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ON THE QUALITATIVE ANALYSIS OF SOLUTIONS OF A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND-ORDER WITH CONSTANT DELAY

ЯКІСНИЙ АНАЛІЗ РОЗВ'ЯЗКІВ НЕЛІНІЙНИХ ДИФЕНЦІАЛЬНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ ЗІ СТАЛИМ ЗАПІЗНЕННЯМ

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A class of nonlinear second-order differential equations with constant delay is considered. Qualitative properties of solutions, namely, global stability of zero solution, eventually uniform boundedness of solutions, existence of periodic solutions and existence of a unique stationary oscillation of the considered equations, are investigated. As techniques of the proofs, the Lyapunov – Krasovskii functional method and the second Lyapunov method are used to prove the main results of this paper. In this paper, we improve and correct some former results, which are available in the literature. Finally, in particular cases, we provide three examples for illustrations and applications of the obtained new results. Hence, we have some contributions to the topic of the paper.

Розглянуто клас нелінійних дифенціальних рівнянь другого порядку зі сталим запізненням. Досліджено якісні властивості розв'язків, а саме глобальну стійкість нульового розв'язку, кінцеву однорідну обмеженість розв'язків, існування періодичних розв'язків і єдиного стаціонарного коливання розглянутих рівнянь. Основні результати роботи отримано з використанням функціонального методу Ляпунова – Красовського та другого методу Ляпунова. Покращено й виправлено деякі відомі результати. Як ілюстрації нових результатів розглянуто три приклади.

1. Introduction. Mathematical models as delay differential equations (DDEs) of second order have many applications in various fields of sciences, engineering and so on. In generally, solving that kind of equations is a hard problem, except numerically. However, during the investigations of that kind of mathematical models, it is needed to have information about behaviors of solutions of that kind of equations without prior information of solutions. In the relevant literature, two methods called Lyapunov – Krasovskii functional (LKF) method and the second method of Lyapunov are stand out and they are very effective to study qualitative properties of solutions delay differential equations of higher order (see the books of Burton [1], Hale [2], Yoshizawa [3] and Krasovskii [4]).

In [5], Peng considered the following nonlinear DDE of second order:

$$x''(t) + f(x(t), x'(t)) + g(x(t), x'(t))\psi(x(t-\tau)) = p(t), \tag{1}$$

where $x \in \mathbb{R},\ \mathbb{R} = (-\infty, \infty),\ t \in \mathbb{R}^+,\ \mathbb{R}^+ = [0, \infty),\ \tau$ is a positive constant, i.e., the constant delay satisfies, $t - \tau \geq 0,\ \psi \in C^1(\mathbb{R}, \mathbb{R}),\ f \in C(\mathbb{R}^2, \mathbb{R}),\ g \in C^1\left(\mathbb{R}^2, \mathbb{R}\right)$ and $p \in C(\mathbb{R}^+, \mathbb{R})$ such that $g(0,0)\psi(0) = 0$ and $f(0,0) = 0$. Under these assumptions, the DDE (1) includes the zero solution when $p(t) = 0$. We should mention that the continuity of the functions $f,\ g,\ \psi$, and p is a sufficient condition for the existence of the solutions of DDE (1). Next, we assume that the functions $f,\ g$ and ψ satisfy the Lipschitz condition with respect to the dependent variable x and its derivative x'. Hence, via this assumption, the uniqueness of the solutions of DDE (1) is guaranteed depending on the proper initial conditions

In [5], Peng proved four new and interesting theorems on the various qualitative properties of the solutions of DDE (1). The mentioned properties include the globally stability of zero solution of DDE (1) when $p(t) = 0$, the eventually uniform boundedness of solutions, existence of the periodic solutions and the existence of unique one stationary oscillation of DDE (1) when $p(t) \neq 0$.

To perform the aim of the paper [5], LKF and Lyapunov function (LF) were defined and then used as basic tools by Peng [5]. Based on that the LKF and the LF, four new and interesting theorems ([5], Theorems 1 – 4) were proved on global stability of the zero solution by the LKF approach, eventually uniform boundedness of solutions, existence of the periodic solutions and existence and uniqueness of the stationary oscillation by the second method of Lyapunov, respectively.

