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EXISTENCE OF POSITIVE SOLUTIONS FOR A POPULATION MODEL INVOLVING NONLOCAL OPERATOR

ІСНУВАННЯ ДОДАТНИХ РОЗВ'ЯЗКІВ МОДЕЛІ ПОПУЛЯЦІЇ, ЯКА МАЄ НЕЛОКАЛЬНИЙ ОПЕРАТОР

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Using the method of sub-supersolutions, we study the existence of positive solutions for a class of singular nonlinear semipositone systems involving nonlocal operator.

З використанням методу суб- і суперрозв'язків досліджено існування додатних розв'язків для одного класу сингулярних нелінійних напівпозитонних систем, які мають нелокальний оператор.

1. Introduction. We consider the existence of positive solutions of singular nonlinear semipositone problem of the form

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p dx\right) \operatorname{div}\left(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u\right) = \\ \qquad = |x|^{-(\alpha+1)p+\beta}\left(au^{p-1} - f(u) - \dfrac{c}{u^\gamma}\right), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N,\ N \geq 3$, with $0 \in \Omega,\ 1 < p < N,\ 0 \leq \alpha < \dfrac{N-p}{p}$, $\gamma \in (0,1)$, and $a,\ c,\ \beta$ are positive constants and $f\colon [0,\infty) \to \mathbb{R}$, are continuous functions and $M\colon [0,\infty] \to \mathbb{R}^+$, aside from being continuous and nondecreasing function and $0 < M_0 \leq \leq M(t) \leq M_\infty$ for all $t \in [0,\infty)$. This model arises in the studies of population biology of one species with u representing the concentration of the species. We use the method of sub-supersolutions to establish our results. We discuss the existence of positive solution when f satisfies certain additional conditions.

We make the following assumptions:

(A1) There exist $L > 0$ and $b > 0$ such that $f(u) < Lu^b$, for all $u \geq 0$.

(A2) There exists a constant $S > 0$ such that $au^{p-1} < f(u) + S$, for all $u \geq 0$.

(A3) There exist $t_2 > t_1 > 0$ such that $\dfrac{M(t_2)}{t_2^{\frac{2}{N-2}}} > \dfrac{M(t_1)}{t_1^{\frac{2}{N-2}}}$ [1].

A typical example of a function satisfying this condition is $M(t) = M_0 + at$ with $a \geq 0$ and for all $t \geq 0$. System (1.1) is related to the stationary problem of a model introduced by Kirchhoff [2]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int\limits_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where $\rho,\ P_0,\ h,\ E$ are all constants. This equation extends the classical d'Alembert wave equation. A distinguishing feature of equation (1.2) is that the equation has a nonlocal coefficient $\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$ which depends on the average $\frac{1}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$, hence the equation is no longer a pointwise identity. We refer to [3] for additional result on Kirchhoff equations. In recent years, there has been considerable progress on the study of nonlocal problems [4–6]. Nonlocal problems can be used for modeling, for example, physical and biological systems for which u describes a process which depends on the average of itself, such as the population density. On the other hand, elliptic problems involving more general operator, such as the degenerate quasilinear elliptic operator given by $-\operatorname{div}\big(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u\big)$, were motivated by Caffarelli, Kohn and Nirenberg's inequality [7–9]. The study of this type of problem is motivated by its various applications, for example, in fluid mechanics, in Newtonian fluids, in flow through porous media and in glaciology [10, 11].

More recently, reaction–diffusion models have been used to describe spatiotemporal phenomena in disciplines other than ecology, such as physics, chemistry, and biology [11–13]. In addition, most ecological systems have some form of predation or harvesting of the population, for example, hunting or fishing is often used as an effective means of wildlife management. This model describes the dynamics of the fish population with predation. In such cases u denotes the population density and the term $\frac{c}{u^\alpha}$ corresponds to predation. So, the study of positive solutions of (1.1) has more practical meanings. We refer to [14–16] for additional results on elliptic problems. So, the study of positive solutions of singular elliptic problems has more practical meanings. Let $\tilde{f}(u) = au^{p-1} - f(u) - \frac{c}{v^\gamma}$, Then $\lim_{u\to\infty}\tilde{f}(u) = -\infty$, and hence we refer to (1.1) as an infinite semipositone system. In [17], the authors discussed the single problem (1.1) when $M_1(t) \equiv 1,\ \alpha = 0,\ p = \beta = 2$, and see [18] for the single equation case when $M_1(t) \equiv 1$. Here we focus on further extending the study in [17, 18] for infinities semipositone Kirchhoff type systems involving singularity. Our approach is based on the method of sub-supersolutions [12, 14].

