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FEATURES OF APPLICATION OF INTEGRAL TRANSFORMATIONS TO THE SOLUTION OF SOME WAVE PROBLEMS

ОСОБЛИВОСТІ ЗАСТОСУВАННЯ ІНТЕГРАЛЬНИХ ПЕРЕТВОРЕНЬ ПРИ РОЗВ'ЯЗАННІ ДЕЯКИХ ХВИЛЬОВИХ ЗАДАЧ

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The solution of wave problems is usually associated with significant difficulties. One of the effective methods for solving such problems is the method of integral transformations. However, even using this method, sometimes, it is impossible to obtain a solution in the final form or at least in a form suitable for further numerical calculations. In this paper, using the example of calculation of the propagation of a wave arising in a plate from a spherical source, we propose a method for overcoming these difficulties. Simplification of the problem is achieved by splitting the total wave into elementary components with respect to wave numbers and complex frequencies. The solution is brought to the possibility of numerical calculations on a computer. The practical application of calculations can be useful, in particular, in the analysis of data obtained by the acoustic emission method.

Розв'язування хвильових задач зазвичай пов'язане зі значними труднощами. Одним із ефективних методів розв'язування таких задач є метод інтегральних перетворень. Але навіть при його використанні не завжди вдається отримати розв'язок у кінцевому вигляді або хоча б у придатному для подальших чисельних розрахунків. У цій роботі на прикладі розрахунку поширення хвилі, що виникає у пластині від джерела сферичної форми, показано один зі способів подолання таких труднощів. Спрощення задачі досягнуто шляхом розбиття сумарної хвилі на елементарні складові за хвильовими числами та комплексними частотами. Розв'язання доведено до можливості чисельних розрахунків на комп'ютері. Практичне застосування розрахунків може бути корисним, зокрема, при аналізі даних, які отримують методом акустичної емісії.

As shown back in the 70s of the 20th century by such experts in the field of studying the destruction of materials as F. McClintock [1], R. McMeeking [2], J. Rise, D. Tracey [3], S. Murakami [4], V. Tvergard [5], A. Gurson [6], D. Broek [7], and some other, one of the essential factors of destruction is the dynamic appearance of voids in materials. As a rule, these voids have a shape very close to spherical, even under conditions of rigid stress concentration [3, 8, 9]. The sharp formation of such voids leads to the appearance of waves in the materials, called acoustic emission (AE) waves. The AE method is widely used to determine the coordinates of developing defects and assess the current state of structural materials. Therefore, the solution of problems on the dynamic occurrence of voids is both theoretical and applied. Knowing the calculated waveforms and comparing them with those obtained by the AE equipment it is possible to estimate the dimensions and features of the location in the material of the defect that caused the wave with the corresponding characteristics.

Quantum fracture mechanics deals with the problems of propagation of AE waves in a material from detected defects [10]. Mathematically, this is modeled using delta functions, but it should be noted that in the equations of wave problems often there is some constant V^*, which describes the volume of the newly formed void.

Studies conducted with the participation of the author, in particular [11, 12], show a relationship between the dynamic processes occurring during the formation and merging of voids and the characteristics of AE signals. Thus, the abstract spherical AE source used in calculations acquires a specific physical meaning.

The calculation will further show which waves types make the most significant contribution to the recorded AE parameters which frequency bands are most suitable for their registration, and how the size of the void formed is related to the amplitude characteristic of the received AE signal. There are two main types of waves that are of practical importance in AE diagnostics-surface waves (Rayleigh) and internal waves (Lamb). The calculation allows their comparative analysis.

We will solve the problem for a plate of infinite dimensions with a thickness δ, bounded above and below by flat surfaces. This will make it possible to calculate the movement of waves over a distance sufficiently large for practice.

Let us assume that a tensile load Q acts on the plate material. The deformation under loading increases monotonically from some initial state. In this case, the medium is considered continuous, and the appearance of a spherical defect occurs at time t_0 in the weakest point of the material, described by a point with coordinates $(0, z_0)$ in a cylindrical coordinate system. In this case, there is a sharp increase in the volume of the newly formed void, which is characterized by a rapid reset of the deformation to zero or to some minimum value for a given period.

