UDC 517.9

PERIODIC-TYPE SOLUTIONS FOR DIFFERENTIAL EQUATIONS WITH POSITIVELY HOMOGENEOUS FUNCTIONALS

ПРО РОЗВ'ЯЗКИ ПЕРІОДИЧНОГО ТИПУ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ІЗ ДОДАТНО ОДНОРІДНИМИ ФУНКЦІОНАЛАМИ

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We establish efficient conditions that guarantee the existence of a solution of the periodic-type boundary-value problem for the two-dimensional system of nonlinear functional-differential equations in the case where the right-hand side of the system is the sum of positively homogeneous terms of degrees λ and $1/\lambda$ and other terms with a relatively slow growth at infinity. The general results are reformulated in the special case of differential equations with maxima.

Розглянуто крайову задачу періодичного типу для двовимірної системи нелінійних функціонально-диференціальних рівнянь у випадку, коли права частина системи є сумою позитивно однорідних доданків зі степенями $\lambda > 0$, $1/\lambda$ та інших доданків із відносно повільним зростанням на нескінченності. Встановлено ефективні умови, що гарантують існування розв'язку такої крайової задачі. Загальні результати застосовано у спеціальному випадку диференціальних рівнянь із максимумами.

Introduction. The concept of differential equations with maxima was introduced in the mathematics in the early 1960s, around 60 years ago [1 – 3]. Since then the theory of this type of functional differential equations has been developed in a series of papers (see [4 – 15] for further references) and monographs [1, 16]. Nowadays this theory provides an adequate framework for analysis of some models and problems appearing in the applied sciences [1 – 3, 5, 15]. The topic of nonlinear oscillations (either periodic [4, 8, 12], or almost periodic [1, 13, 17], or even chaotic [15]) is

* R. Hakl was supported by RVO 67985840.
** S. Trofimchuk was partially supported by FONDECYT (Chile), project 1190712.

central for the theory. In the present paper, we show how a rather abstract Fredholm-type result from [10] can be successfully applied to study ω-periodic solutions to the following second order differential equation with a $(\lambda+1)$-Laplacian and maxima:

$$\Big(|u'(t)|^{\lambda}\operatorname{sgn}u'(t)\Big)' = g(t)\max\Big\{|u(s)|^{\lambda}\operatorname{sgn}u(s)\colon \mu(t)\le s\le \tau(t)\Big\} + f_0(t), \tag{1.1}$$

where $f_0, g \in L\big([0,\omega];\mathbb{R}\big)$, $\lambda > 0$, and $\mu,\tau\colon [0,\omega]\to[0,\omega]$ are measurable functions satisfying $\mu(t)\le\tau(t)$ for almost all t belonging to the period segment $[0,\omega]$. Two of our main results stated in Section 2, Corollaries 2.3 and 2.4, present easily verifiable conditions for the existence of at least one ω-periodic solution for ω-periodic equation (1.1) for each periodic perturbation $f_0(t)$, cf. with [4, 17]. Importantly, the leading coefficient $g(t)$ in (1.1) can oscillate: in such a case, we will assume that either positive or negative part of $g(t)$ dominates the part of $g(t)$ having the opposite sign, see Corollaries 2.3 and 2.4 for the precise formulations. Note that the uniqueness of periodic solutions is not analysed in the present work. Nevertheless, it is known from [12, 15] that even the first order periodic equation with the right-hand side as in (1.1) and constant coefficient $g(t)$ can have multiple (or even infinite number of) subharmonic periodic solutions for a class of sine-like forcing terms $f_0(t)$. We leave the aforementioned uniqueness problem for equation (1.1) as an interesting open question.

Now, our approach allows to consider more general objects in the form of two-dimensional system of functional differential equations

$$u_1'(t) = f_1(u_1,u_2)(t), \tag{1.2}$$

$$u_2'(t) = f_2(u_1,u_2)(t), \quad t\in[0,\omega], \tag{1.3}$$

subjected to the periodic-type boundary value conditions

$$u_1(\omega)-u_1(0) = h_1(u_1,u_2), \qquad u_2(\omega)-u_2(0) = h_2(u_1,u_2). \tag{1.4}$$

Here $f_i\colon C\big([0,\omega];\mathbb{R}\big)\times C\big([0,\omega];\mathbb{R}\big)\to L\big([0,\omega];\mathbb{R}\big)$, $i=1,2$, are continuous operators satisfying Carathéodory conditions, i.e., for every $r>0$ there exists $q_r\in L\big([0,\omega];\mathbb{R}_+\big)$ such that

$$|f_1(u_1,u_2)(t)| + |f_2(u_1,u_2)(t)| \le q_r(t) \quad \text{for a.e.} \quad t\in[0,\omega] \quad \text{whenever} \quad \|u_1\|_C + \|u_2\|_C \le r,$$

and $h_i\colon C\big([0,\omega];\mathbb{R}\big)\times C\big([0,\omega];\mathbb{R}\big)\to\mathbb{R}$, $i=1,2$, are continuous functionals bounded on every ball by a constant, i.e., for every $r>0$ there exists $M_r>0$ such that

$$|h_1(u_1,u_2)| + |h_2(u_1,u_2)| \le M_r \quad \text{whenever} \quad \|u_1\|_C + \|u_2\|_C \le r.$$

By a solution to the system (1.2), (1.3) we understand a vector-valued function $(u_1,u_2)\in C\big([0,\omega];\mathbb{R}\big)\times C\big([0,\omega];\mathbb{R}\big)$ with absolutely continuous components that satisfy the equalities (1.2) and (1.3) almost everywhere in $[0,\omega]$. By a solution to the problem (1.2)–(1.4) we understand a solution to (1.2), (1.3) which satisfies (1.4).