2. Preliminaries and existing result. In this paper, we denote $W_0^{1,p}\left(\Omega, |x|^{-\alpha p}\right)$, the completion of $C_0^\infty(\Omega)$, with respect to the norm $\|u\| = \left(\int_\Omega |x|^{-\alpha p}|\nabla u|^p dx\right)^{\frac{1}{p}}$. To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -\operatorname{div}\big(|x|^{-\alpha p}|\nabla\phi|^{p-2}\nabla\phi\big) = \lambda|x|^{-(\alpha+1)p+\beta}|\phi|^{p-2}\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Let $\phi_{1,p}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,p}$ of (2.1) such that $\phi_{1,p}(x) > 0$ in Ω and $\|\phi_{1,p}\|_\infty = 1$ [19, 20]. It can be shown that $\frac{\partial\phi_{1,p}}{\partial n} < 0$ on $\partial\Omega$. Here n is the outward normal. We will also consider the unique solution $\zeta_p(x) \in W_0^{1,p}\left(\Omega, |x|^{-\alpha p}\right)$ for the problem

$$\begin{cases} -\operatorname{div}\big(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u\big) = |x|^{-(\alpha+1)p+\beta}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result. It is well known that $\zeta_p(x) > 0$ in Ω and $\dfrac{\partial \zeta_p(x)}{\partial n} < 0$ on $\partial\Omega$ [19].

Now, we give the definition of weak solution and sub-supersolution of (1.1). A nonnegative function ψ is called a sub-solution of (1.1) if it satisfies $\psi \le 0$ on $\partial\Omega$ and

$$M\left(\int\limits_\Omega |x|^{-\alpha p}|\nabla\psi|^p dx\right)\int\limits_\Omega |x|^{-\alpha p}|\nabla\psi|^{p-2}\nabla\psi\cdot\nabla w dx \le$$

$$\le \int\limits_\Omega |x|^{-(\alpha+1)p+\beta}\left(a\psi^{p-1} - f(\psi) - \frac{c}{\psi^\gamma}\right) w dx,$$

and a nonnegative function z is called a super-solution of (1.1) if it satisfies $z \ge 0$ on $\partial\Omega$ and

$$M\left(\int\limits_\Omega |x|^{-\alpha p}|\nabla z|^p dx\right)\int\limits_\Omega |x|^{-\alpha p}|\nabla z|^{p-2}\nabla z\cdot\nabla w dx \ge$$

$$\ge \int\limits_\Omega |x|^{-(\alpha+1)p+\beta}\left(a z^{p-1} - f(z) - \frac{c}{z^\gamma}\right) w dx.$$

for all $w \in W = \{w \in C_0^\infty(\Omega) \mid w \ge 0,\ x \in \Omega\}$.

A key role in our arguments will be played by the following auxiliary result. Its proof is similar to those presented in [13], the reader can consult further the papers [16, 21].

Lemma 2.1. *Assume that $M : R^+ \to R^+$ is a continuous and increasing function satisfying*

$$M(t) \ge M_0 > 0 \quad \textit{for all} \quad t \in R^+.$$

If the functions $u, v \in W_0^{1,p}(\Omega)$ satisfies

$$M\left(\int\limits_\Omega |\nabla u|^p\, dx\right)\int\limits_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\, dx \le$$

$$\le M\left(\int\limits_\Omega |\nabla v|^p\, dx\right)\int\limits_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\, dx \tag{2.2}$$

for all $\varphi \in W_0^{1,p}(\Omega),\ \varphi \ge 0,$ then $u \le v$ in Ω.

