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ON THE HYBRID CAPUTO-PROPORTIONAL FRACTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES*

ПРО ГИБРИДНІ ДРОБОВІ ДИФЕРЕНЦІАЛЬНІ ВКЛЮЧЕННЯ, ПРОПОРЦІЙНІ ЗА КАПУТО У ПРОСТОРАХ БАНАХА

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The purpose of this article is to investigate the existence of solutions for a nonlinear fractional differential inclusion in the sense of hybrid Caputo-proportional fractional derivatives (HCPFDs) in Banach space. The main result is discussed using the set-valued concern from the Mönch fixed point theorem along with the Kuratowski measure of non-compactness. An example is used to demonstrate theoretical findings.

Метою цієї роботи є дослідження існування розв'язків нелінійних дробових диференціальних включень у сенсі дробових похідних, пропорційних за Капуто у просторі Банаха. З використанням теореми Мьонха про нерухому точку й міри некомпактності Куратовського проаналізовано головний результат. Наведено приклад, яких ілюструє отримані теоретичні результати.

1. Introduction. In the past two decades, fractional calculus has become widespread due to its great relevance to reality and their dignified influence in describing several real-world problems in physics, mechanics and engineering. For intance, we refer the reader to the works [1 – 11].

For fractional derivatives and fractional integrals, there are a variety of methods to define them, including the Riemann – Liouville method, Caputo method, Marchand method, tempered method, Hilfer method, and Atangana – Baleanu method [9, 12 – 14]. According to their structure and qualities, these various definitions can be grouped together into general classes [15].

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The so-called Caputo fractional derivative is of special interest when dealing with fractional differential equations (FDEs). When applied with FDEs, the Caputo fractional derivative requires more natural initial conditions than the Riemann – Liouville fractional derivative [10]. When other fractional derivatives are made from the fractional integral operators, they are called "Riemann – Liouville type" or "Caputo type." These two are so important that many other fractional derivatives are made from them. While the standard (N-order) derivative of the fractional integral is used to derive the Riemann – Liouville fractional derivative (RLFD), that used to derive the Caputo fractional derivative (CFD) is derived by applying the fractional integral to a standard function derivative.

In 2014, [16] introduced the concept of conformable derivatives as a local, limit-based definition. In the beginning, it was thought of as a conformable fractional derivative. However, it doesn't have all of the properties that make fractional derivatives useful. Other studies have extensively researched this operator, its features, and applications, with [17] being one of the most notable.

Recently, a new type of fractional derivative, called proportional Caputo fractional derivative is introduced in [18]. In the case of constant kernals this type of derivative is applied for model and study investigate the HIV epidemic [19], in [20] for fractional building heating and cooling model, in [21] for study the unsteady and incompressible viscous fluid flow, in [22] to discussed the fractional model of Brinkman type fluid holding hybrid nanoparticles, in [23] to study the unsteady and an incompressible magnetohydrodynamic viscous fluid with heat transfer. In the case of general kernels this type of derivative is applied in [24, 25] to obtain existence results and the Ulam stability.

Due to the importance of fractional differential inclusions (FDIs) in mathematical modeling of problems in game theory, stability, optimal control, and so on. For this reason, many contributions have been investigated by some researchers [26 – 32].

On the other hand, the theory of measure of noncompactness is an essential tool in investigating the existence of solutions for nonlinear integral and differential equations, see, for example, the recent papers [33 – 37] and the references existing therein.

In [38], Benchohra et al. studied the existence of solutions for the FDIs with boundary conditions

$$\begin{cases} {}^{C}\mathcal{D}^{r}y(t) \in G(t, y(t)), \quad \text{a.e. on} \quad [0,T], \quad 1 < r < 2, \\ y(0) = y_0, \quad y(T) = y_T, \end{cases}$$

where ${}^{C}\mathcal{D}^{r}$ is the CFD, $G\colon [0,T] \times \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ is a multi-valued map, $y_0, y_T \in \mathbf{E}$ and $(\mathbf{E}, |\cdot|)$ is a Banach space.