Before presenting our main results in Section 2 and the proofs of these results in Sections 3, 4, let us introduce basic notation used in this work:

$\mathbb{R}$ is a set of all real numbers;

$C\big([0,\omega];\mathbb{R}\big)$ is a Banach space of continuous functions $u\colon[0,\omega]\to\mathbb{R}$ endowed with the norm

$$\|u\|_C=\max\{|u(t)|\colon t\in[0,\omega]\};$$

$L\big([0,\omega];\mathbb{R}\big)$ is a Banach space of Lebesgue integrable functions $u\colon[0,\omega]\to\mathbb{R}$ endowed with the norm

$$\|u\|_L=\int\limits_0^\omega|u(t)|dt;$$

if $g\in L([0,\omega];\mathbb{R})$, then $[g]_+$, resp. $[g]_-$, denotes the non-negative, resp. non-positive, part of the function g, i.e.,

$$[g]_+(t)\stackrel{\mathrm{df}}{=}\frac{|g(t)|+g(t)}{2},\qquad [g]_-(t)\stackrel{\mathrm{df}}{=}\frac{|g(t)|-g(t)}{2}\quad\text{for a.e.}\quad t\in[0,\omega];$$

$\mathcal{P}(\lambda)$, where $\lambda>0$, is a set of all continuous nondecreasing operators $p\colon C\big([0,\omega];\mathbb{R}\big)\to$ $\to L\big([0,\omega];\mathbb{R}\big)$ satisfying Carathéodory conditions which are positively homogeneous with a degree λ, i.e., for every $c>0$ and $u\in C\big([0,\omega];\mathbb{R}\big)$ the following identity holds:

$$p(cu)(t)=c^\lambda p(u)(t)\quad\text{for a.e.}\quad t\in[0,\omega].$$

Let $\mu,\tau\colon[0,\omega]\to[0,\omega]$ be measurable functions. Then, for every $t\in[0,\omega]$, we put $I(\mu(t),\tau(t))=[\mu(t),\tau(t)]$ if $\mu(t)\le\tau(t)$ and $I(\mu(t),\tau(t))=\varnothing$ otherwise.

$\mathcal{S}$ is a set of all mappings $S\colon[0,\omega]\to 2^{[0,\omega]}$ such that $S(t)$ is a union of at most countable number of intervals $I(\mu_k(t),\tau_k(t))$, where $\mu_k,\tau_k\colon[0,\omega]\to[0,\omega]$ are measurable functions satisfying $\mu_k(t)\le\tau_k(t)$ for almost all $t\in[0,\omega]$.

Note that the function $t\mapsto\sup\big\{|u(s)|^\lambda\operatorname{sgn}u(s)\colon s\in S(t)\big\}$ is measurable whenever $u\in$ $\in C\big([0,\omega];\mathbb{R}\big)$, $S\in\mathcal{S}$, and $\lambda>0$ (we put $\sup\varnothing=-\infty$).

For given $p\in\mathcal{P}(\lambda)$ and a number $\delta\in[0,1]$ we define the operator $p(\cdot;\delta)\colon C\big([0,\omega];\mathbb{R}\big)\to$ $\to L\big([0,\omega];\mathbb{R}\big)$ and a non-negative numbers $\widehat{P}(\delta)$ and $P(\delta)$ in the following way:

$$p(u;\delta)(t)\stackrel{\mathrm{df}}{=}(1-\delta)p(u)(t)-\delta p(-u)(t)\quad\text{for a.e.}\quad t\in[0,\omega],\qquad \widehat{P}(\delta)\stackrel{\mathrm{df}}{=}\int\limits_0^\omega p(1;\delta)(t)dt,$$

$$P(\delta)\stackrel{\mathrm{df}}{=}\max\left\{\int\limits_x^y p(1;\delta)(t)dt+\int\limits_y^{x+\omega}p(1;1-\delta)(t)dt\colon x\in[0,\omega],y\in[x,x+\omega]\right\},$$

where

$$p(1;\nu)(t)=p(1;\nu)(t-\omega)\quad\text{for a.e.}\quad t\in(\omega,2\omega],\quad \nu=\delta,1-\delta.$$

Obviously, $\widehat{P}(\delta)\le P(\delta)$ and $-p(-u;\delta)\equiv p(u;1-\delta)$ for every $u\in C\big([0,\omega];\mathbb{R}\big)$ and $\delta\in[0,1]$. It can be also easily verified that

$$P(\delta)=P(1-\delta)\quad\text{for}\quad\delta\in[0,1].\tag{1.5}$$

Furthermore, for given $p_0 \in \mathcal{P}(\lambda_1)$ and $p_1, p_2 \in \mathcal{P}(\lambda_2)$ we define the following functions:

$$q_1(t,\rho) \overset{\mathrm{df}}{=} \sup\Big\{|f_1(u_1,u_2)(t) - p_0(u_2)(t)|: \\ \|u_1\|_C \le \rho, \|u_2\|_C \le \rho^{\lambda_2}\Big\} \quad \text{for a.e.} \quad t \in [0,\omega], \tag{1.6}$$

$$q_2(t,\rho) \overset{\mathrm{df}}{=} \sup\Big\{|f_2(u_1,u_2)(t) - p_1(u_1)(t) + p_2(u_1)(t)|: \\ \|u_1\|_C \le \rho^{\lambda_1}, \|u_2\|_C \le \rho\Big\} \quad \text{for a.e.} \quad t \in [0,\omega], \tag{1.7}$$

$$\eta_k(\rho) \overset{\mathrm{df}}{=} \sup\Big\{|h_k(u_1,u_2)|: \|u_k\|_C \le \rho, \|u_{3-k}\|_C \le \rho^{\lambda_{3-k}}\Big\}, \quad k = 1,2. \tag{1.8}$$

2. Main results. Now we can formulate our main results. The proofs of the results slightly differ depending on the values of λ_i. Therefore it is convenient formulate assertions for two separate cases. Thus, Theorem 2.1 deals with the case when $\lambda_2 \ge 1$, Theorem 2.2 can be applied in the case when $\lambda_2 < 1$.