A new approach in compiling the deformation equation is to provide an unambiguous relationship between the wave frequency p and its wavenumber α through the wave propagation velocity C:

$$\delta_+(p \pm i\alpha C). \tag{1}$$

Expression (1) allows one to linearize the problem being solved.

For the calculation, the critical time t_0 is important, when the deformation abruptly changes its value. The process of material deformation up to the moment t_0 is monotonic. Dynamics arises at the t_0 moment and can be described by the impulse function $\delta_+(t - t_0)$. If moving the origin of coordinates in time to the t_0 point, then the distribution of the volume V_0^* suddenly changed at a depth V_0^* from the surface of the plate along the coordinate and time axes can be represented as the product of δ_+ functions:

$$\varepsilon^* = \frac{V_0^*}{2\pi r}\delta_+(r)\delta_+(t)\delta_+(z - z_0)\delta_+(p \pm i\alpha C), \tag{2}$$

where ε^* — relative initial strain (m^3), ${V_0}^*$ — newly emerged volume of empty space inside the plate, $\delta_+(r)$, $\delta_+(z - z_0)$ — infinitesimal positive increments of the specified space in the direction of coordinates r and z, $\delta_+(t - t_0)$ — an infinitesimal positive increment of time during which the described deformation impulse occurs.

The problem can be solved by the method of integral transformations which allows summing up solutions expressed by infinitesimal elementary waves and satisfying boundary, initial, and physical conditions. The sum of such elementary waves determines the total common wave propagating in the plate. Note that in this way it is possible to obtain the sum of only several individual elementary waves that satisfy the given conditions, and, if necessary, one elementary wave. This provides wide opportunities for the analysis of wave processes of varying complexity.

Let us derive the differential equations of the wave problem under the assumption that at time t_0 there are initial deformations ε^*.

Representation of an internal defect in the form of δ_+ functions product makes it possible to obtain dynamic damage of the plate material in the form of a spherical discontinuity.

Let us assume that the source-defect appeared in the plate in the form of a microexplosion with volume V_0^* during a very short time interval with elastic wave motion symmetric in all directions. Then the equations describing the propagation of waves can be written as:

$$\begin{aligned} &\nabla^2\varphi - \frac{1}{C_1^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{1+\nu}{1-\nu}V_0^*\frac{\delta(r)}{2\pi r}\delta(z-z_0)\delta(t)\delta(p\pm i\alpha C_\alpha), \\ &\nabla^2\psi - \frac{1}{C_2^2}\frac{\partial^2\psi}{\partial t^2} = 0. \end{aligned} \tag{3}$$

where φ and ψ are functions defining displacements in the plate; t — time, ν — Poisson's ratio, $C_1,\ C_2$ — propagation velocities of longitudinal and transverse waves, respectively, $C_\alpha,\ p,\ \alpha$ — parameters of elementary waves in a packet forming a wave (velocity, frequency, wavenumber),

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}.$$

Let's solve system (3) under the assumption that the body is bounded from two sides and the conditions $\sigma_z = \tau_{rz} = 0$ at $z = 0,\ \delta$ are met on the boundary.

In this case, the initial conditions of the problem are determined by the function $\delta(t-t_0)$ what indicates that at $t = t_0$ a radiation source has appeared, and at $t = t_0 + 0$ the radiation source has disappeared.

Applying the Laplace transformation by time t (with parameter p) and the Hankel transformation by coordinate r (with parameter α) gives:

$$\begin{aligned} &\frac{\partial^2\overline{\overline{\varphi}}}{\partial z^2} - \left(\alpha^2 + \frac{p^2}{C_1^2}\right)\overline{\overline{\varphi}} = -\frac{1+\nu}{1-\nu}\frac{V_0^*}{2\pi}\delta(z-z_0)\delta(p\pm i\alpha C_\alpha), \\ &\frac{\partial^2\overline{\overline{\psi}}}{\partial z^2} - \left(\alpha^2 + \frac{p^2}{C_2^2}\right)\overline{\overline{\psi}} = 0. \end{aligned} \tag{4}$$

Here and below, an integral transformation performed once on some expression will be denoted by one horizontal line above it, a two-fold one by two lines.