As for the motivation of this paper, it comes from the results of Peng [5] (Theorems 1 – 4).

To the best of information of the authors of this paper, in the general cases, the results of [5] (Theorems 1 – 4) are not correct for DDE (1). Here, our aim is to show the mistakes in [5] and to correct the results of Peng [2] (Theorems 1 – 4) for some particular cases.

Indeed, for the results of [5] to be correct, the function g in DDE (1) must be independent from the second variable, x', i.e., it is needed that $g(x(t), x'(t)) = g(x(t))$ must be satisfied. We assume that $y(t) = \dot{x}(t)$. Hence, DDE (1) can be transformed to the following system:

$$\begin{aligned}
&\dot{x}(t) = y(t),\\
&\dot{y}(t) = -f(x(t), y(t)) - g(x(t), y(t))\psi(x(t)) +\\
&\qquad\qquad + g(x(t), y(t)) \int\limits_{-\tau}^{0} \psi'_x(x(t+\eta))y(t+\eta)d\eta + p(t).
\end{aligned} \tag{2}$$

Through this paper, when we need without mention, $x(t)$ and $y(t)$ will be represented by x and y, respectively.

2. Qualitative results of DDE (1). We now present the first theorem of Peng [2] (Theorem 1) on the globally stability of the zero solution of DDE (1), which has been proved by the LKF approach (see Burton [1], Graef and Tunç [6] (Theorem 2.1), Hale [2], Sinha [7] (Lemma 1), Yoshizawa [3, p. 202] (Theorem 35.4), Krasovskii [4]).

Theorem 1 ([5], Theorem 1)**.** *Let* $p(t) \equiv 0,\ f(0,0) = 0,\ g(0,0)\psi(0) = 0$ *and the following conditions are satisfied:*

(A1) $g(x,y)\psi(x) \neq 0$ *if* $x \neq 0,\ y \neq 0,$
$g(x,y)\psi(x)x > 0$ *for all* $x \neq 0$ *as* $y \in \mathbb{R}$;

(A2) $\int_0^x g(\xi,y)\psi(\xi)d\xi \to +\infty$ *for all* $y \in \mathbb{R}$ *as* $|x| \to +\infty$;

(A3) *there exist a positive constant* C *such that*

$$\left|g(x,y)\psi'(z)\right| \leq C \quad \textit{for all} \quad x,y,z \in \mathbb{R};$$

(A4) $C\tau y^2 < f(x,y)$ *for all* $y \neq 0$ *as* $x \in \mathbb{R}$, *where* C *is the positive constant from (A3).*

Then, the zero solution of DDE (1) is globally stable.

Remark 1. To prove Theorem 1, the second method of Lyapunov is used by Peng [5]. For this case, the author defined the following LKF $V = V(x_t,y_t)$:

$$V(x_t,y_t) = \frac{1}{2}y^2(t) + \int\limits_0^{x(t)} g(\xi,y(t))\psi(\xi)d\xi + \frac{1}{2}C\int\limits_{-\tau}^{0}\int\limits_{t+\eta}^{t} y^2(\xi)d\xi d\eta. \tag{3}$$

As for the next step, it is clear from (3) that

$$V(x_t,y_t) \geq 0, V(x_t,y_t) = 0 \quad \text{if and only if} \quad x(t) = y(t) \equiv 0.$$

We can also obtain from condition (A2) that

$$V(x_t,y_t) \to \infty \quad \text{as} \quad x^2 + y^2 \to \infty.$$

For the case $p(t) \equiv 0$ in DDE (1), Peng [5] (Theorem 1) calculated the time derivative of the LKF $V(x_t,y_t)$ along the solutions of the system (2) and the author obtained the following relation:

$$\begin{aligned}
\frac{d}{dt}V(x_t,y_t) = y(t)\left[-f(x(t),y(t)) - g(x(t),y(t))\psi(x(t))\right] + \\
+ y(t)g(x(t),y(t))\int\limits_{-\tau}^{0}\psi'_x(x(t+\eta))y(t+\eta)d\eta + \\
+ g(x(t),y(t))\psi(x(t))y(t) + \frac{1}{2}C\int\limits_{-\tau}^{0}\left[y^2(t) - y^2(t+\eta)\right]d\eta = \\
= -f(x(t),y(t))y(t) - g(x(t),y(t))\psi(x(t))y(t) + \\
+ y(t)g(x(t),y(t))\int\limits_{-\tau}^{0}\psi'_x(x(t+\eta))y(t+\eta)d\eta + \\
+ g(x(t),y(t))\psi(x(t))y(t) + \frac{1}{2}C\int\limits_{-\tau}^{0}\left[y^2(t) - y^2(t+\eta)\right]d\eta =
\end{aligned}$$