Proof. Our proof is based on the arguments presented in [22, 23]. Define the functional $\Phi : W_0^{1,p}(\Omega) \to R$ by the formula

$$\Phi(u) := \frac{1}{p}\,\widehat{M}\left(\int\limits_{\Omega} |\nabla u|^p\,dx\right), \quad u \in W_0^{1,p}(\Omega).$$

It is obviously that the functional Φ is a continuously Gáteaux differentiable whose Gáteaux derivative at the point $u \in W_0^{1,p}(\Omega)$ is the functional $\Phi' \in W_0^{-1,p}(\Omega),$ given by

$$\Phi'(u)(\varphi) = M\left(\int\limits_{\Omega} |\nabla u|^p\,dx\right)\int\limits_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dx, \quad \varphi \in W_0^{1,p}(\Omega).$$

It is obvious that Φ' is continuous and bounded since the function M is continuous. We will show that Φ' is strictly monotone in $W_0^{1,p}(\Omega)$. Indeed, for any $u, v \in W_0^{1,p}(\Omega),\ u \neq v,$ without loss of generality, we may assume that

$$\int\limits_{\Omega} |\nabla u|^p\,dx \geq \int\limits_{\Omega} |\nabla v|^p\,dx$$

(otherwise, changing the role of u and v in the following proof). Therefore, we have

$$M\left(\int\limits_{\Omega} |\nabla u|^p\,dx\right) \geq M\left(\int\limits_{\Omega} |\nabla v|^p\,dx\right) \tag{2.3}$$

since $M(t)$ is a monotone function. By using Cauchy's inequality, we have

$$\nabla u \cdot \nabla v \leq |\nabla u||\nabla v| \leq \frac{1}{2}\left(|\nabla u|^2 + |\nabla v|^2\right). \tag{2.4}$$

By using (2.4), we get

$$\int\limits_{\Omega} |\nabla u|^p\,dx - \int\limits_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx \geq \frac{1}{2}\int\limits_{\Omega} |\nabla u|^{p-2}\left(|\nabla u|^2 - |\nabla v|^2\right)dx \tag{2.5}$$

and

$$\int\limits_{\Omega} |\nabla v|^p\,dx - \int\limits_{\Omega} |\nabla v|^{p-2}\nabla v \cdot \nabla u\,dx \geq \frac{1}{2}\int\limits_{\Omega} |\nabla v|^{p-2}\left(|\nabla v|^2 - |\nabla u|^2\right)dx. \tag{2.6}$$

If $|\nabla u| \geq |\nabla v|$, by using (2.3) – (2.6), we obtain

$$I_1 := \Phi'(u)(u) - \Phi'(u)(v) - \Phi'(v)(u) + \Phi'(v)(v) =$$

$$= M\left(\int_\Omega |\nabla u|^p\, dx\right)\left(\int_\Omega |\nabla u|^p\, dx - \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\, dx\right) -$$

$$- M\left(\int_\Omega |\nabla v|^p\, dx\right)\left(\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u\, dx - \int_\Omega |\nabla v|^p\, dx\right) \geq$$

$$\geq \frac{1}{2} M\left(\int_\Omega |\nabla u|^p\, dx\right)\int_\Omega |\nabla u|^{p-2}\left(|\nabla u|^2 - |\nabla v|^2\right) dx -$$

$$- \frac{1}{2} M\left(\int_\Omega |\nabla v|^p\, dx\right)\int_\Omega |\nabla u|^{p-2}\left(|\nabla u|^2 - |\nabla v|^2\right) dx =$$

$$= \frac{1}{2} M\left(\int_\Omega |\nabla v|^p\, dx\right)\int_\Omega \left(|\nabla u|^{p-2} - |\nabla v|^{p-2}\right)\left(|\nabla u|^2 - |\nabla v|^2\right) dx \geq$$

$$\geq \frac{M_0}{2}\int_\Omega \left(|\nabla u|^{p-2} - |\nabla v|^{p-2}\right)\left(|\nabla u|^2 - |\nabla v|^2\right) dx. \tag{2.7}$$

If $|\nabla v| \geq |\nabla u|$, changing the role of u and v in (2.3) – (2.6), we have

$$I_2 := \Phi'(v)(v) - \Phi'(v)(u) - \Phi'(u)(v) + \Phi'(u)(u) =$$

$$= M\left(\int_\Omega |\nabla v|^p\, dx\right)\left(\int_\Omega |\nabla v|^p\, dx - \int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u\, dx\right) -$$

$$- M\left(\int_\Omega |\nabla u|^p\, dx\right)\left(\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\, dx - \int_\Omega |\nabla u|^p\, dx\right) \geq$$

$$\geq \frac{1}{2} M\left(\int_\Omega |\nabla v|^p\, dx\right)\int_\Omega |\nabla v|^{p-2}\left(|\nabla v|^2 - |\nabla u|^2\right) dx -$$