Theorem 2.1. *Let $\lambda_1, \lambda_2 > 0$, $\lambda_1\lambda_2 = 1$, and let there exist $p_0 \in \mathcal{P}(\lambda_1)$ and $p_1, p_2 \in \mathcal{P}(\lambda_2)$ such that*

$$\lim_{\rho\to+\infty} \int_0^\omega \frac{q_k(s,\rho)}{\rho} ds = 0, \qquad \lim_{\rho\to+\infty} \frac{\eta_k(\rho)}{\rho} = 0, \quad k = 1,2, \tag{2.1}$$

where q_k and η_k are given by (1.6) – (1.8). Let, moreover, $\lambda_2 \ge 1$, $p_0(1) \not\equiv 0$, $p_0(-1) \not\equiv 0$, and let there exist $i \in \{1,2\}$ such that, for every $\delta \in [0,1]$, the following inequalities hold:

$$\frac{P_0(\delta)}{2^{1+\lambda_1}} P_i^{\lambda_1}(\delta) < 1, \qquad \widehat{P}_i^{\lambda_1}(\delta) < \left(1 - \frac{P_0(\delta)}{2^{1+\lambda_1}} \widehat{P}_i^{\lambda_1}(\delta)\right) \widehat{P}_{3-i}^{\lambda_1}(\delta), \tag{2.2}$$

$$\frac{P_0^{\lambda_2}(\delta)}{2^{2+\lambda_2}} P_{3-i}(\delta) < 2^{\lambda_2} - 1 + \sqrt{1 - \frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}} P_i(\delta)}. \tag{2.3}$$

Then the problem (1.2) – (1.4) has at least one solution.

Theorem 2.2. *Let $\lambda_1, \lambda_2 > 0$, $\lambda_1\lambda_2 = 1$, and let there exist $p_0 \in \mathcal{P}(\lambda_1)$ and $p_1, p_2 \in \mathcal{P}(\lambda_2)$ such that (2.1) is fulfilled where q_k and η_k are given by (1.6) – (1.8). Let, moreover, $\lambda_2 < 1$, $p_0(1) \not\equiv 0$, $p_0(-1) \not\equiv 0$, and let there exist $i \in \{1,2\}$ such that, for every $\delta \in [0,1]$, the following inequalities hold:*

$$\frac{P_0(\delta)}{4} P_i^{\lambda_1}(\delta) < 1, \qquad \widehat{P}_i^{\lambda_1}(\delta) < \left(1 - \frac{P_0(\delta)}{2^{1+\lambda_1}} \widehat{P}_i^{\lambda_1}(\delta)\right) \widehat{P}_{3-i}^{\lambda_1}(\delta), \tag{2.4}$$

$$\frac{P_0^{\lambda_2}(\delta)}{2^{2\lambda_2+1}} P_{3-i}(\delta) < 1 + \sqrt{1 - \frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}} P_i(\delta)}. \tag{2.5}$$

Then the problem (1.2) – (1.4) has at least one solution.

In the case when the operator $p \in \mathcal{P}(\lambda)$ is homogeneous on the constant functions, i.e., if $p(-1) \equiv -p(1)$, then the numbers $\widehat{P}(\delta),\ P(\delta)$ take more simple form. More precisely, they do not depend on δ anymore and

$$\widehat{P}(\delta) = P(\delta) = \int\limits_0^\omega p(1)(t)\,dt.$$

The typical operator having the above-described property is an operator defined by means of suprema of the function u over certain subsets of its domain:

$$p(u)(t) \stackrel{\text{df}}{=} g(t) \sup\Big\{|u(s)|^\lambda \operatorname{sgn} u(s) : s \in S(t)\Big\},$$

where $g \in L([0,\omega];\mathbb{R})$ and $S \in \mathcal{S}$. Therefore, considering the system

$$u_1'(t) = g_0(t) \sup\Big\{|u_2(s)|^{\lambda_1} \operatorname{sgn} u_2(s) : s \in S_0(t)\Big\} + \widetilde{f}_1(u_1,u_2)(t), \tag{2.6}$$

$$u_2'(t) = g_1(t) \sup\Big\{|u_1(s)|^{\lambda_2} \operatorname{sgn} u_1(s) : s \in S_1(t)\Big\} -$$
$$- g_2(t) \sup\Big\{|u_1(s)|^{\lambda_2} \operatorname{sgn} u_1(s) : s \in S_2(t)\Big\} + \widetilde{f}_2(u_1,u_2)(t), \tag{2.7}$$

where $g_i \in L([0,\omega]\mathbb{R}_+),\ S_i \in \mathcal{S},\ i = 0,1,2,$ and $\widetilde{f}_1, \widetilde{f}_2 : C([0,\omega];\mathbb{R}) \times C([0,\omega];\mathbb{R}) \to L([0,\omega];\mathbb{R})$ are continuous operators satisfying Carathéodory conditions, from Theorems 2.1 and 2.2 we derive the following assertions:

Corollary 2.1. *Let $\lambda_1, \lambda_2 > 0,\ \lambda_1\lambda_2 = 1,$ and let (2.1) be fulfilled where*

$$q_k(t,\rho) \stackrel{\text{df}}{=} \sup\Big\{\Big|\widetilde{f}_k(u_1,u_2)(t)\Big| : \|u_k\|_C \le \rho, \|u_{3-k}\|_C \le \rho^{\lambda_{3-k}}\Big\} \quad \text{for a.e.} \quad t \in [0,\omega] \tag{2.8}$$

and η_k are given by (1.8). Let, moreover, $\lambda_2 \ge 1$ and $g_i(t) \ge 0,\ i = 0,1,2,$ for almost every $t \in [0,\omega],\ g_0 \not\equiv 0,$ and let there exist $i \in \{1,2\}$ such that the following inequalities hold:

$$\frac{\|g_0\|_L}{2^{1+\lambda_1}} \|g_i\|_L^{\lambda_1} < 1, \qquad \|g_i\|_L^{\lambda_1} < \left(1 - \frac{\|g_0\|_L}{2^{1+\lambda_1}} \|g_i\|_L^{\lambda_1}\right) \|g_{3-i}\|_L^{\lambda_1},$$

$$\frac{\|g_0\|_L^{\lambda_2}}{2^{2+\lambda_2}} \|g_{3-i}\|_L < 2^{\lambda_2} - 1 + \sqrt{1 - \frac{\|g_0\|_L^{\lambda_2}}{2^{1+\lambda_2}} \|g_i\|_L}.$$

Then the problem (2.6), (2.7), (1.4) has at least one solution.