$$= -f(x(t), y(t))y(t) + y(t)g(x(t), y(t)) \int_{-\tau}^{0} {\psi'}_x(x(t+\eta))y(t+\eta)d\eta +$$

$$+ \frac{1}{2} C \int_{-\tau}^{0} \left[y^2(t) - y^2(t+\eta) \right] d\eta. \tag{4}$$

Now, it can be noted that the above time derivative (4) of the LKF $V(x_t, y_t)$ along solutions of the system (2) is not correct. Namely, let us consider the term

$$\int_0^{x(t)} g(\xi, y(t))\psi(\xi)d\xi,$$

which is contained in (3).

If we calculate the time derivative of this term, then it follows that

$$\frac{d}{dt} \int_0^{x(t)} g(\xi, y(t))\psi(\xi)d\xi = g(x(t), y(t))\psi(x(t))y(t) + \int_0^{x} \frac{\partial g(\xi, y)}{\partial y} \frac{dy}{dt} \psi(\xi)d\xi,$$

where

$$\frac{dy}{dt} = -f(x(t), y(t)) - g(x(t), y(t))\psi(x(t)) + g(x(t), y(t)) \int_{-\tau}^{0} {\psi'}_x(x(t+\eta))y(t+\eta)d\eta. \tag{5}$$

From this point of view, it follows that the three terms of (5) are included in the derivative of the integral term

$$\int_0^{x(t)} \frac{\partial g(\xi, y(t))}{\partial y} \frac{dy}{dt} \psi(\xi)d\xi.$$

However, instead of this fact, Peng [5] calculated the time derivative of the above integral term as the following:

$$\frac{d}{dt} \int_0^{x(t)} g(\xi, y(t))\psi(\xi)d\xi = g(x(t), y(t))\psi(x(t))y(t). \tag{6}$$

Indeed, the equality (6) is not correct, and not complete, too. That is, the time derivative of the LKF $V(x_t, y_t)$, which given by (4), does not include the term $\int_0^{x(t)} \frac{\partial g(\xi, y(t))}{\partial y} \frac{dy}{dt} \psi(\xi)d\xi$ with the three terms of $\frac{dy}{dt}$ in (5). We think that probably during the calculation of the time derivative of the integral term in (6), Peng [5] considered the variable y in the integral $\int_0^{x(t)} g(\xi, y(t))\psi(\xi)d\xi$ as a constant. This idea can lead a wrong and lack calculation of the time derivative of the

LKF $V(x_t, y_t)$. Hence, the result and idea of Theorem 1 is not correct for the general cases, when $g = g(x, y)$.

In view of the above discussion, the result of Theorem 1 can be corrected for the particular case when $g = g(x)$ in DDE (1). Hence, it is correct for the particular case of DDE (1) given by

$$x''(t) + f(x(t), x'(t)) + g(x(t))\psi(x(t-\tau)) = 0. \tag{7}$$

In this case, the first result of Peng [5] (Theorem 1) can be updated and corrected by Theorem 1* as the following:

Theorem 1*. *Let* $p(t) \equiv 0,\ f(0,0) = 0,\ g(0)\psi(0) = 0$ *and the following conditions are satisfied:*

(A1)′ $g(x)\psi(x)x > 0$ *for all* $x \neq 0$;

(A2)′ $\int_0^x g(\xi)\psi(\xi)d\xi \to +\infty$ *as* $|x| \to +\infty$;

(A3)′ *there exist a positive constant* C *such that*

$$\left|g(x)\psi'(z)\right| \le C \quad \textit{for all} \quad x, z \in \mathbb{R};$$

(A4)′ $C\tau y^2 < f(x,y)$ *for all* $y \neq 0$ *as* $x \in \mathbb{R}$, *where* C *is a positive constant from* $(A3)'$

Then, the zero solution of DDE (7) is globally stable.