$$- \frac{1}{2} M\left(\int_\Omega |\nabla u|^p\, dx\right)\int_\Omega |\nabla u|^{p-2}\left(|\nabla v|^2 - |\nabla u|^2\right) dx =$$

$$= \frac{1}{2} M\left(\int\limits_\Omega |\nabla v|^p\,dx\right)\int\limits_\Omega \left(|\nabla v|^{p-2} - |\nabla u|^{p-2}\right)\left(|\nabla v|^2 - |\nabla u|^2\right)dx \geq$$

$$\geq \frac{M_0}{2}\int\limits_\Omega \left(|\nabla v|^{p-2} - |\nabla u|^{p-2}\right)\left(|\nabla v|^2 - |\nabla u|^2\right)dx. \tag{2.8}$$

From (2.7) and (2.8) we get

$$\left(\Phi'(u) - \Phi'(v)\right)(u - v) = I_1 = I_2 \geq 0 \quad \forall u, v \in W_0^{1,p}(\Omega).$$

Moreover, if $u \neq v$ and $(\Phi'(u) - \Phi'(v))(u - v) = 0,$ then we obtain

$$\int\limits_\Omega \left(|\nabla u|^{p-2} - |\nabla v|^{p-2}\right)\left(|\nabla u|^2 - |\nabla v|^2\right)dx = 0,$$

so $|\nabla u| = |\nabla v|$ in Ω. Thus, we deduce that

$$\left(\Phi'(u) - \Phi'(v)\right)(u - v) = \Phi'(u)(u - v) - \Phi'(v)(u - v) =$$

$$= M\left(\int\limits_\Omega |\nabla u|^p\,dx\right)\int\limits_\Omega |\nabla u|^{p-2}|\nabla u - \nabla v|^2\,dx = 0, \tag{2.9}$$

i.e., $u - v$ is a constant. In view of $u = v = 0$ on $\partial\Omega$ we have $u \equiv v$ which is contrary with $u \neq v$. Therefore, $(\Phi'(u) - \Phi'(v))(u - v) > 0$ and Φ' is strictly monotone in $W_0^{1,p}(\Omega)$.

Let $u,\ v$ be two functions such that (2.2) is verified. Taking $\varphi = (u - v)^+,$ the positive part of $u - v,$ as a test function of (2.2), we have

$$\left(\Phi'(u) - \Phi'(v)\right)(\varphi) = M\left(\int\limits_\Omega |\nabla u|^p\,dx\right)\int\limits_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx -$$

$$- M\left(\int\limits_\Omega |\nabla v|^p\,dx\right)\int\limits_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx \leq 0. \tag{2.10}$$

Relations (2.9) and (2.10) mean that $u \leq v.$

Then the following result holds:

Lemma 2.2. *Suppose there exist sub and super-solutions ψ and z, respectively, of (1.1) such that $\psi \leq z$. Then (1.1) has a solution u such that $\psi \leq u \leq z$.*

We are now ready to give our existence result.

Theorem 2.1. *Assume (A1) – (A3) hold, that if $\dfrac{a}{M_\infty} > \left(\dfrac{p}{p-1+\gamma}\right)^{p-1}\lambda_{1,p},$ then there exists $c_0 > 0$ such that if $0 < c < c_0,$ then the system (1.1) admits a positive solution.*

Proof. We start with the construction of a positive subsolution for (1.1). To get a positive subsolution, we can apply an anti-maximum principle [15], from which we know that there exist a $\delta_1 > 0$ and a solution z_λ of

$$\begin{cases} -\operatorname{div}\big(|x|^{-\alpha p}|\nabla z|^{p-2}\nabla z\big) = |x|^{-(\alpha+1)p+\beta}(\lambda z^{p-1} - 1), & x \in \Omega, \\ z = 0, & x \in \partial\Omega, \end{cases}$$

for $\lambda \in (\lambda_{1,p}, \lambda_{1,p} + \delta_1)$. Fix

$$\hat{\lambda} \in \left(\lambda_{1,p}, \min\left\{\left(\frac{p-1+\gamma}{p}\right)a, \lambda_{1,p} + \delta_1\right\}\right).$$