The propagation velocities of longitudinal and transverse waves C_1 and C_2 are material constants:

$$C_1 = \sqrt{\frac{E}{\rho}\frac{1-\nu}{(1+\nu)(1-2\nu)}}, \qquad C_2 = \sqrt{\frac{E}{\rho}\frac{1}{2(1+\nu)}},$$

where E is the modulus of elasticity, ν is the transverse strain coefficient (Poisson's ratio), ρ is the density of the material.

The boundary conditions after transformations will take the form at $z = 0,\ \delta$:

$$\begin{aligned} \frac{1}{G}\,\overline{\overline{\tau}}_{rz} &= 2\frac{\partial\overline{\overline{\varphi}}}{\partial z} + \frac{\partial^2\overline{\overline{\psi}}}{\partial z^2} + \alpha^2\overline{\overline{\psi}} = 0, \\ \frac{1}{G}\,\overline{\overline{\sigma}}_z &= \left(\frac{p^2}{2C_2^2} + \alpha^2\right)\overline{\overline{\varphi}} + \alpha^2\,\frac{\partial\overline{\overline{\psi}}}{\partial z} = 0. \end{aligned} \tag{5}$$

The solution of equations (5) is a rather complicated problem. Earlier calculations for somewhat simpler equations of this type have shown that in the process of solution, frequently repeating multipliers arise, like $\gamma_2 = \sqrt{1 - \dfrac{C_\alpha^2}{C_2^2}}$. The presence of such multipliers dictates the search for a solution to equations (5) in two ways, since for a speed C_2 greater than C_α, the expression becomes a complex number and requires a different way of solving the equations.

We will look for a solution for each part separately. Let us first consider the case when the wave velocity in the plates is less than the velocity C_2. The solution of the boundary value problem (4), (5) will be sought in the form:

$$\begin{aligned} \overline{\overline{\varphi}} &= Ce^{z\beta_1} + \frac{P_1}{\beta_1}\,e^{-|z-z_0|\beta_1}, \\ \overline{\overline{\psi}} &= Ae^{-z\beta_2} + Be^{z\beta_2}. \end{aligned} \tag{6}$$

where

$$\beta_1 = \sqrt{\alpha^2 + p^2/C_1^2}, \qquad \beta_2 = \sqrt{\alpha^2 + p^2/C_2^2}, \qquad P_1 = \frac{1+\nu}{1-\nu}\,\frac{V_0^*}{4\pi}\,\delta(p \pm i\alpha C_\alpha),$$

and ν — Poisson's ratio, $A,\ B$ and C — constants to be determined. The remaining quantities included in the equations are described above.

Substituting the values of the functions $\overline{\overline{\varphi}},\ \overline{\overline{\psi}}$ from (6) into equations (5), we arrive at a system of algebraic equations for determining the constants $A,\ B,\ C,$ and the relationship between $p,$ $\alpha,$ and C:

$$\begin{aligned} &A\left(\alpha^2 + \beta_2^2\right) + B\left(\alpha^2 + \beta_2^2\right) + C\,2\beta_1 = -2P_1e^{-z_0\beta_1}, \\ &A\left(\alpha^2 + \beta_2^2\right)e^{-\delta\beta_2} + B\left(\alpha^2 + \beta_2^2\right)e^{\delta\beta_2} + C2\beta_1e^{\delta\beta_1} = 2P_1e^{-(\delta-z_0)\beta_1}, \\ &A\left(-\alpha^2\beta_2\right) + B\alpha^2\beta_2 + C\left(\frac{p^2}{2C_2^2} + \alpha^2\right) = -\frac{P_1}{\beta_1}\left(\frac{p^2}{2C_2^2} + \alpha^2\right)e^{-z_0\beta_1}, \\ &A\left(-\alpha^2\beta_2\right)e^{-\delta\beta_2} + B\alpha^2\beta_2e^{\delta\beta_2} + C\left(\frac{p^2}{2C_2^2} + \alpha^2\right)e^{\delta\beta_1} = -\frac{P_1}{\beta_1}\left(\frac{p^2}{2C_2^2} + \alpha^2\right)e^{-(\delta-z_0)\beta_1}. \end{aligned} \tag{7}$$