Remark 2. It should be noted that to prove this theorem, we can use the following LKF:

$$V(t) = \frac{1}{2}y^2(t) + \int\limits_0^{x(t)} g(\xi)\psi(\xi)d\xi + \frac{1}{2}C\int\limits_{-\tau}^{0}\int\limits_{t+\eta}^{t} y^2(\xi)d\xi d\eta,$$

which is obtained from the LKF (3) for the case $g(x(t), y(t)) = g(x(t))$. In view of the Routh–Hurwitz stability conditions, (see, Ahmad and Rama Mohana Rao [8, p. 89, 90], which are related to differential equations of second order and the conditions of Theorem 1*, we can assume that $\dfrac{g(x(t))\psi(x(t))}{x(t)} \ge a_2 > 0,\ a_2 \in \mathbb{R},\ x(t) \neq 0.$ Then, it is clear that

$$\int\limits_0^{x(t)} g(\xi)\psi(\xi)d\xi = \int\limits_0^{x(t)} \frac{g(\xi)\psi(\xi)}{\xi}\xi d\xi \ge \frac{1}{2}\,a_2x^2(t).$$

Hence, we derive that

$$V(x_t, y_t) \ge \frac{1}{2}\,y^2(t) + \frac{1}{2}\,a_2x^2(t).$$

From this point of view, the terms $\frac{1}{2}\,y^2(t) + \frac{1}{2}\,a_2x^2(t)$ can be taken as lower bound of the LKF $V(x_t, y_t)$. Similarly, for the next step, it can be shown that the LKF $V(x_t, y_t)$ has an upper bound. Here, we omit the details of mathematical calculations for the sake of the brevity.

Example 1. We consider the following DDE of second order:

$$x''(t) + \left(10x'(t) + \frac{x^2(t)}{2(1+x^2(t))}\sin x'(t)\right) + \frac{2\exp\left(x^2(t)\right)}{1+\exp\left(x^2(t)\right)}\,x\left(t - \frac{1}{3}\right) = 0. \tag{8}$$

From DDE (8), we have

$$\begin{aligned}
x'(t) &= y(t), \\
y'(t) &= -\left(10y(t) + \frac{x^2(t)}{2\left(1+x^2(t)\right)}\,\sin y(t)\right) - \\
&\qquad - \frac{2x(t)\exp\left(x^2(t)\right)}{1+\exp\left(x^2(t)\right)} + \frac{2\exp\left(x^2(t)\right)}{1+\exp\left(x^2(t)\right)} \int\limits_{t-\frac{1}{3}}^{t} y(s)ds.
\end{aligned} \tag{9}$$

By comparing the system (9) with the system (2) and taking into account the assumptions of Theorem 1*, we derive the following relations, respectively:

$$f(x,y) = 10y + \frac{x^2}{2\left(1+x^2\right)}\,\sin y, \quad f(0,0)=0,$$

$$yf(x,y) = 10y^2 + \frac{x^2}{2\left(1+x^2\right)}\,y\sin y \geq 10y^2 > \frac{29}{3}\left(y^2\right), \quad y \neq 0, \quad \tau = \frac{1}{3}, \quad C = 29,$$

$$g(x) = \frac{2\exp(x^2)}{1+\exp(x^2)}, \quad \psi(x) = x, \quad \psi'(x) = 1,$$

$$g(0)\psi(0) = 0, \quad \text{i.e.,} \quad g(x)\psi(x) = 0 \Leftrightarrow x = 0,$$

$$g(x)\psi(x)x = \frac{2x^2\exp(x^2)}{1+\exp(x^2)} > 0 \quad \text{for all} \quad x \neq 0,$$

$$\left|g(x)\psi'(z)\right| = \frac{2\exp(x^2)}{1+\exp(x^2)} \leq 2 = C,$$

$$\int\limits_0^x g(\xi)\psi(\xi)d\xi = \int\limits_0^x \frac{2\xi\exp\left(\xi^2\right)}{1+\exp\left(\xi^2\right)}\,d\xi = \ln\left[\frac{1+\exp(x^2)}{2}\right].$$

If $|x| \to +\infty$, then $\ln\left[\dfrac{1+\exp(x^2)}{2}\right] \to \infty$, i.e., $\displaystyle\int_0^x g(\xi)\psi(\xi)d\xi \to \infty$.