In the present work, we are interested in studying the existence of solutions for the following nonlinear FDIs with the HCPFDs:

$$\begin{cases} {}^{PC}_{\ 0}\mathcal{D}^{\alpha}_{t}x(t) \in F(t, x(t)), \quad \text{a.e. on} \quad \mathbf{J} := [0,b], \quad 0 < \alpha < 1, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where ${}^{PC}_{\ 0}\mathcal{D}^{\alpha}_{t}$ denotes the HPCFD of order α, $(\mathbf{E}, |\cdot|)$ is a Banach space, $\mathfrak{P}(\mathbf{E})$ is the family of all nonempty subsets of $\mathbf{E}$, $x_0 \in \mathbf{E}$ and $F\colon \mathbf{J} \times \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ is a given multi-valued map. By using the set-valued issue of Mönch fixed point theorem along with the Kuratowski measure of noncompactness, we investigate the inclusion problem (1.1) in the case when the right hand side is convex-valued.

It is worth noting that the relevant results of FDIs with the HPCFDs are scarce. So the main goal of the present work is to contribute to the development of this area.

2. Preliminaries. We explore the new definitions of the generalized HPCFD.

Definition 2.1 [18]**.** *The HPCFD of order* $\alpha \in (0,1)$ *of a differentiable function* $g(t)$ *is given by*

$$ {}_{0}^{PC}\mathcal{D}_t^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \big(k_1(\alpha,\tau)g(t) + k_0(\alpha,\tau)g'(t)\big)(t-\tau)^{-\alpha}\, d\tau, $$

where the domain of the function is determined by requiring that g *be differentiable and that both* g *and* g' *be locally* L^1 *functions on the positive reals,* $k_0, k_1 : [0,1]\times\mathbb{R} \to [0,\infty)$ *are continuous functions of the variable* t *and the parameter* $\alpha \in [0,1]$ *which satisfy the following conditions for all* $t \in \mathbb{R}$*:*

$$ \lim_{\alpha\to 0^+} k_0(\alpha,t) = 0, \qquad \lim_{\alpha\to 1^-} k_0(\alpha,t) = 1, \quad k_0(\alpha,t) \neq 0, \quad \alpha \in (0,1], $$

$$ \lim_{\alpha\to 0^+} k_1(\alpha,t) = 1, \qquad \lim_{\alpha\to 1^-} k_1(\alpha,t) = 0, \quad k_1(\alpha,t) \neq 0, \quad \alpha \in [0,1). $$

Definition 2.2 [18]**.** *The inverse operator of the HPCFD of order is given by*

$$ {}_{0}^{PC}\mathcal{I}_t^{\alpha} g(t) = \int\limits_0^t \exp\left(-\int\limits_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) \frac{{}_{0}^{RL}\mathcal{D}_u^{1-\alpha} g(u)}{k_0(\alpha,u)}\, du, \tag{2.1} $$

where ${}_{0}^{RL}\mathcal{D}_u^{1-\alpha}$ *denotes the RLFD of order* $1-\alpha$ *and is given by*

$$ {}_{0}^{RL}\mathcal{D}_u^{1-\alpha} g(u) = \frac{1}{\Gamma(\alpha)} \frac{d}{du} \int\limits_0^u (u-s)^{\alpha-1} g(s)\, ds. \tag{2.2} $$

We suggest the reader to Kilbas et al. [5] for further information.

Proposition 2.1 [18]**.** *The following inversion relations:*

$$ {}_{0}^{PC}\mathcal{D}_t^{\alpha}\ {}_{0}^{PC}\mathcal{I}_t^{\alpha} g(t) = g(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \lim_{t\to 0}\ {}_{0}^{RL}\mathcal{I}_t^{\alpha} g(t), $$

$$ {}_{0}^{PC}\mathcal{I}_t^{\alpha}\ {}_{0}^{PC}\mathcal{D}_t^{\alpha} g(t) = g(t) - \exp\left(-\int\limits_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds \right) g(0) $$

are satisfied.