The fourth equation of system (7) defines the relation between wave numbers $\alpha,$ elementary wave velocities C and their frequency characteristics $p.$ This is the so-called *spectral equation* of the plate. It determines the existence of certain waves.

We solve the system of the first three equations for $A,\ B,\ C$ and substitute the found coefficients into the fourth equation. Having then done the inverse Laplace transformation (integrating over p in the range from $-i\infty$ to $i\infty$), we obtain, considering that

$$\gamma_1 = \sqrt{1 - \frac{C_\alpha^2}{C_1^2}}, \qquad \gamma_2 = \sqrt{1 - \frac{C_\alpha^2}{C_2^2}}, \qquad P_0 = \frac{1+\nu}{1-\nu}\,\frac{V_0^*}{4\pi^2}.$$

relationship equation between α and C_α/C_2:

$$\frac{2\gamma_2}{\Delta_1}\cosh[\alpha\gamma_1(\delta - z_0)]\left\{\left[\left(1+\gamma_2^2\right)^2 - 4\gamma_1\gamma_2\right]e^{-\alpha\gamma_2\delta} + \left[\left(1+\gamma_2^2\right)^2 + 4\gamma_1\gamma_2\right]e^{\alpha\gamma_2\delta}\right\} -$$
$$- \frac{1}{2\Delta_1}\left\{\frac{1}{\gamma_1}\left(1+\gamma_2^2\right)^4 e^{\alpha\gamma_1(\delta - z_0)}\sinh(\alpha\gamma_2\delta) + \right.$$
$$\left. + 4\gamma_2\left(1+\gamma_2^2\right)^2 e^{\alpha\gamma_1(\delta - z_0)}\cosh(\alpha\gamma_2\delta) + 4\gamma_2\left(1+\gamma_2^2\right)^2 e^{\alpha\gamma_1 z_0}\right\} -$$
$$- \frac{1}{2\gamma_1}\left(1+\gamma_2^2\right)e^{-\alpha\gamma_1(\delta - z_0)} = 0, \tag{8}$$

where

$$\Delta_1 = -\left(1+\gamma_2^2\right)^3\sinh(\alpha\gamma_2\delta) - 4\gamma_1\gamma_2\left(1+\gamma_2^2\right)\left[\cosh(\alpha\gamma_2\delta) - e^{\alpha\gamma_1\delta}\right].$$

Letting δ tend to infinity in the spectral equation (8) and carrying out simple transformations, we obtain a simpler equation for determining the wave propagation velocity in thick plates:

$$4\gamma_1\gamma_2 - \left(1+\gamma_2^2\right)^2 = 0. \tag{9}$$

Substituting C_1^2 from the ratio $\dfrac{C_2^2}{C_1^2} = \dfrac{1-2\nu}{2(1-\nu)}$ to γ_1, with $\nu = 0.3$ we get

$$\gamma_1 = \sqrt{1 - 0.286 C_\alpha^2/C_2^2}.$$

Solving equation (9) under these conditions, we obtain the velocity of the propagating wave for a plate whose thickness tends to infinity. This speed $C_3 = 0.927C_2$, in contrast to thin plates, is constant over the entire range of wave numbers from 0 to infinity.