Corollary 2.2. *Let $\lambda_1, \lambda_2 > 0,\ \lambda_1\lambda_2 = 1,$ and let (2.1) be fulfilled where q_k and η_k are given by (2.8) and (1.8), respectively. Let, moreover, $\lambda_2 < 1$ and $g_i(t) \ge 0,\ i = 0,1,2,$ for almost every $t \in [0,\omega],\ g_0 \not\equiv 0,$ and let there exist $i \in \{1,2\}$ such that the following inequalities hold:*

$$\frac{\|g_0\|_L}{4} \|g_i\|_L^{\lambda_1} < 1, \qquad \|g_i\|_L^{\lambda_1} < \left(1 - \frac{\|g_0\|_L}{2^{1+\lambda_1}} \|g_i\|_L^{\lambda_1}\right) \|g_{3-i}\|_L^{\lambda_1},$$

$$\frac{\|g_0\|_L^{\lambda_2}}{2^{2\lambda_2+1}} \|g_{3-i}\|_L < 1 + \sqrt{1 - \frac{\|g_0\|_L^{\lambda_2}}{4^{\lambda_2}} \|g_i\|_L}.$$

Then the problem (2.6), (2.7), (1.4) has at least one solution.

Corollaries 2.1, 2.2 immediately follows from Theorems 2.1, 2.2 and their proofs are omitted.

Now, consider the particular case of equation (1.1) where $f_0, g \in L([0,\omega];\mathbb{R}),\ \lambda > 0,$ and $\mu, \tau : [0,\omega] \to [0,\omega]$ are measurable functions satisfying $\mu(t) \le \tau(t)$ for almost all $t \in [0,\omega]$. Obviously, in such a case, we can invoke our previous results setting $g_0 \equiv 1,\ g_1 \equiv [g]_+,\ g_2 \equiv [g]_-,$ $\lambda_1 = 1/\lambda,\ \lambda_2 = \lambda,$ and $S_0(t) = \{t\},\ S_1(t) = S_2(t) = [\mu(t), \tau(t)]$ for almost all $t \in [0,\omega]$. Thus, Corollaries 2.1 and 2.2 yields the following assertions dealing with the equation (1.1).

Corollary 2.3. *Let* $\lambda \ge 1$ *and let there exist* $\sigma \in \{-1, 1\}$ *such that*

$$\|[\sigma g]_+\|_L < \frac{2^{1+\lambda}}{\omega^\lambda},$$

$$\frac{\|[\sigma g]_+\|_L}{\left(1 - \dfrac{\omega}{2^{1+1/\lambda}} \|[\sigma g]_+\|_L^{1/\lambda}\right)^\lambda} < \|[\sigma g]_-\|_L < \frac{2^{2+\lambda}}{\omega^\lambda}\left(2^\lambda - 1 + \sqrt{1 - \frac{\omega^\lambda}{2^{1+\lambda}} \|[\sigma g]_+\|_L}\right).$$

Then the equation (1.1) has at least one solution u *that satisfies* $u(0) = u(\omega),\ u'(0) = u'(\omega)$.

Corollary 2.4. *Let* $0 < \lambda < 1$ *and let there exist* $\sigma \in \{-1, 1\}$ *such that*

$$\|[\sigma g]_+\|_L < \left(\frac{4}{\omega}\right)^\lambda,$$

$$\frac{\|[\sigma g]_+\|_L}{\left(1 - \dfrac{\omega}{2^{1+1/\lambda}} \|[\sigma g]_+\|_L^{1/\lambda}\right)^\lambda} < \|[\sigma g]_-\|_L < \frac{2^{2\lambda+1}}{\omega^\lambda}\left(1 + \sqrt{1 - \left(\frac{\omega}{4}\right)^\lambda \|[\sigma g]_+\|_L}\right).$$

Then the equation (1.1) has at least one solution u *that satisfies* $u(0) = u(\omega),\ u'(0) = u'(\omega)$.

3. Fredholm-type core theorem and three lemmas. The proofs of the main results are based on the following theorem which can be found in [10] (Theorem 1). We formulate it in a form suitable for us.

Theorem 3.1. *Let* $\lambda_1, \lambda_2 > 0,\ \lambda_1\lambda_2 = 1,$ *and let there exist* $p_0 \in \mathcal{P}(\lambda_1)$ *and* $p_1, p_2 \in \mathcal{P}(\lambda_2)$ *such that (2.1) is fulfilled, where* q_k *and* η_k *are given by (1.6) – (1.8). Let, moreover, the problem*

$$u_1'(t) = p_0(u_2;\delta)(t), \tag{3.1}$$

$$u_2'(t) = p_1(u_1;\delta)(t) - p_2(u_1;\delta)(t), \tag{3.2}$$

$$u_1(0) = u_1(\omega), \quad u_2(0) = u_2(\omega) \tag{3.3}$$

has only the trivial solution for every $\delta \in [0,1]$. *Then the problem (1.2) – (1.4) has at least one solution.*

Remark 3.1. Note that if (u_1, u_2) is a solution to (3.1) – (3.3) for some $\delta = \delta_0 \in [0,1]$, then $(\widetilde{u}_1, \widetilde{u}_2) \overset{\mathrm{df}}{=} (-u_1, -u_2)$ is a solution to (3.1) – (3.3) with $\delta = 1 - \delta_0$.