Proposition 2.2 [18]**.** *The HPCFD operator* ${}_{0}^{PC}\mathcal{D}_t^{\alpha}$ *is nonlocal and singular.*

Remark 2.1 [18]**.** We obtain the following exceptional instances from the limiting cases $\alpha \to 0$ and $\alpha \to 1$:

$$ \lim_{\alpha\to 0}\ {}_{0}^{PC}\mathcal{D}_t^{\alpha} g(t) = \int\limits_0^t g(\tau)\, d\tau, $$

$$ \lim_{\alpha\to 1}\ {}_{0}^{PC}\mathcal{D}_t^{\alpha} g(t) = g'(t). $$

Denote by $C(\mathbf{J},\mathbf{E})$ the Banach space of all continuous functions from $\mathbf{J}$ to $\mathbf{E}$ with the norm $\|x\| = \sup_{t\in \mathbf{J}} |x(t)|$. By $L^1(\mathbf{J},\mathbf{E})$, we indicate the space of Bochner integrable functions from $\mathbf{J}$ to $\mathbf{E}$ with the norm $\|x\|_1 = \displaystyle\int_0^b |x(t)|\,dt$.

2.1. Multi-valued maps analysis. For the Banach space $(\mathbf{E}, |\cdot|)$, let $\mathfrak{P}_{\mathbf{cl}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E})\colon Z \text{ is closed}\}$, $\mathfrak{P}_{\mathbf{bd}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E})\colon Z \text{ is bounded}\}$, $\mathfrak{P}_{\mathbf{cp}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E})\colon Z \text{ is compact}\}$, and $\mathfrak{P}_{\mathbf{cvx}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E})\colon Z \text{ is convex}\}$.

– A multi-valued map $\mathfrak{U}\colon \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ is convex (closed) valued, if $\mathfrak{U}(x)$ is convex (closed) for all $x \in \mathbf{E}$.

– $\mathfrak{U}$ is bounded on bounded sets if $\mathfrak{U}(B) = \cup_{x\in B}\mathfrak{U}(x)$ is bounded in $\mathbf{E}$ for any $B \in \mathfrak{P}_{\mathbf{bd}}(\mathbf{E})$, i.e. $\sup_{x\in B}\{\sup\{\|y\|\colon y \in \mathfrak{U}(x)\}\} < \infty$.

– $\mathfrak{U}$ is called upper semi-continuous on $\mathbf{E}$ if for each $x^* \in \mathbf{E}$, the set $\mathfrak{U}(x^*)$ is nonempty, closed subset of $\mathbf{E}$, and if for each open set N of $\mathbf{E}$ containing $\mathfrak{U}(x^*)$, there exists an open neighborhood N^* of x^* such that $\mathfrak{U}(N^*) \subset N$.

– $\mathfrak{U}$ is completely continuous if $\mathfrak{U}(B)$ is relatively compact for each $B \in \mathfrak{P}_{\mathbf{bd}}(\mathbf{E})$.

– If $\mathfrak{U}$ is a multi-valued map that is completely continuous with nonempty compact values, then $\mathfrak{U}$ is u.s.c. if and only if $\mathfrak{U}$ has a closed graph (that is, if $x_n \to x_0,\ y_n \to y_0$, and $y_n \in \mathfrak{U}(x_n)$, then $y_0 \in \mathfrak{U}(x_0)$.

For more details about multi-valued maps, we refer to the book of Deimling [39].

Definition 2.3. *A multi-valued map* $F\colon \mathbf{J}\times\mathbf{E} \to \mathfrak{P}(\mathbf{E})$ *is said to be Carathéodory if*

(i) $t \mapsto F(t,x)$ *is measurable for each* $u \in \mathbf{E}$*;*

(ii) $x \mapsto F(t,x)$ *is upper semi-continuous for almost all* $t \in \mathbf{J}$.