Let us determine the displacements of the plate surface. We express the displacement w as:

$$\overline{\overline{w}}(z) = \frac{\partial\overline{\overline{\varphi}}}{\partial z} + \alpha^2\overline{\overline{\psi}}.$$

Using the first relation (6), we rewrite the expression in the following form:

$$\overline{\overline{w}}(z) = \beta_1 C e^{z\beta_1} + P_1 e^{-(z_0 - z)\beta_1} + \alpha^2\left[Ae^{-z\beta_2} + Be^{z\beta_2}\right] \quad \text{for} \quad z \le z_0. \tag{10}$$

Applying the inverse Laplace and Hankel transformations to expression (10), at $z = 0$ we obtain:

$$w = \int\limits_0^\infty \alpha\left[\alpha\gamma_1 C + P_0 e^{-\alpha\gamma_1 z_0} + \alpha^2(A+B)\right]J_0(\alpha r)\sin(\alpha C_\alpha t)\,d\alpha, \tag{11}$$

where

$$P_0 = \frac{1+\nu}{1-\nu}\frac{V_0^*}{4\pi^2}.$$

The displacement of value w on the plate surface $(z = 0)$ is determined after substituting the values of the coefficients A, B, and C into expression (11) and integrating within the indicated limits.

It should be noted that equation (8) in the range of wave velocities from 0 to C_2 in thick plates allows the existence of one wave propagating at a speed of $C_3 = 0.927C_2$.

Then, considering what has been said, for large values of $\alpha\gamma_1\delta$, expression (11) can be rewritten as:

$$w \approx -\frac{1+\nu}{1-\nu}\frac{V_0^*}{4\pi^2}\frac{1-\gamma_2^2}{1+\gamma_2^2}\int\limits_{\alpha_n}^{\infty} \alpha \exp(-\alpha\gamma_1 z_0) J_0(\alpha r)\sin(\alpha C_3 t)\, d\alpha. \tag{12}$$

Integration is limited from the lower side by the limiting value of the wavenumber α_n, at which the wave propagation velocity reaches the calculated value of $0.927C_2$. If we set the plate thickness to infinity (which in practice is approximately 2.6 cm or more), the lower limit of integration can be taken equal to zero. Then integration within the range from 0 to infinity in formula (12) becomes possible. After integration, we get:

$$w = -\frac{1+\nu}{1-\nu}\frac{V_0^*}{4\pi^2}\frac{1-\gamma_2^2}{1+\gamma_2^2}\left\{\frac{\sqrt{(\gamma_1 z_0)^3 + (C_3 t)^2}}{\left\{[(\gamma_1 z_0)^2 - (C_3 t)^2 + r^2]^2 + (2\gamma_1 z_0 C_3 t)^2\right\}^{3/4}} \times\right.$$
$$\left.\times \sin\left\{\arctan\left(\frac{C_3 t}{\gamma_1 z_0}\right) - \frac{3}{2}\arctan\left[\frac{2\gamma_1 z_0 C_3 t}{(\gamma_1 z_0)^2 - (C_3 t)^2 + r^2}\right]\right\}\right\}, \tag{13}$$

where

$$\gamma_1 = \sqrt{1 - \frac{C_3^2}{C_1^2}}, \qquad \gamma_2 = \sqrt{1 - \frac{C_3^2}{C_2^2}}, \qquad C_3 = 0.927\, C_2.$$

It should be noted that in formula (11) each elementary wave is determined by its propagation velocity and its limiting wavenumber α. Thus, in the general case, the calculation of waves is carried out according to the same formula (11) but at different values of velocities and wavenumbers.

So, for large values of $\alpha\gamma_1\delta$ formula (11) has been simplified. The value of the lower limit of integration in (12) α_n is the value of the wave number, at which, for a given plate thickness, zero is provided in the spectral equation (6).

The displacement of the plates outer surfaces consists of two components: in the direction of the z axis — the w component caused by transverse waves, and in the direction of the r axis — the u component caused by the action of the longitudinal wave.