Let $\theta = \|z_{\hat{\lambda}}\|$. It is well known that $z_{\hat{\lambda}} > 0$ in Ω and $\dfrac{\partial z_{\hat{\lambda}}}{\partial n} < 0$ on $\partial\Omega$, where n is the outer unit normal to Ω. Hence, there exist positive constants $\epsilon,\ \delta,\ \sigma$ such that

$$|x|^{-\alpha p}|\nabla z_{\hat{\lambda}}|^p \geq \epsilon, \quad x \in \overline{\Omega_\delta}, \tag{2.11}$$

$$z_{\hat{\lambda}} \geq \sigma, \quad x \in \Omega_0 = \Omega \backslash \overline{\Omega_\delta},$$

where $\overline{\Omega_\delta} = \big\{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\big\}$. Choose $\eta_1, \eta_2 > 0$ such that $\eta_1 \leq \min|x|^{-(\alpha+1)p+\beta}$ and $\eta_2 \geq \max|x|^{-(\alpha+1)p+\beta}$ in $\overline{\Omega}_\delta$. We construct a subsolution ψ of (1.1), by using $z_{\hat{\lambda}}$. Define

$$\psi = M\left(\frac{p-1+\gamma}{p}\right) z_{\hat{\lambda}}^{\frac{p}{p-1+\gamma}},$$

where

$$M = \min\left\{\left(\frac{M_\infty\left(\dfrac{p}{p-1+\gamma}\right)^b \theta^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}}}{L\theta^{\frac{pb}{p-1+\gamma}}}\right)^{\frac{1}{b-p+1}},\right.$$

$$\left.\left(\frac{\left(\dfrac{p-1}{Lp}\right)\theta^{\frac{p(p-1)}{p-1+\gamma}}\left[\left(\dfrac{p-1+\gamma}{p}\right)^{p-1} a_1 - M_\infty\hat{\lambda}\right]}{\left(\dfrac{p-1+\gamma}{p}\right)^b \theta^{\frac{pb}{p-1+\gamma}}}\right)^{\frac{1}{b-p+1}}\right\}.$$

Let $w \in W$. Then a calculation shows that

$$\nabla\psi = M z_{\hat{\lambda}}^{\frac{1-\gamma}{p-1+\gamma}} \nabla z_{\hat{\lambda}},$$

$$M\left(\int_\Omega |\nabla\psi|^p dx\right)\int_\Omega |x|^{-\alpha p}|\nabla\psi|^{p-2}\nabla\psi\cdot\nabla w dx \le$$

$$\le M_\infty M^{p-1}\int_\Omega |x|^{-\alpha p} z_{\hat\lambda}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}}\left|\nabla z_{\hat\lambda}\right|^{p-2}\nabla z_{\hat\lambda}\nabla w dx =$$

$$= M_\infty M^{p-1}\int_\Omega |x|^{-\alpha p}|\nabla z_{\hat\lambda}|^{p-2}\nabla z_{\hat\lambda}\left[\nabla\left(z_{\hat\lambda}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}} w\right)-\left(\nabla z_{\hat\lambda}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}}\right)w\right]dx =$$

$$= M_\infty M^{p-1}\int_\Omega \left[|x|^{-(\alpha+1)p+\beta} z_{\hat\lambda}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}}\left(\hat\lambda z_{\hat\lambda}^{p-1}-1\right)-\right.$$

$$\left.-|x|^{-\alpha p}\frac{(1-\gamma)(p-1)}{p-1+\gamma}\frac{|\nabla z_{\hat\lambda}|^p}{z_{\hat\lambda}^{\frac{\gamma p}{p-1+\gamma}}}\right]w dx =$$

$$= M_\infty\int_\Omega\left[|x|^{-(\alpha+1)p+\beta}M^{p-1}\hat\lambda z_{\hat\lambda}^{\frac{p(p-1)}{p-1+\gamma}}-|x|^{-(\alpha+1)p+\beta}M^{p-1}z_{\hat\lambda}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}}-\right.$$

$$\left.-|x|^{-\alpha p}M^{p-1}\frac{(1-\gamma)(p-1)}{p-1+\gamma}\frac{|\nabla z_{\hat\lambda}|^p}{z_{\hat\lambda}^{\frac{\gamma p}{p-1+\gamma}}}\right]w dx,$$

and

$$\int_\Omega |x|^{-(\alpha+1)p+\beta}\left[a\psi^{p-1}-f(\psi)-\frac{c}{\psi^\gamma}\right]w dx =$$