Lemma 3.1. *Assume that* $\lambda_1 > 0$ *and* $\lambda_2 = 1/\lambda_1,\ p_0 \in \mathcal{P}(\lambda_1)$ *and* $p_1, p_2 \in \mathcal{P}(\lambda_2)$. *Let there exist* $\delta \in [0,1]$ *such that the problem (3.1) – (3.3) has a nontrivial solution* (u_1, u_2), *where*

$$u_1(t) \ge 0 \quad \textit{for} \quad t \in [0,\omega].$$

Let, moreover, $p_0(1;\delta) \not\equiv 0$ *and* $p_0(-1;\delta) \not\equiv 0$. *Then*

$$\widehat{P}_{3-i}^{\lambda_1}(\delta)\left(1 - \frac{P_0(\delta)}{2^{1+\lambda_1}}\,\widehat{P}_i^{\lambda_1}(\delta)\right) \le \widehat{P}_i^{\lambda_1}(\delta), \quad i = 1, 2. \tag{3.4}$$

Proof. Obviously, if $2^{1+\lambda_1} \le P_0(\delta)\widehat{P}_i^{\lambda_1}(\delta)$ for some $i \in \{1, 2\}$, then the corresponding inequality holds trivially. Assume therefore that

$$\frac{P_0(\delta)}{2^{1+\lambda_1}}\,\widehat{P}_i^{\lambda_1}(\delta) < 1, \quad i = 1, 2. \tag{3.5}$$

Suppose first that u_2 is still non-positive or still non-negative. Then from (3.1) it follows that u_1 is a monotone function, which together with (3.3) implies that u_1 is a constant function, i.e., $u_1(t) = u_0$ for $t \in [0, \omega]$. Consequently, integrating (3.2) over $[0, \omega]$ results in

$$u_0\widehat{P}_1(\delta) = u_0\widehat{P}_2(\delta).$$

If $u_0 \ne 0$, then we have $\widehat{P}_1(\delta) = \widehat{P}_2(\delta)$, and so (3.4) is valid. In the case when $u_0 = 0$, from (3.2) we obtain that u_2 is a (nonzero) constant function. Consequently, (3.1) implies $p_0(1;\delta) \equiv 0$ if u_2 is positive and $p_0(-1;\delta) \equiv 0$ if u_2 is negative. However, both cases contradicts our assumptions.

Assume now that u_2 changes its sign. Put

$$M_1 = \max\{u_1(t) : t \in [0, \omega]\}, \qquad m_1 = \min\{u_1(t) : t \in [0, \omega]\}, \tag{3.6}$$

$$M_2 = \max\{u_2(t) : t \in [0, \omega]\}, \qquad m_2 = \max\{-u_2(t) : t \in [0, \omega]\}. \tag{3.7}$$

Then $M_1 > 0,\ m_1 \ge 0,\ M_2 > 0,\ m_2 > 0$.

Now we prolong the functions u_1 and u_2 ω-periodically to the interval $[0, 2\omega]$ and we put

$$p_0(u_2;\delta)(t) = p_0(u_2;\delta)(t-\omega) \quad \text{for a.e.} \quad t \in (\omega, 2\omega],$$

$$p_1(u_1;\delta)(t) = p_1(u_1;\delta)(t-\omega), \qquad p_2(u_1;\delta)(t) = p_2(u_1;\delta)(t-\omega) \quad \text{for a.e.} \quad t \in (\omega, 2\omega].$$

Then, obviously, the equalities (3.1) and (3.2) hold for almost every $t \in [0, 2\omega]$. Let $t_m \in [0, \omega)$, $t_M \in (t_m, t_m + \omega),\ s_m \in [0, \omega),\ s_M \in (s_m, s_m + \omega)$ be such that

$$u_1(t_m) = m_1, \qquad u_1(t_M) = M_1, \qquad u_2(s_m) = -m_2, \qquad u_2(s_M) = M_2. \tag{3.8}$$

Then the integration of (3.1) from t_m to t_M and from t_M to $t_m + \omega$, respectively, yields

$$M_1 - m_1 \le M_2^{\lambda_1}\int_{t_m}^{t_M} p_0(1;\delta)(t)dt, \tag{3.9}$$

$$M_1 - m_1 \le -m_2^{\lambda_1}\int_{t_M}^{t_m+\omega} p_0(-1;\delta)(t)\,dt = m_2^{\lambda_1}\int_{t_M}^{t_m+\omega} p_0(1;1-\delta)(t)\,dt. \tag{3.10}$$

Moreover, multiplying the corresponding sides of (3.9) and (3.10), applying the inequality $4AB \leq \leq (A+B)^2$, we arrive at

$$M_1 - m_1 \leq \frac{(M_2+m_2)^{\lambda_1}}{2^{1+\lambda_1}} \left(\int\limits_{t_m}^{t_M} p_0(1;\delta)(t)dt + \int\limits_{t_M}^{t_m+\omega} p_0(1;1-\delta)(t)dt \right) \leq$$

$$\leq \frac{(M_2+m_2)^{\lambda_1}}{2^{1+\lambda_1}} P_0(\delta). \tag{3.11}$$

Furthermore, the integration of (3.2) on $[s_m, s_M]$ and on $[s_M, s_m+\omega]$ results in

$$M_2 + m_2 \leq M_1^{\lambda_2} \int\limits_{s_m}^{s_M} p_1(1;\delta)(t)\,dt \leq M_1^{\lambda_2} \widehat{P}_1(\delta), \tag{3.12}$$

$$M_2 + m_2 \leq M_1^{\lambda_2} \int\limits_{s_M}^{s_m+\omega} p_2(1;\delta)(t)\,dt \leq M_1^{\lambda_2} \widehat{P}_2(\delta). \tag{3.13}$$

Consequently, since $\lambda_1\lambda_2 = 1$, from (3.11) – (3.13), on account of (3.5), we obtain

$$0 < M_1 \left(1 - \frac{P_0(\delta)}{2^{1+\lambda_1}} \widehat{P}_i^{\lambda_1}(\delta) \right) \leq m_1, \quad i = 1, 2. \tag{3.14}$$

On the other hand, the integration of (3.2) on $[0, \omega]$, with respect to (3.3), results in

$$m_1 \widehat{P}_{3-i}^{\lambda_1}(\delta) \leq \left(\int\limits_0^\omega p_{3-i}(u_1;\delta)(t)\,dt \right)^{\lambda_1} =$$

$$= \left(\int\limits_0^\omega p_i(u_1;\delta)(t)\,dt \right)^{\lambda_1} \leq M_1 \widehat{P}_i^{\lambda_1}(\delta), \quad i = 1, 2. \tag{3.15}$$

Now, the inequalities (3.14) and (3.15) imply (3.4).