It can be shown that the second component for waves moving at velocities less than C_2 can be neglected due to its smallness compared to the first. So, if the longitudinal displacements in the plate from the action of an instantaneous radiation source in the converted form can be written as

$$\overline{\overline{u}} = \frac{\partial}{\partial r}\left(\overline{\overline{\varphi}} + \frac{\partial \overline{\overline{\psi}}}{\partial z}\right),$$

then the formula for determining the displacements u of the upper surface of the plate in the direction of the r axis can be written as:

$$u = \int_{\alpha_n}^{\infty} \alpha^2 \left[C + \frac{P_0}{\alpha\gamma_1} e^{-\alpha\gamma_1 z_0} - \alpha\gamma_2(A - B)\right] \sin(\alpha C_\alpha t) J_1(\alpha r) d\alpha, \tag{14}$$

and for plates with large values of $\alpha\gamma_1\delta$ after substituting the values of the coefficients $A,\ B,$ and C we get:

$$u = \frac{1+\nu}{1-\nu}\frac{V_0^*}{4\pi^2}\frac{1-\gamma_2^2}{2\gamma_1}\int_{\alpha_n}^{\infty} \alpha J_1(\alpha r)\sin(\alpha C_3 t) e^{-\alpha\gamma_1 z_0}\, d\alpha.$$

After integrating this expression from zero to infinity, we get:

$$u = \frac{1+\nu}{1-\nu}\frac{V_0^*}{4\pi^2}\frac{1-\gamma_2^2}{2\gamma_1}\left\{\frac{r\sin\left[\frac{3}{2}\arctan\frac{2\gamma_1 z_0 C_3 t}{(\gamma_1 z_0)^2 + r^2 - (C_3 t)^2}\right]}{\left\{\left[(\gamma_1 z_0)^2 + r^2 - (C_3 t)^2\right]^2 + (2\gamma_1 z_0 C_3 t)^2\right\}^{3/4}}\right\}. \tag{15}$$

Calculations of displacements u and w show that the magnitude of displacements for a thick plate in the direction of the r axis caused by the action of a longitudinal wave is approximately 2 times less than the displacements caused by a transverse wave.

Let us now consider the case when the wave velocity in the plates exceeds the velocity C_2.

We can write the solution of equations (4), (5) for this case in the form:

$$\overline{\overline{\varphi}} = Ce^{z\beta_1} + \frac{1+\nu}{1-\nu}\frac{V_0^*}{4\pi}\frac{e^{-|z-z_0|\beta_1}}{\beta_1}\delta_+(p \pm i\alpha C_\alpha),$$

$$\overline{\overline{\psi}} = A\sin(z\beta_2) + B\cos(z\beta_2),$$

where ν is Poisson's ratio, $A,\ B$ and C are constants that need to be determined,

$$P_1 = \frac{1+\nu}{1-\nu}\frac{V_0^*}{4\pi}\delta(p - i\alpha C_\alpha), \qquad \beta_1 = \sqrt{\alpha^2 + \frac{p^2}{C_1^2}}, \qquad \beta_2 = \sqrt{\frac{p^2}{C_2^2} + \alpha^2}.$$

The rest of the quantities included in the equations are described above.

Substituting the expressions for φ and ψ into the equations of boundary conditions (5), we obtain:

$$B\left(\alpha^2 - \beta_2^2\right) + C\,2\beta_1 + 2P_1 e^{-z_0\beta_1} = 0,$$
$$A\left(\alpha^2 - \beta_2^2\right)\sin(\delta\beta_2) + B\left(\alpha^2 - \beta_2^2\right)\cos(\delta\beta_2) + 2C\beta_1 e^{\delta\beta_1} - 2P_1 e^{-(\delta - z_0)\beta_1} = 0,$$
$$-A\alpha^2\beta_2 - C\left(\frac{p^2}{2C_2^2} + \alpha^2\right) - \frac{P_1}{\beta_1}\left(\frac{p^2}{2C_2^2} + \alpha^2\right)e^{-z_0\beta_1} = 0,$$

$$- A\alpha^2\beta_2\cos(\delta\beta_2) + B\alpha^2\beta_2\sin(\delta\beta_2) -$$

$$- C\left(\frac{p^2}{2C_2^2} + \alpha^2\right)e^{\delta\beta_1} - \frac{P_1}{\beta_1}\left(\frac{p^2}{2C_2^2} + \alpha^2\right)e^{-(\delta - z_0)\beta_1} = 0. \tag{16}$$