$$= \int_\Omega\left[|x|^{-(\alpha+1)p+\beta}aM^{p-1}\left(\frac{p-1+\gamma}{p}\right)^{p-1} z_{\hat\lambda}^{\frac{p(p-1)}{p-1+\gamma}}-\right.$$

$$-|x|^{-(\alpha+1)p+\beta}f_1\left(M\left(\frac{p-1+\gamma}{p}\right)z_{\hat\lambda}^{\frac{p}{p-1+\gamma}}\right)-$$

$$\left.-|x|^{-(\alpha+1)p+\beta}\frac{c}{M^\gamma\left(\frac{p-1+\gamma}{p}\right)^\gamma z_{\hat\lambda}^{\frac{\gamma p}{p-1+\gamma}}}\right]w\,dx.$$

Let

$$c_0 = M^{p-1+\gamma}\min\left\{\frac{M_\infty(1-\gamma)(p-1)}{p-1+\gamma}\left(\frac{p-1+\gamma}{p}\right)^\gamma\frac{\epsilon}{\eta_2},\right.$$

$$\left.\frac{1}{p}\left(\frac{p-1+\gamma}{p}\right)^\gamma\sigma^p\left[\left(\frac{p-1+\gamma}{p}\right)a-M_\infty\widehat\lambda\right]\right\}.$$

First we consider the case when $x \in \overline{\Omega}_\delta$. We have $|x|^{-\alpha p}|\nabla z_{\hat{\lambda}}| \geq \epsilon$ on $\overline{\Omega}_\delta$. Since $M_\infty\left(\dfrac{p}{p-1+\gamma}\right)^{p-1}\hat{\lambda} \leq a,$ we get

$$|x|^{-(\alpha+1)p+\beta} M_\infty M^{p-1}\hat{\lambda} z_{\hat{\lambda}}^{\frac{p(p-1)}{p-1+\gamma}} \leq$$

$$\leq |x|^{-(\alpha+1)p+\beta} a M^{p-1}\left(\frac{p-1+\gamma}{p}\right)^{p-1} z_{\hat{\lambda}}^{\frac{p(p-1)}{p-1+\gamma}}, \tag{2.12}$$

and from the choice of $M,$ we arrive at

$$L M^{b-p+1}\theta^{\frac{pb}{p-1+\gamma}} \leq M_\infty\left(\frac{p}{p-1+\gamma}\right)^{b}\theta^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}}. \tag{2.13}$$

By (2.13) and $(A_1),$ we find

$$-\,|x|^{-(\alpha+1)p+\beta} M_\infty M^{p-1} z_{\hat{\lambda}}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}} \leq$$

$$\leq -|x|^{-(\alpha+1)p+\beta} L M^{b}\left(\frac{p-1+\gamma}{p}\right)^{b} z_{\hat{\lambda}}^{\frac{pb}{p-1+\gamma}} \leq$$

$$\leq -|x|^{-(\alpha+1)p+\beta} f_1\left(M\left(\frac{p-1+\gamma}{p}\right) z_{\hat{\lambda}}^{\frac{p}{p-1+\gamma}}\right). \tag{2.14}$$

Next, from (2.11) and definition of $c_0,$ we obtain

$$|x|^{-\alpha p} M_\infty M^{p-1}\frac{(1-\gamma)(p-1)}{p-1+\gamma}|\nabla z_{\hat{\lambda}}|^{p} \geq |x|^{-(\alpha+1)p+\beta}\frac{c}{M^{\gamma}\left(\dfrac{p-1+\gamma}{p}\right)^{\gamma}}$$

and

$$-\,|x|^{-\alpha p} M_\infty M^{p-1}\frac{(1-\gamma)(p-1)}{p-1+\gamma}\frac{|\nabla z_{\hat{\lambda}}|^{p}}{z_{\hat{\lambda}}^{\frac{\gamma p}{p-1+\gamma}}} \leq$$

$$\leq -|x|^{-(\alpha+1)p+\beta}\frac{c}{M^{\gamma}\left(\dfrac{p-1+\gamma}{p}\right)^{\gamma} z_{\hat{\lambda}}^{\frac{\gamma p}{p-1+\gamma}}}. \tag{2.15}$$