Lemma 3.2. *Assume that* $\lambda_1 > 0$ *and* $\lambda_2 = 1/\lambda_1$, $p_0 \in \mathcal{P}(\lambda_1)$ *and* $p_1, p_2 \in \mathcal{P}(\lambda_2)$. *Let there exist* $\delta \in [0,1]$ *such that the problem (3.1) – (3.3) has a nontrivial solution* (u_1, u_2). *Let, moreover,* u_1 *attain both positive and negative values. Then there exist* $c \in (0,1)$, $s_0 \in [0,\omega)$, *and* $s_1 \in (s_0, s_0+\omega)$ *such that*

$$1 \leq \frac{c^{\lambda_2}}{2^{1+\lambda_2}} P_0^{\lambda_2}(\delta) \int\limits_{s_0}^{s_1} p_1(1;\delta)(t)\,dt + \frac{(1-c)^{\lambda_2}}{2^{1+\lambda_2}} P_0^{\lambda_2}(\delta) \int\limits_{s_0}^{s_1} p_2(1;1-\delta)(t)\,dt, \tag{3.16}$$

$$1 \leq \frac{(1-c)^{\lambda_2}}{2^{1+\lambda_2}} P_0^{\lambda_2}(\delta) \int\limits_{s_1}^{s_0+\omega} p_1(1;1-\delta)(t)\,dt + \frac{c^{\lambda_2}}{2^{1+\lambda_2}} P_0^{\lambda_2}(\delta) \int\limits_{s_1}^{s_0+\omega} p_2(1;\delta)(t)\,dt, \tag{3.17}$$

where

$$p_k(1;\nu)(t) = p_k(1;\nu)(t-\omega) \quad \textit{for a.e.} \quad t \in (\omega, 2\omega], \quad k = 1, 2, \quad \nu = \delta, 1-\delta.$$

Proof. Obviously, the function u_2 has to change its sign. Define M_2 and m_2 by (3.7) and put

$$M_1 = \max\{u_1(t) : t \in [0, \omega]\}, \qquad m_1 = \max\{-u_1(t) : t \in [0, \omega]\}. \tag{3.18}$$

Then $M_1 > 0,\ m_1 > 0,\ M_2 > 0$, and $m_2 > 0$.

Now we prolong the functions u_1 and u_2 ω-periodically to the interval $[0, 2\omega]$ and we put

$$p_0(u_2; \delta)(t) = p_0(u_2; \delta)(t - \omega) \quad \text{for a.e.} \quad t \in (\omega, 2\omega],$$

$$p_1(u_1; \delta)(t) = p_1(u_1; \delta)(t - \omega), \qquad p_2(u_1; \delta)(t) = p_2(u_1; \delta)(t - \omega) \quad \text{for a.e.} \quad t \in (\omega, 2\omega].$$

Then, obviously, the equalities (3.1) and (3.2) hold for almost every $t \in [0, 2\omega]$. Let $t_m \in [0, \omega)$ and $t_M \in (t_m, t_m + \omega)$ be such that

$$u_1(t_m) = -m_1, \qquad u_1(t_M) = M_1. \tag{3.19}$$

Choose $t_0 \in (t_m, t_M)$ and $t_1 \in (t_M, t_m + \omega)$ such that

$$u_1(t_0) = 0, \qquad u_1(t_1) = 0. \tag{3.20}$$

Moreover, let $s_0 \in [0, \omega)$ and $s_1 \in (s_0, s_0 + \omega)$ be such that

$$u_2(s_0) = -m_2, \qquad u_2(s_1) = M_2. \tag{3.21}$$

Then the integration of (3.1) from t_1 to $t_m + \omega$ and from t_m to t_0 yields

$$m_1 \le -m_2^{\lambda_1} \int_{t_1}^{t_m+\omega} p_0(-1; \delta)(t)\, dt = m_2^{\lambda_1} \int_{t_1}^{t_m+\omega} p_0(1; 1-\delta)(t)\, dt, \tag{3.22}$$

$$m_1 \le M_2^{\lambda_1} \int_{t_m}^{t_0} p_0(1; \delta)(t)\, dt. \tag{3.23}$$

Now the multiplication of the corresponding sides of (3.22) and (3.23), on account of the inequality $4AB \le (A + B)^2$, results in

$$0 < m_1 \le \frac{(M_2 + m_2)^{\lambda_1}}{2^{\lambda_1}} \left(\int_{t_m}^{t_0} p_0(1; \delta)(t)\, dt \int_{t_1}^{t_m+\omega} p_0(1; 1-\delta)(t)\, dt \right)^{1/2}. \tag{3.24}$$

Analogously, on the intervals $[t_0, t_M]$ and $[t_M, t_1]$ we obtain

$$M_1 \le M_2^{\lambda_1} \int_{t_0}^{t_M} p_0(1; \delta)(t) dt, \qquad M_1 \le m_2^{\lambda_1} \int_{t_M}^{t_1} p_0(1; 1-\delta)(t) dt,$$

and, consequently,

$$0 < M_1 \le \frac{(M_2 + m_2)^{\lambda_1}}{2^{\lambda_1}} \left(\int_{t_0}^{t_M} p_0(1; \delta)(t)\, dt \int_{t_M}^{t_1} p_0(1; 1-\delta)(t)\, dt \right)^{1/2}. \tag{3.25}$$

Moreover, there exist $c_1, c_2 \in (0,1)$ such that

$$\int\limits_{t_m}^{t_0} p_0(1;\delta)(t)\,dt = c_1 \int\limits_{t_m}^{t_M} p_0(1;\delta)(t)\,dt, \tag{3.26}$$

$$\int\limits_{t_M}^{t_1} p_0(1;1-\delta)(t)\,dt = c_2 \int\limits_{t_M}^{t_m+\omega} p_0(1;1-\delta)(t)\,dt. \tag{3.27}$$

Obviously,

$$\int\limits_{t_0}^{t_M} p_0(1;\delta)(t)\,dt = (1-c_1) \int\limits_{t_m}^{t_M} p_0(1;\delta)(t)\,dt, \tag{3.28}$$

$$\int\limits_{t_1}^{t_m+\omega} p_0(1;1-\delta)(t)\,dt = (1-c_2) \int\limits_{t_M}^{t_m+\omega} p_0(1;1-\delta)(t)\,dt. \tag{3.29}$$