After performing the inverse Laplace transformation for the first three equations (16) and solving the system, we find expressions for the coefficients $A,\ B,\ C,$ and after substituting the values of the coefficients $A,\ B,\ C$ into the fourth equation (16), we obtain the spectral equation of the plate for the case when $C > C_2$:

$$\frac{e^{\alpha\gamma_1\delta}}{2\gamma_1}\frac{\left(1-\gamma_2^2\right)^2\sin(\alpha\delta\gamma_2)e^{-\alpha\gamma_1 z_0} + 4\gamma_1\gamma_2\left[e^{-\alpha\gamma_1(\delta - z_0)} + \cos(\alpha\delta\gamma_2)e^{-\alpha\gamma_1 z_0}\right]}{4\gamma_1\gamma_2\left[\cos(\alpha\delta\gamma_2) - e^{\alpha\delta\gamma_1}\right] + \left(1-\gamma_2^2\right)^2\sin(\alpha\delta\gamma_2)}\left(1-\gamma_2^2\right) -$$

$$- \frac{4\gamma_2\left(1-\gamma_2^2\right)\cos(\alpha\delta\gamma_2)\cosh\left[\alpha\gamma_1(\delta - z_0)\right]}{4\gamma_1\gamma_2\left[\cos(\alpha\delta\gamma_2) - e^{\alpha\delta\gamma_1}\right] + \left(1-\gamma_2^2\right)^2\sin(\alpha\delta\gamma_2)} +$$

$$+ \frac{16\gamma_1\gamma_2^2\sin(\alpha\delta\gamma_2)\cosh\left[\alpha\gamma_1(\delta - z_0)\right]}{\left(1-\gamma_2^2\right)\left[4\gamma_1\gamma_2\left(\cos(\alpha\delta\gamma_2) - e^{\alpha\delta\gamma_1}\right) + \left(1-\gamma_2^2\right)^2\sin(\alpha\delta\gamma_2)\right]} -$$

$$- \frac{1-\gamma_2^2}{2\gamma_1}e^{-\alpha\gamma_1(\delta - z_0)} = 0. \tag{17}$$

Knowing the spectral equation of the plate (17), i.e., relationship between the wavenumber α and the propagation velocity of elementary waves C, we can determine the displacements w that occur in the plate when waves move with velocities exceeding C_2. Thus, displacements on the plate surface at $z = 0$ with the accepted initial conditions after performing the inverse Hankel transformation by α and Laplace by p will have the form:

$$w = \int_0^\infty P_0\Bigg\{ - \frac{\left(1-\gamma_2^2\right)^2\sin(\alpha\delta\gamma_2)e^{-\alpha\gamma_1 z_0} + 4\gamma_1\gamma_2\left[e^{-\alpha\gamma_1(\delta - z_0)} + \cos(\alpha\delta\gamma_2)e^{-\alpha\gamma_1 z_0}\right]}{4\gamma_1\gamma_2\left(\cos(\alpha\delta\gamma_2) - e^{\alpha\delta\gamma_1}\right) + \left(1-\gamma_2^2\right)^2\sin(\alpha\delta\gamma_2)} +$$

$$+ \frac{16\gamma_1\gamma_2\cosh\left[\alpha\gamma_1(\delta - z_0)\right]}{\left(1-\gamma_2^2\right)\left[4\gamma_1\gamma_2\left(\cos(\alpha\delta\gamma_2) - e^{\alpha\delta\gamma_1}\right) + \left(1-\gamma_2^2\right)^2\sin(\alpha\delta\gamma_2)\right]} + e^{-\alpha\gamma_1 z_0}\Bigg\} \times$$

$$\times\ \alpha J_0(\alpha r)\sin(\alpha C_\alpha t)\,d\alpha. \tag{18}$$

Displacements of the plate surface during wave propagation will consist of the sum of displacements determined by formulas (11) (Rayleigh wave) and (18) (Lamb wave).