Thus, all the conditions of Theorem 1* are held. By virtue of this result, we reach that the zero solution of DDE (8) is globally stable.

We now present the second theorem of Peng [5] (Theorem 2) on the eventually uniform boundedness of solutions of DDE (1) by the second method of Lyapunov.

Theorem 2 ([5], Theorem 2)**.** *Assume that $p(t)$ is a bounded function and the following conditions are satisfied:*

(C1) *there exist constants $m > 0,\ M > 0$ such that*

$$m \leq g(x,y) \leq M \quad \textit{for all} \quad x, y \in \mathbb{R};$$

(C2) *there exist a constant $C_1 > 0$ such that*

$$\left|g(x,y)\psi'(z)\right| \leq C_1 \quad \textit{for all} \quad x, y, z \in \mathbb{R};$$

(C3) *there are constants* $C_2 > 0,\ H > 0$ *such that*

$$\frac{f(x,y)}{y} \geq C_2 \quad \textit{for all} \quad x \in \mathbb{R} \quad \textit{as} \quad |y| \geq H;$$

(C4) $C_1\tau\sqrt{\dfrac{M}{m}} < C_2$, *where* $m,\ M,\ C_1$ *and* C_2 *are taken from (C1) – (C3);*

(C5) $\int_0^x g(\xi, y)\psi(\xi)d\xi \to +\infty$ *for all* $y \in \mathbb{R}$ *as* $|x| \to +\infty$.

Then, every solution of DDE (1) is eventually uniform bounded.

Remark 3. To prove Theorem 2, Peng [5] (Theorem 2) benefited from the direct Lyapunov method and, hence, the author defined a new Lyapunov function (LF) $V(x(t), y(t))$ by

$$V(x(t), y(t)) = \frac{1}{2}y^2(t) + \int\limits_0^{x(t)} g(\xi, y(t))\psi(\xi)d\xi. \tag{10}$$

Subject to conditions (C1) – (C5) of Theorem 2, it can be shown that there exists a positive monotonously increasing continuous function $a(x)$ such that $V(x(t), y(t)) \leq a(x)$. Further, there exists a non-negative monotonously increasing continuous function $b(x)$ such that

$$b(r) \leq V(x(t), y(t)), \qquad \lim_{r\to+\infty} b(r) = +\infty \quad \text{with} \quad r = \sqrt{x^2 + y^2}.$$

Next, for the case $p(t) \neq 0$ [5] (Theorem 2), Peng [5] calculated the time derivative of the LF $V(x(t),\ y(t))$ in (10) along the solutions of the system (2) and obtained the following relation:

$$\begin{aligned}
\frac{d}{dt}V(x(t), y(t)) &= y(t)[-f(x(t), y(t)) - g(x(t), y(t))\psi(x(t))] + \\
&\quad + y(t)g(x(t), y(t)) \int\limits_{-\tau}^{0} \psi'_x(x(t+\eta))y(t+\eta)d\eta + \\
&\quad + g(x(t), y(t))\psi(x(t))y(t) + y(t)p(t) = \\
&= -y(t)f(x(t), y(t)) + y(t)p(t) + \\
&\quad + y(t)g(x(t), y(t)) \int\limits_{-\tau}^{0} \psi'_x(x(t+\eta))y(t+\eta)d\eta.
\end{aligned} \tag{11}$$

When we consider the LF $V(x(t), y(t))$, it follows that the LF in (10) includes the integral term $\int_0^{x(t)} g(\xi, y(t))\psi(\xi)d\xi$. The time derivative of this integral term with respect to the variables $y = y(t)$ and t were calculated lack, i.e., wrong. Indeed, during the calculations of the time derivative of this integral term, it is omitted the term

$$\int\limits_0^{x(t)} \frac{\partial g(\xi, y(t))}{\partial y}\frac{dy}{dt}\psi(\xi)\,d\xi$$

with

$$\frac{dy}{dt} = -f(x(t), y(t)) - g(x(t), y(t))\psi(x(t)) + g(x(t), y(t)) \int_{-\tau}^{0} \psi'_x(x(t+\eta))y(t+\eta)\, d\eta.$$