Put $c = (1-c_1+c_2)/2$. Then $c \in (0,1)$ and

$$(1-c_1)c_2 \le \frac{(1-c_1+c_2)^2}{4} = c^2, \qquad c_1(1-c_2) \le \frac{(1-c_2+c_1)^2}{4} = (1-c)^2. \tag{3.30}$$

Consequently, from (3.24) and (3.25), with respect to (3.26) – (3.30) and the identity $\lambda_1\lambda_2 = 1$, it follows that

$$M_1^{\lambda_2} \le \frac{M_2+m_2}{2^{1+\lambda_2}}\, c^{\lambda_2} P_0^{\lambda_2}(\delta), \qquad m_1^{\lambda_2} \le \frac{M_2+m_2}{2^{1+\lambda_2}}\, (1-c)^{\lambda_2} P_0^{\lambda_2}(\delta). \tag{3.31}$$

On the other hand, the integration of (3.2) from s_0 to s_1 and from s_1 to $s_0+\omega$, respectively, in view of (3.21), yields

$$M_2 + m_2 \le M_1^{\lambda_2} \int\limits_{s_0}^{s_1} p_1(1;\delta)(t)\,dt + m_1^{\lambda_2} \int\limits_{s_0}^{s_1} p_2(1;1-\delta)(t)\,dt, \tag{3.32}$$

$$M_2 + m_2 \le m_1^{\lambda_2} \int\limits_{s_1}^{s_0+\omega} p_1(1;1-\delta)(t)\,dt + M_1^{\lambda_2} \int\limits_{s_1}^{s_0+\omega} p_2(1;\delta)(t)\,dt. \tag{3.33}$$

Thus, by using (3.31) in (3.32) and (3.33) we get (3.16) and (3.17).

Lemma 3.3. *Let* $\lambda \ge 1$ *and*

$$\varphi(x) = \frac{1-x^\lambda}{(1-x)^\lambda} + \frac{1-(1-x)^\lambda}{x^\lambda} \quad \textit{for} \quad x \in (0,1).$$

Then

$$\min\{\varphi(x) : x \in (0,1)\} = \varphi(1/2) = 2\big(2^\lambda - 1\big). \tag{3.34}$$

Proof. Set $u = 0.5 - x \in (-0.5, 0.5)$, then, by the Cauchy inequality and the binomial formula, $\psi(u) = \varphi(0.5 - u)$ satisfies

$$\psi(u) \geq 2\sqrt{[(0.5-u)^{-\lambda} - 1]\,[(0.5+u)^{-\lambda} - 1]} =$$

$$= 2\sqrt{1 + (0.25 - u^2)^{-\lambda} - (0.5-u)^{-\lambda} - (0.5+u)^{-\lambda}} =$$

$$= 2\sqrt{(2^\lambda - 1)^2 + 2^{\lambda+1}\sum_{k=1}^{+\infty} c_k\,\frac{\lambda(\lambda+1)\ldots(\lambda+k-1)}{k!}(4u^2)^k} \geq 2\bigl(2^\lambda - 1\bigr),$$

$$c_k := 2^{\lambda-1} - \left(1 + \frac{\lambda-1}{k+1}\right)\ldots\left(1 + \frac{\lambda-1}{2k}\right) \geq 2^{\lambda-1} -$$

$$- \left(1 + \frac{\lambda-1}{k}\left(\frac{1}{k+1} + \ldots + \frac{1}{2k}\right)\right)^k \geq$$

$$\geq 2^{\lambda-1} - \left(1 + \frac{\lambda-1}{k}\int\limits_k^{2k}\frac{dx}{x}\right)^k = 2^{\lambda-1} - \left(1 + \frac{\lambda-1}{k}\ln 2\right)^k \geq 0,$$

where we have used the AM-GM inequality and the inequality $2^{x/k} \geq 1 + x\ln 2/k,\ x \in \mathbb{R}$.

4. Proofs of the main results.

Proof of Theorem 2.1. According to Theorem 3.1 it is sufficient to show that the problem (3.1)–(3.3) has only the trivial solution for every $\delta \in [0,1]$. Therefore, assume on the contrary that for some $\delta \in [0,1]$ there exists a nontrivial solution (u_1, u_2) to the problem (3.1)–(3.3).

First assume that u_1 is still non-positive or still non-negative. With respect to Remark 3.1, without loss of generality we can assume that $u_1(t) \geq 0$ for $t \in [0,\omega]$. Therefore, Lemma 3.1 yields (3.4) which contradicts (2.2).

Assume therefore that u_1 attains both positive and negative values. Thus, according to Lemma 3.2 there exist $c \in (0,1)$, $s_0 \in [0,\omega)$, and $s_1 \in (s_0, s_0+\omega)$ such that (3.16) and (3.17) are fulfilled. Consequently, if (2.2) holds for $i = 1$ then we obtain

$$0 < c^{\lambda_2}\left(1 - \frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}\int\limits_{s_0}^{s_1} p_1(1;\delta)(t)\,dt\right) \leq$$

$$\leq (1-c)^{\lambda_2}\left(\frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}\int\limits_{s_0}^{s_1} p_2(1;1-\delta)(t)\,dt - \frac{1-c^{\lambda_2}}{(1-c)^{\lambda_2}}\right), \tag{4.1}$$

$$0 < (1-c)^{\lambda_2}\left(1 - \frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}\int\limits_{s_1}^{s_0+\omega} p_1(1;1-\delta)(t)\,dt\right) \leq$$

$$\leq c^{\lambda_2}\left(\frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}\int\limits_{s_1}^{s_0+\omega} p_2(1;\delta)(t)\,dt-\frac{1-(1-c)^{\lambda_2}}{c^{\lambda_2}}\right). \tag{4.2}$$