Hence, by using (2.12), (2.14), and (2.15), for $c \leq c_0,$ we go to

$$M\left(\int\limits_{\overline{\Omega}_\delta}|\nabla\psi|^{p}dx\right)\int\limits_{\overline{\Omega}_\delta}|x|^{-\alpha p}|\nabla\psi|^{p-2}\nabla\psi\cdot\nabla w\,dx \leq$$

$$\leq \int\limits_{\overline{\Omega}_\delta}\Bigg[|x|^{-(\alpha+1)p+\beta} a M^{p-1}\left(\frac{p-1+\gamma}{p}\right)^{p-1} z_{\hat{\lambda}}^{\frac{p(p-1)}{p-1+\gamma}} -$$

$$- |x|^{-(\alpha+1)p+\beta} f\left(M\left(\frac{p-1+\gamma}{p}\right) z_{\hat\lambda}^{\frac{p}{p-1+\gamma}}\right) -$$

$$- |x|^{-(\alpha+1)p+\beta} \frac{c}{M^\gamma \left(\frac{p-1+\gamma}{p}\right)^\gamma z_{\hat\lambda_1}^{\frac{\gamma p}{p-1+\gamma}}}\Bigg] w dx =$$

$$= \int\limits_{\overline{\Omega}_\delta} |x|^{-(\alpha+1)p+\beta} \left[a\psi^{p-1} - f(\psi) - \frac{c}{\psi^\gamma}\right] w dx. \tag{2.16}$$

On the other hand, on $\Omega_0 = \Omega \backslash \overline{\Omega}_\delta,$ we have $z_{\hat\lambda} \geq \sigma$ for some $0 < \sigma < 1.$ From the definition of $c_0,$ for $c \leq c_0,$ we get

$$\frac{c}{M^\gamma \left(\frac{p-1+\gamma}{p}\right)^\gamma} \leq \frac{1}{p} M^{p-1} \sigma^p \left[\left(\frac{p-1+\gamma}{p}\right)^{p-1} a - M_\infty \hat\lambda\right] \leq$$

$$\leq \frac{1}{p} M^{p-1} z_{\hat\lambda}^p \left[\left(\frac{p-1+\gamma}{p}\right)^{p-1} a - M_\infty \hat\lambda\right]. \tag{2.17}$$

Also, from the choice of $M,$ we obtain

$$LM^{b-p+1} \left(\frac{p-1+\gamma}{p}\right)^b z_{\hat\lambda}^{\frac{pb}{p-1+\gamma}} \leq z_{\hat\lambda}^{\frac{p(p-1)}{p-1+\gamma}} \frac{p-1}{p} \left[\left(\frac{p-1+\gamma}{p}\right)^{p-1} a - M_\infty \hat\lambda\right]. \tag{2.18}$$

Hence, from (2.17) and (2.18), we have

$$M\left(\int\limits_{\Omega_0} |\nabla\psi|^p dx\right) \int\limits_{\Omega_0} |x|^{-\alpha p} |\nabla\psi|^{p-2} \nabla\psi \nabla w\, dx \leq$$

$$\leq M_\infty \int\limits_{\Omega_0} \Bigg[|x|^{-(\alpha+1)p+\beta} M^{p-1} \hat\lambda z_{\hat\lambda}^{\frac{p(p-1)}{p-1+\gamma}} - |x|^{-(\alpha+1)p+\beta} M^{p-1} z_{\hat\lambda}^{\frac{(1-\gamma)(p-1)}{p-1+\gamma}} -$$

$$- |x|^{-\alpha p} M^{p-1} \frac{(1-\gamma)(p-1)}{p-1+\gamma} \frac{|\nabla z_{\hat\lambda}|^p}{z_{\hat\lambda}^{\frac{\gamma p}{p-1+\gamma}}}\Bigg] w\, dx \leq$$

$$\leq M_\infty \int\limits_{\Omega_0} |x|^{-(\alpha+1)p+\beta} M^{p-1} \hat\lambda z_{\hat\lambda}^{\frac{p(p-1)}{p-1+\gamma}} w\, dx =$$

$$= M_\infty \int\limits_{\Omega_0} |x|^{-(\alpha+1)p+\beta} \frac{1}{z_{\hat\lambda}^{\frac{\gamma p}{p-1+\gamma}}} \left[\frac{1}{p} \hat\lambda M^{p-1} z_{\hat\lambda}^p + \frac{p-1}{p} \hat\lambda M^{p-1} z_{\hat\lambda}^p\right] w\, dx \leq$$