We define the set of the selections of a multi-valued map F by

$$\mathcal{S}_{F,x} := \big\{f \in L^1(\mathbf{J},\mathbf{E})\colon f(t) \in F(t,x(t)) \text{ for a.e. } t \in \mathbf{J}\big\}.$$

Lemma 2.1 [40]**.** *Let* $\mathbf{J}$ *be a compact real interval and* $\mathbf{E}$ *be a Banach space. Let* F *be a multi-valued map satisfying the Carathèodory conditions with the set of* L^1*-selections* $\mathcal{S}_{F,u}$ *nonempty, and let* $\Theta\colon L^1(\mathbf{J},\mathbf{E}) \to C(\mathbf{J},\mathbf{E})$ *be a linear continuous mapping. Then the operator*

$$\Theta\circ\mathcal{S}_{F,x}\colon C(\mathbf{J},\mathbf{E}) \to \mathfrak{P}_{\mathbf{bd,cl,cvx}}(C(\mathbf{J},\mathbf{E})), \quad x \mapsto (\Theta\circ\mathcal{S}_{F,x})(x) := \Theta(\mathcal{S}_{F,x})$$

is a closed graph operator in $C(\mathbf{J},\mathbf{E})\times C(\mathbf{J},\mathbf{E})$.

2.2. Measure of noncompactness. We specify this part of the paper to explore some important details of the Kuratowski measure of noncompactness.

Definition 2.4 [41]**.** *Let* $\Lambda_{\mathbf{E}}$ *be the family of bounded subsets of a Banach space* $\mathbf{E}$*. We define the Kuratowski measure of noncompactness* $\kappa\colon \Lambda_{\mathbf{E}} \to [0,\infty]$ *of* $\mathbf{B} \in \Lambda_{\mathbf{E}}$ *as*

$$\kappa(\mathbf{B}) = \inf\left\{\epsilon > 0\colon \mathbf{B} \subset \bigcup_{j=1}^{m} \mathbf{B}_j \ \textit{and}\ \operatorname{diam}(\mathbf{B}_j) \le \epsilon\right\}.$$

Lemma 2.2 [41]**.** *Let* $\mathbf{C},\mathbf{D} \subset \mathbf{E}$ *be bounded, the Kuratowski measure of noncompactness possesses the next characteristics:*

i) $\kappa(\mathbf{C}) = 0 \Leftrightarrow \mathbf{C}$ *is relatively compact;*

ii) $\mathbf{C} \subset \mathbf{D} \Rightarrow \kappa(\mathbf{C}) \le \kappa(\mathbf{D})$*;*

iii) $\kappa(\mathbf{C}) = \kappa(\overline{\mathbf{C}})$, *where* $\overline{\mathbf{C}}$ *is the closure of* $\mathbf{C}$*;*

iv) $\kappa(\mathbf{C}) = \kappa(\operatorname{conv}(\mathbf{C}))$, *where* $\operatorname{conv}(\mathbf{C})$ *is the convex hull of* $\mathbf{C}$*;*
v) $\kappa(\mathbf{C}+\mathbf{D}) \le \kappa(\mathbf{C}) + \kappa(\mathbf{D})$, *where* $\mathbf{C}+\mathbf{D} = \{u+v : u \in \mathbf{C}, v \in \mathbf{D}\}$*;*
vi) $\kappa(\nu\mathbf{C}) = |\nu|\kappa(\mathbf{C})$, *for any* $\nu \in \mathbb{R}$.

Theorem 2.1 (Mönch's fixed point theorem [42])**.** *Let* Ω *be a closed and convex subset of a Banach space* $\mathbf{E}$*;* $\mathcal{U}$ *a relatively open subset of* Ω, *and* $\mathcal{N} : \overline{\mathcal{U}} \to \mathfrak{P}(\Omega)$. *Assume that graph* $\mathcal{N}$ *is closed,* $\mathcal{N}$ *maps compact sets into relatively compact sets and, for some* $x_0 \in \mathcal{U}$, *the following two conditions are satisfied:*

(i) $G \subset \overline{\mathcal{U}}$, $G \subset \operatorname{conv}(x_0 \cup \mathcal{N}(G))$, $\overline{G} = \overline{C}$ *implies* $\overline{G}$ *is compact, where* C *is a countable subset of* G*;*

(ii) $x \notin (1-\mu)x_0 + \mu\mathcal{N}(x)$ $\forall u \in \overline{\mathcal{U}} \backslash \mathcal{U}$, $\mu \in (0,1)$.