For a plate of infinite dimensions, the number of roots of the spectral equation will be infinite. However, the numerical method for calculating w requires limiting the upper limit of integration to a specific value. The necessary accuracy of the calculations was achieved by limiting the number α so that its further increase would not lead to noticeable errors in the calculations of w.

It can be noted that all orders of waves obtained from the spectral equation single out one wave in the entire frequency range, moving at a speed of 0.4433 cm/μs. This wave represents the main, largest displacement of the plate surface, which is typical for Lamb waves.

To implement the calculation of displacements $w,$ we will perform integration in expression (18) considering the discreteness of C values for continuous values of the wavenumber $\alpha.$

In this case, we will assume that the values of the integrand will be determined as the sum of its values for each C_i determined from expression (17). Then:

$$w = \int\limits_0^\infty \sum_{i=1}^{k} P_0 \Biggl\{ -\frac{\left(1-\gamma_2^2\right)^2 \sin(\alpha\delta\gamma_2) e^{-\alpha\gamma_1 z_0} + 4\gamma_1\gamma_2 \left[e^{-\alpha\gamma_1(\delta - z_0)} + \cos(\alpha\delta\gamma_2) e^{-\alpha\gamma_1 z_0}\right]}{4\gamma_1\gamma_2 \left[\cos(\alpha\delta\gamma_2) - e^{\alpha\delta\gamma_1}\right] + \left(1-\gamma_2^2\right)^2 \sin(\alpha\delta\gamma_2)} +$$

$$+ \frac{16\gamma_1\gamma_2 \cosh\left[\alpha\gamma_1(\delta - z_0)\right]}{\left(1-\gamma_2^2\right)\left[4\gamma_1\gamma_2 \left[\cos(\alpha\delta\gamma_2) - e^{\alpha\delta\gamma_1}\right] + \left(1-\gamma_2^2\right)^2 \sin(\alpha\delta\gamma_2)\right]} + e^{-\alpha\gamma_1 z_0} \Biggr\} \times$$

$$\times\, \alpha J_0(\alpha r) \sin(\alpha C_\alpha t)\, d\alpha, \tag{19}$$

where

$$\gamma_1 = \sqrt{1 - \frac{C_i^2}{C_1^2}}, \qquad \gamma_2 = \sqrt{\frac{C_i^2}{C_2^2} - 1}, \qquad P_0 = \frac{1+\nu}{1-\nu}\,\frac{V_0^*}{4\pi^2}.$$

Here the values γ_1 and γ_2 also have the i-th index. In the above formula, these indices are not affixed to avoid unnecessary difficulty in reading the expression. Integration (19) is possible only by a numerical method and is performed as follows:

1. From the spectral equation (19) find the values of velocities C_i for each given value of the wavenumber α_i.

2. For the selected r_j, the total value of the integrand over k is plotted along the α_i axis for all k orders of the roots C_i.

3. Integrate expression (19) for the obtained values of the integrand and proceed to the next value r_{j+1}.

4. Repeat steps 1 – 3 for all values of r_j.

Thus, the necessary procedures for calculating wave processes on a computer were prepared and programs were created that calculate wave fields in plates by summing elementary waves determined by given boundary and initial conditions.

Useful ways to apply the results of calculations are the optimization of the measured parameters of equipment based on the AE principle and the analysis of AE data.

Conclusions. An original method for calculating wave fields is proposed, based on the relationship between wave speed, frequency, and wave number, using the summation of elementary waves in the spectrum.

The spectral equation makes it possible to determine the elementary waves that exist under given conditions.

The advantage of the method is the possibility of obtaining various wave parameters depending on the established boundary and initial conditions determined by the spectral equation.

The equations obtained as a result make it possible to calculate the wave fields for the Rayleigh and Lamb waves with sufficient accuracy for practice.

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