$$\leq \int\limits_{\Omega_0} |x|^{-(\alpha+1)p+\beta} \frac{1}{z_{\hat{\lambda}}^{\frac{\gamma p}{p-1+\gamma}}} \left[\left(\frac{1}{p} M^{p-1} \left(\frac{p-1+\gamma}{p} \right)^{p-1} a z_{\hat{\lambda}}^{p} - \frac{c}{M^{\gamma} \left(\frac{p-1+\gamma}{p} \right)^{\gamma}} \right) + \right.$$

$$+ M^{p-1} z_{\hat{\lambda}}^{p} \left(\frac{p-1+\gamma}{p} \right)^{p-1} \times$$

$$\left. \times \left(\frac{(p-1)a}{p} - L M^{b-p+1} \left(\frac{p-1+\gamma}{p} \right)^{b} \left(\frac{p-1+\gamma}{p} \right)^{1-p} \frac{z_{\hat{\lambda}}^{\frac{pb}{p-1+\gamma}}}{z_{\hat{\lambda}}^{\frac{p(p-1)}{p-1+\gamma}}} \right) \right] w dx =$$

$$= \int\limits_{\Omega_0} |x|^{-(\alpha+1)p+\beta} \left[a M^{p-1} \left(\frac{p-1+\gamma}{p} \right)^{p-1} z_{\hat{\lambda}}^{\frac{p(p-1)}{p-1+\gamma}} - \right.$$

$$\left. - L M^{b} \left(\frac{p-1+\gamma}{p} \right)^{b} z_{\hat{\lambda}}^{\frac{pb}{p-1+\gamma}} - \frac{b_1 z_{\hat{\lambda}}^{\frac{-\gamma p}{p-1+\gamma}}}{M^{\gamma} \left(\frac{p-1+\gamma}{p} \right)^{\gamma}} \right] w\, dx \leq$$

$$\leq \int\limits_{\Omega_0} |x|^{-(\alpha+1)p+\beta} \left[a M^{p-1} \left(\frac{p-1+\gamma}{p} \right)^{p-1} z_{\hat{\lambda}}^{\frac{p(p-1)}{p-1+\gamma}} - \right.$$

$$\left. - f \left(M \left(\frac{p-1+\gamma}{q} \right) z_{\hat{\lambda}}^{\frac{p}{p-1+\gamma}} \right) - \frac{c}{M^{\gamma} \left(\frac{p-1+\gamma}{p} \right)^{\gamma} z_{\hat{\lambda}}^{\frac{\gamma p}{p-1+\gamma}}} \right] w\, dx =$$

$$= \int\limits_{\Omega_0} |x|^{-(\alpha+1)p+\beta} \left[a\psi^{p-1} - f(\psi) - \frac{c}{\psi^{\gamma}} \right] w\, dx. \tag{2.19}$$

By using (2.16) and (2.19), we see that ψ is a sub-solution of (1.1).

Next, we construct a super-solution z of (1.1) such that $z \geq \psi$. By (A_2) and choose a large constant S^*, such that $au^{p-1} - f(u) - \dfrac{c}{u^{\gamma}} \leq S^*$, for all $u > 0$. Let

$$z = \left(\frac{S^*}{M_0} \right)^{\frac{1}{p-1}} \zeta_p(x).$$

We shall verify that z is a super-solution of (1.1). To this end, let $w \in W$. Then we have

$$M \left(\int\limits_{\Omega} |\nabla z|^p dx \right) \int\limits_{\Omega} |x|^{-\alpha p} |\nabla z|^{p-2} \nabla z \nabla w\, dx \geq$$

$$\geq S^* \int_\Omega |x|^{-(\alpha+1)p+\beta} w\,dx \geq$$

$$\geq \int_\Omega |x|^{-(\alpha+1)p+\beta} \left[az^{p-1} - f(z) - \frac{c}{z^\gamma}\right] w\,dx.$$

Thus, z is a super-solution of (1.1). Finally, we can choose $S^* \gg 1$ such that $\psi \leq z$ in Ω. Hence, if $c \leq c_0$, by Lemma 2.1, there exists a positive solution u of (1.1) such that $\psi \leq z$.

This completes the proof of Theorem 2.1.

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