Then there exists $x \in \overline{\mathcal{U}}$ *with* $x \in \mathcal{N}(x)$.

Theorem 2.2 [43]**.** *Let* $\mathbf{E}$ *be a Banach space and* $C \subset L^1(\mathbf{J}, \mathbf{E})$ *countable with* $|u(t)| \le h(t)$ *for a.e.* $t \in \mathbf{J}$, *and every* $u \in C$*; where* $h \in L^1(\mathbf{J}, \mathbb{R}_+)$. *Then the function* $z(t) = \kappa(C(t))$ *belongs to* $L^1(\mathbf{J}, \mathbb{R}_+)$ *and satisfies*

$$\kappa\left(\left\{\int_0^b u(\tau)\,d\tau : u \in C\right\}\right) \le 2\int_0^b \kappa(C(\tau))\,d\tau.$$

3. Main results. This section begins with a definition of an inclusion problem solution (1.1).

Definition 3.1. *A function* $x \in C(\mathbf{J}, \mathbf{E})$ *is said to be a solution of the inclusion problem (1.1) if there exist a function* $f \in L^1(\mathbf{J}, \mathbf{E})$ *with* $f(t) \in F(t, x(t))$ *for a.e.* $t \in \mathbf{J}$, *such that* ${}_0^{PC}D_t^{\alpha} x(t) = f(t)$ *on* $\mathbf{J}$, *and the condition* $x(0) = x_0$ *is satisfied.*

Lemma 3.1. *For* $0 < \alpha < 1$ *and* $h \in C(\mathbf{J}, \mathbb{R})$ *the solution* x *of the linear hybrid Caputo-proportional FDE*

$$\begin{cases} {}_0^{PC}D_t^{\alpha} x(t) = h(t), & t \in \mathbf{J}, \\ x(0) = x_0, \end{cases} \tag{3.1}$$

is given by the following integral equation:

$$x(t) = \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0 + \\ + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}h(\tau)\,d\tau\,du, \quad t \in \mathbf{J}.$$

Proof. Applying the operator ${}_0^{PC}I_t^{\alpha}(\cdot)$ on both sides of (3.1), we get

$${}_0^{PC}I_t^{\alpha}\,{}_0^{PC}D_t^{\alpha} x(t) = {}_0^{PC}I_t^{\alpha} h(t).$$

By using (2.1) and (2.2) together with Proposition 2.1, we get

$$x(t) - \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x(0) = \int_0^t \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{{}_0^{RL}D_u^{1-\alpha}h(u)}{k_0(\alpha,u)}\,du =$$

$$= \frac{1}{\Gamma(\alpha)} \int\limits_0^t \exp\left(-\int\limits_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) \frac{1}{k_0(\alpha,u)} \frac{d}{du} \int\limits_0^u (u-\tau)^{\alpha-1} h(\tau)\,d\tau\,du. \tag{3.2}$$

By using the following Leibniz's rule:

$$\frac{d}{du} \int\limits_{a_1(u)}^{a_2(u)} w(u,\tau)\,d\tau = \int\limits_{a_1(u)}^{a_2(u)} \frac{\partial}{\partial u} w(u,\tau)\,d\tau + w(u,a_2(u))a_2'(u) - w(u,a_1(u))a_1'(u),$$

where $w(u,\tau) = (u-\tau)^{\alpha-1}h(\tau),\ a_1(u) = 0,$ and $a_2(u) = u,$ we obtain that

$$\frac{d}{du} \int\limits_0^u (u-\tau)^{\alpha-1} h(\tau)\,d\tau = (\alpha-1) \int\limits_0^u (u-\tau)^{\alpha-2} h(\tau)\,d\tau. \tag{3.3}$$

Therefore, the substitution from (3.3) in (3.2), we get

$$x(t) = \exp\left(-\int\limits_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) x_0 +$$
$$+ \frac{1}{\Gamma(\alpha-1)} \int\limits_0^t \int\limits_0^u \exp\left(-\int\limits_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} h(\tau)\,d\tau\,du.$$

This completes the proof.