In fact, if the time derivative of the term $\int_0^{x(t)} g(\xi, y(t))\psi(\xi)d\xi$ is calculated correctly, then we will have

$$\int_0^{x(t)} \frac{\partial g(\xi, y(t))}{\partial y} \frac{dy}{dt} \psi(\xi)\, d\xi = -\int_0^{x(t)} \frac{\partial g(\xi, y(t))}{\partial y} f(x(t), y(t))\psi(\xi)d\xi -$$

$$- \int_0^{x(t)} \frac{\partial g(\xi, y(t))}{\partial y} g(x(t), y(t))\psi(x(t))\psi(\xi)d\xi +$$

$$+ \int_0^{x(t)} \frac{\partial g(\xi, y(t))}{\partial y} \left[g(x(t), y(t)) \int_{-\tau}^{0} \psi'_x(x(t+\eta))y(t+\eta)d\eta \right] \psi(\xi)d\xi.$$

However, these terms are not involved in (11). Now, it is seen that the time derivative $\frac{d}{dt}V(x(t), y(t))$ was not calculated correctly by Peng [5] (Theorem 2).

Therefore, it is worth to mention that Remark 2 can also be updated for Theorem 2. Here, we would not like to give the details of the discussion for the sake of the brevity.

In this case, the correct form of Theorem 2, i.e., that of Peng [5] (Theorem 2) can be updated for DDE (7) and corrected by Theorem 2* as the following.

Theorem 2*. *Assume that $p(t)$ is a bounded function and the following conditions are held:*

(C1)′ *there are constants $m > 0,\ M > 0$ such that*

$$m \le g(x) \le M \quad \textit{for all} \quad x \in \mathbb{R};$$

(C2)′ *there exist a constant $C_1 > 0$ such that*

$$\left|g(x)\psi'(z)\right| \le C_1 \quad \textit{for all} \quad x, z \in \mathbb{R};$$

(C3)′ *there are constants $C_2 > 0,\ H > 0$ such that*

$$\frac{f(x, y)}{y} \ge C_2 \quad \textit{for all} \quad x, y \in \mathbb{R} \quad \textit{as} \quad |y| \ge H;$$

(C4)′ $C_1\tau\sqrt{\frac{M}{m}} < C_2$, *where $m,\ M,\ C_1$ and C_2 are given in (C1)–(C3), τ is the constant delay;*

(C5)′

$$\int_0^x g(\xi)\psi(\xi)d\xi \to +\infty, \quad |x| \to +\infty.$$

Then, every solution of DDE (7) is eventually uniformly bounded.

In particular case of the DDE (7), we now give an example which satisfies all the conditions of Theorem 2*.

Example 2. Assume that $p(t)$ is a bounded function. Then, we consider the following DDE of second order:

$$x''(t) + \left(25x'(t) + \frac{x^2(t)}{2(1+x^2(t))}\sin x'(t)\right) + \frac{3+x^2(t)}{1+x^2(t)}\,x\left(t-\frac{1}{3}\right) = e^{-t}\sin t. \tag{12}$$

From DDE (12), we have

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\left(25y(t) + \frac{x^2(t)}{2\left(1+x^2(t)\right)}\sin y(t)\right) - \frac{\left(3+x^2(t)\right)x(t)}{1+x^2(t)} + \\ &\quad + \frac{3+x^2(t)}{1+x^2(t)}\int\limits_{t-\frac{1}{3}}^{t} y(s)\,ds + e^{-t}\sin t. \end{aligned} \tag{13}$$

By comparing the systems (13) and (2) and taking into account the conditions of Theorem 2*, we obtain the following relations:

$$f(x,y) = \left(25y + \frac{x^2}{2\left(1+x^2\right)}\sin y\right), \quad f(0,0) = 0,$$

$$\frac{f(x,y)}{y} = 25 + \frac{x^2}{2\left(1+x^2\right)}\frac{\sin y}{y} \geq 24 = C_2 \quad \text{for all} \quad x \in \mathbb{R} \quad \text{as} \quad |y| \geq |H|,$$

$$g(x) = \frac{3+x^2}{1+x^2}, \quad \psi'(x) = 1, \quad g(0)\psi(0) = 3 \times 0 = 0,$$

$$1 = m \leq \frac{3+x^2}{1+x^2} = g(x) \leq 3 = M \quad \text{for all} \quad x \in \mathbb{R},$$

$$\left|g(x)\psi'(z)\right| = \frac{3+x^2}{1+x^2} = 1 + \frac{2}{1+x^2} \leq 3 = C_1,$$

$$\int\limits_0^x g(\xi)\psi(\xi)d\xi = \int\limits_0^x \frac{3\xi+\xi^3}{1+\xi^2}\,d\xi = \ln\left(1+x^2\right) + \frac{1}{2}x^2.$$

