. Let us introduce the assumption

(H) There are $c, \beta > 0$ such that

$$|f(k+N) - f(k)| \le ck^{-\beta} \quad \forall k \in \mathbb{N}.$$

Assuming (H), Lemma 2.3 yields

$$|G_1(k)| \le c \sum_{l=1}^{k} l^{\mu-1}(k-l-\mu+1)^{-\beta} =$$

$$= \frac{c}{k^{1-\mu}(1-\mu)^\beta} + c \sum_{l=1}^{k-1} l^{\mu-1}(k-l-\mu+1)^{-\beta}.$$

We had to split the sum because of what follows. We are going to estimate $\sum_{l=1}^{k-1}$ by using $\int_0^{k-1}$. Function $x \mapsto x^{\mu-1}$ is decreasing on $(0, k-1]$ and greater than or equal to the piece-wise constant function $l \mapsto \lceil l \rceil^{\mu-1}$ where $\lceil \cdot \rceil$ is a ceiling function defined as $\lceil x \rceil = \min\{y \in \mathbb{Z} \mid y \ge x\}$ for any $x \in \mathbb{R}$. As an upper bound for the non-decreasing function $l \mapsto \lceil k-l-\mu+1 \rceil^{-\beta}$, $l \in [0, k-1]$ we use $x \mapsto (k-x-\mu)^{-\beta}$. So we obtain

$$|G_1(k)| \le \frac{c}{k^{1-\mu}(1-\mu)^\beta} + c \int_0^{k-1} x^{\mu-1}(k-x-\mu)^{-\beta} dx \le$$

$$\le \frac{c}{k^{1-\mu}(1-\mu)^\beta} + c \int_0^{k-\mu} x^{\mu-1}(k-x-\mu)^{-\beta} dx.$$

Substituting $x = (k-\mu)y$, we arrive at

$$|G_1(k)| \le \frac{c}{k^{1-\mu}(1-\mu)^\beta} + c(k-\mu)^{\mu-\beta} \int_0^1 y^{\mu-1}(1-y)^{-\beta} dy =$$

$$= \frac{c}{k^{1-\mu}(1-\mu)^\beta} + c(k-\mu)^{\mu-\beta} B(\mu, 1-\beta) \tag{4.3}$$

for each $k \in \mathbb{N}$, where $B(t, s)$ is the Euler beta function. Now we can formulate the main result of this section.

Theorem 4.1. *Let $N \in \mathbb{N}_2$. If f satisfies the assumption (H) with $\mu < \beta < 1$, then a solution u to (1.1) is S-asymptotically N-periodic. More precisely, there is a constant $C > 0$ such that*

$$|u(k+N) - u(k)| \le \frac{C}{(k-\mu)^{\beta-\mu}}$$

for all $k \in \mathbb{N}$ sufficiently large.

Proof. Applying estimations (4.3), (4.2) to (4.1) results in

$$|u(k+N) - u(k)| \le \frac{c}{\Gamma(\mu) k^{1-\mu}(1-\mu)^\beta} + \frac{cB(\mu, 1-\beta)}{\Gamma(\mu)(k-\mu)^{\beta-\mu}} +$$

$$+\frac{N\max_{j=1-\mu,\ldots,N-\mu}|f(j)|}{\Gamma(\mu)(k+1)^{1-\mu}}.$$

Clearly, the right-hand side tends to 0 if $k\to\infty$ and $\mu<\beta<1$. Finally, $(k-\mu)^{\mu-\beta}$ converges to 0 in the slowest way.

Theorem 4.1 is proved.

References

1. M. Fečkan, *Note on periodic solutions of fractional differential equations*, Math. Methods Appl. Sci., **41**, № 13, 5065 – 5073 (2018).
2. J. Diblík, M. Fečkan, M. Pospíšil, *Nonexistence of periodic solutions and S-asymptotically periodic solutions in fractional difference equations*, Appl. Math. Comput., **257**, 230 – 240 (2015).
3. G. A. Anastassiou, *Discrete fractional calculus and inequalities* (2009); http://arxiv.org/abs/0911.3370.
4. K. S. Miller, M. Ross, *Fractional difference calculus*, Univalent functions, fractional calculus, and their applications (Kōriyama, 1988), Ellis Horwood Ser. Math. Appl., Horwood, Chichester, 139 – 152 (1989).
5. F. M. Atici, P. W. Eloe, *A transform method in discrete fractional calculus*, Int. J. Difference Equ., **2**, № 2, 165 – 176 (2007).
6. F. M. Atici, P. W. Eloe, *Initial value problems in discrete fractional calculus*, Proc. Amer. Math. Soc., **137**, № 3, 981 – 989 (2009).
7. F. Chen, X. Luo, Y. Zhou, *Existence results for nonlinear fractional difference equation*, Adv. Difference Equ., **2011**, 1 – 12 (2011).
8. H. R. Henríquez, M. Pierri, P. Táboas, *On S-asymptotically ω-periodic functions on Banach spaces and applications*, J. Math. Anal. Appl., **343**, № 2, 1119 – 1130 (2008).
9. F. Qi, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl., **2010**, Art. ID 493058, 84 p. (2010).
10. J. G. Wendel, *Note on the gamma function*, Amer. Math. Monthly, **55**, 563 – 564 (1948).

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