On the other hand, if (2.2) is fulfilled with $i=2$ then we get

$$0<(1-c)^{\lambda_2}\left(1-\frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}\int\limits_{s_0}^{s_1} p_2(1;1-\delta)(t)\,dt\right)\leq$$

$$\leq c^{\lambda_2}\left(\frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}\int\limits_{s_0}^{s_1} p_1(1;\delta)(t)\,dt-\frac{1-(1-c)^{\lambda_2}}{c^{\lambda_2}}\right), \tag{4.3}$$

$$0<c^{\lambda_2}\left(1-\frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}\int\limits_{s_1}^{s_0+\omega} p_2(1;\delta)(t)\,dt\right)\leq$$

$$\leq (1-c)^{\lambda_2}\left(\frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}\int\limits_{s_1}^{s_0+\omega} p_1(1;1-\delta)(t)\,dt-\frac{1-c^{\lambda_2}}{(1-c)^{\lambda_2}}\right). \tag{4.4}$$

Multiplying the corresponding sides of the inequalities (4.1) and (4.2), respectively of the inequalities (4.3) and (4.4), by using the estimates $(1-A)(1-B)\geq 1-(A+B)$ on the left-hand side and $4AB\leq (A+B)^2$ on the right-hand side, on account of (1.5) we arrive at

$$4\left(1-\frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}P_i(\delta)\right)\leq\left(\frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}P_{3-i}(\delta)-\left(\frac{1-c^{\lambda_2}}{(1-c)^{\lambda_2}}+\frac{1-(1-c)^{\lambda_2}}{c^{\lambda_2}}\right)\right)^2. \tag{4.5}$$

Note that in view of (4.1), (4.2) and (4.3), (4.4), respectively, we have

$$\frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}P_{3-i}(\delta)>\frac{1-c^{\lambda_2}}{(1-c)^{\lambda_2}}+\frac{1-(1-c)^{\lambda_2}}{c^{\lambda_2}}.$$

Therefore, by using Lemma 3.3, from (4.5) it follows that

$$2\sqrt{1-\frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}P_i(\delta)}\leq\frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}P_{3-i}(\delta)-2(2^{\lambda_2}-1).$$

However, the latter inequality contradicts (2.3).

Proof of Theorem 2.2. According to Theorem 3.1 it is sufficient to show that the problem (3.1) – (3.3) has only the trivial solution for every $\delta\in[0,1]$. Therefore, assume on the contrary that for some $\delta\in[0,1]$ there exists a nontrivial solution (u_1,u_2) to the problem (3.1) – (3.3).

First assume that u_1 is still non-positive or still non-negative. With respect to Remark 3.1, without loss of generality we can assume that $u_1(t)\geq 0$ for $t\in[0,\omega]$. Therefore, Lemma 3.1 yields (3.4) which contradicts (2.4).

Assume therefore that u_1 attains both positive and negative values. Thus, according to Lemma 3.2 there exist $c \in (0,1)$, $s_0 \in [0,\omega)$, and $s_1 \in (s_0, s_0+\omega)$ such that (3.16) and (3.17) are fulfilled. Note that, because of $\lambda_2 < 1$, the function $x \mapsto (xA + (1-x)B)^{\lambda_2}$ for $x \in [0,1]$ is concave, and so we have

$$\left(\frac{A}{2} + \frac{B}{2}\right)^{\lambda_2} \geq \frac{A^{\lambda_2}}{2} + \frac{B^{\lambda_2}}{2}.$$

Consequently, $1 \geq 2^{\lambda_2 - 1}\big((1-c)^{\lambda_2} + c^{\lambda_2}\big)$, and so, if (2.4) holds for $i = 1$ then we obtain

$$0 < c^{\lambda_2}\left(1 - \frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}} \int\limits_{s_0}^{s_1} p_1(1;\delta)(t)\,dt\right) \leq$$

$$\leq (1-c)^{\lambda_2}\left(\frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}} \int_{s_0}^{s_1} p_2(1;1-\delta)(t)\,dt - 1\right), \tag{4.6}$$

$$0 < (1-c)^{\lambda_2}\left(1 - \frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}} \int\limits_{s_1}^{s_0+\omega} p_1(1;1-\delta)(t)\,dt\right) \leq$$

$$\leq c^{\lambda_2}\left(\frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}} \int\limits_{s_1}^{s_0+\omega} p_2(1;\delta)(t)\,dt - 1\right). \tag{4.7}$$

On the other hand, if (2.4) is fulfilled with $i = 2$ then we get

$$0 < (1-c)^{\lambda_2}\left(1 - \frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}} \int\limits_{s_0}^{s_1} p_2(1;1-\delta)(t)\,dt\right) \leq$$

$$\leq c^{\lambda_2}\left(\frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}} \int\limits_{s_0}^{s_1} p_1(1;\delta)(t)\,dt - 1\right), \tag{4.8}$$

$$0 < c^{\lambda_2}\left(1 - \frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}} \int\limits_{s_1}^{s_0+\omega} p_2(1;\delta)(t)\,dt\right) \leq$$

$$\leq (1-c)^{\lambda_2}\left(\frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}} \int\limits_{s_1}^{s_0+\omega} p_1(1;1-\delta)(t)\,dt - 1\right). \tag{4.9}$$

Multiplying the corresponding sides of the inequalities (4.6) and (4.7), respectively of the inequalities (4.8) and (4.9), using the estimates $(1-A)(1-B) \geq 1-(A+B)$ on the left-hand side and $4AB \leq (A+B)^2$ on the right-hand side, on account of (1.5) we arrive at

$$4\left(1 - \frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}}\, P_i(\delta)\right) \leq \left(\frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}}\, P_{3-i}(\delta) - 2\right)^2. \tag{4.10}$$

Note that in view of (4.6), (4.7) and (4.8), (4.9), respectively, we have

$$\frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}}\,P_{3-i}(\delta) > 2.$$

Therefore, from (4.10) it follows that

$$2\sqrt{1-\frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}}\,P_i(\delta)} \le \frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}}\,P_{3-i}(\delta) - 2.$$

However, the latter inequality contradicts (2.5).

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Received 30.01.22