Hence, if $|x| \to +\infty$, then $\ln\left(1+x^2\right) + \left[\frac{1}{2}x^2\right] \to \infty$,

i.e.,

$$\int\limits_0^x g(\xi)\psi(\xi)d\xi \to \infty \quad \text{as} \quad |x| \to +\infty,$$

$$3 \times \frac{1}{3}\sqrt{\frac{3}{1}} = \sqrt{3} = C_1\tau\sqrt{\frac{M}{m}} < C_2 = 25.$$

$$p(t) = e^{-t}\sin t \le e^{-t} \le 1,$$

i.e., $p(t)$ is bounded.

Thus, the conditions of Theorem 2* are satisfied. Hence, every solution of DDE (12) is eventually uniformly bounded.

We now present the third theorem of Peng [5] (Theorem 3) on the existence of the periodic solutions of T-periodicity of DDE (1).

Theorem 3. *Suppose that conditions (C1) – (C5) of Theorem 2 hold, and, in addition,* $p(t)$ *is a periodic function of* T*-periodicity, i.e.,* $p(t+T) = p(t)$*. Then, DDE (1) has some periodic solutions of* T*-periodicity.*

Remark 4. In [5] (Theorem 3), the proof of the existence of the periodic solutions of DDE (1) is based on the boundedness of the solutions of this equation for all $t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$. To prove Theorem 3, Peng [5] (Theorem 3) used the second method of Lyapunov and, hence, in the proof of this theorem, the author used the LF $V(x(t), y(t))$, which is given by (10). Because of the reason above, Remark 3 is valid for Theorem 3. In this case, the conditions of Theorem 3 are held true for the particular case when $g(x, y) = g(x)$ in DDE (1), i.e., for DDE (7).

In the light of Remark 4, the third result of Peng [5] (Theorem 3)can be updated and corrected by Theorem 3* as the following:

Theorem 3*. *Suppose that conditions (C1)$'$ – (C5)$'$ of Theorem 3* are satisfied and, in addition,* $p(t)$ *is a periodic function of* T*-periodicity, i.e.,* $p(t+T) = p(t)$*. Then, DDE (7) has some periodic solutions of* T*-periodicity.*

We now introduce the fourth theorem of Peng [5] (Theorem 4) on the existence of the unique one stationary oscillation in DDE (1).

Theorem 4. *Assume that the conditions of Theorem 3 are satisfied. In addition, if* $x_1(t)$ *and* $x_2(t)$ *are any two solutions of DDE (1) such that* $x_1(t) - x_2(t) \to 0$ *and* $x'_1(t) - x'_2(t) \to 0$ *as* $t \to \infty$, *then there exists unique one stationary oscillation in DDE (1).*

Remark 5. Since the proof of Theorem 4 (see, also, [5], Theorem 4) depends on the conditions of Theorem 3, then the conditions of Theorem 4 are valid for the particular case when $g(x, y) = g(x)$ in DDE (1), i.e., for DDE (7). Because of this reason, Remark 4 is valid for Theorem 4.

In the light of Remark 5, the fourth result Peng [5] (Theorem 4) can be updated and corrected by Theorem 4* as the next theorem at the following.

Theorem 4*. *Assume that the conditions of Theorem 3* are satisfied. In addition, if* $x_1(t)$ *and* $x_2(t)$ *are any two solutions of DDE (7) such that* $x_1(t) - x_2(t) \to 0$ *and* $x'_1(t) - x'_2(t) \to 0$ *as* $t \to \infty$, *then there exists unique one stationary oscillation in DDE (7).*

Here, we would not like to give the proof of this theorem for the sake of the brevity (see [5], Theorem 4).

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