Remark 3.1. The result of Lemma 3.1 is true not only for real valued functions $x \in C(\mathbf{J},\mathbb{R})$ but also for a Banach space functions $x \in C(\mathbf{J},\mathbf{E})$.

Lemma 3.2. *Assume that $F\colon \mathbf{J}\times\mathbf{E} \to \mathfrak{P}(\mathbf{E})$ satisfies Caratheodory conditions, i.e., $t \mapsto$ $\mapsto F(t,x)$ is measurable for every $x \in \mathbf{E}$ and $x \mapsto F(t,x)$ is continuous for every $t \in \mathbf{J}$. A function $x \in C(\mathbf{J},\mathbf{E})$ is a solution of the inclusion problem (1.1) if and only if it satisfies the integral equation*

$$x(t) = \exp\left(-\int\limits_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) x_0 +$$
$$+ \frac{1}{\Gamma(\alpha-1)} \int\limits_0^t \int\limits_0^u \exp\left(-\int\limits_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} f(\tau)\,d\tau\,du,$$

where $f \in L^1(\mathbf{J},\mathbf{E})$ with $f(t) \in F(t,x(t))$ for a.e. $t \in \mathbf{J}$.

The major outcome of the current study is now ready to be presented.

Theorem 3.1. *Let $\varrho > 0,\ \mathcal{K} = \{x \in \mathbf{E}\colon \|x\| \le \varrho\},\ \mathcal{U} = \{x \in C(\mathbf{J},\mathbf{E})\colon \|x\| < \varrho\},$ and suppose that:*

(H1) *The multi-valued map $F\colon \mathbf{J}\times\mathbf{E} \to \mathfrak{P}_{\mathbf{cp,cvx}}(\mathbf{E})$ is Caratheodory,* (H2) *For each* $\varrho > 0,$ *there exists a function* $\varphi \in L^1(\mathbf{J}, \mathbb{R}_+)$ *such that*

$$\|F(t,x)\|_{\mathfrak{P}} = \{|f| \colon f(t) \in F(t,x)\} \le \varphi(t),$$

for a.e. $t \in \mathbf{J}$ *and* $x \in \mathbf{E}$ *with* $|x| \le \varrho,$ *and*

$$\lim_{\varrho \to \infty} \inf \frac{\int_0^b \varphi(t)dt}{\varrho} = \ell < \infty.$$

(H3) *There is a Carathèodory function* $\vartheta \colon \mathbf{J} \times [0, 2\varrho] \to \mathbb{R}_+$ *such that*

$$\kappa(F(t,G)) \le \vartheta(t, \kappa(G)),$$

a.e. $t \in \mathbf{J}$ *and each* $G \subset \mathcal{K}$, *and the unique solution* $\theta \in C(\mathbf{J}, [0, 2\varrho])$ *of the inequality*

$$\theta(t) \le 2\left\{\frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} \vartheta(\tau, \kappa(G(\tau)))\, d\tau\, du\right\}, \quad t \in \mathbf{J},$$

is $\theta \equiv 0.$

Then the inclusion problem (1.1) possesses at least one solution, provided that

$$\ell < \frac{\Gamma(\alpha) M_{k_0}}{b}, \tag{3.4}$$

where $M_{k_0} := \inf_{t \in \mathbf{J}} |k_0(\alpha, t)| \neq 0.$

Proof. Define the multi-valued map $\mathcal{N} \colon C(\mathbf{J}, \mathbf{E}) \to \mathfrak{P}(C(\mathbf{J}, \mathbf{E}))$ by

$$(\mathcal{N}x)(t) = \begin{cases} f \in C(\mathbf{J}, \mathbf{E}) \colon \\ f(t) = \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds\right) x_0 + \\ \qquad + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} w(\tau)\, d\tau\, du, \ w \in \mathcal{S}_{F,x}. \end{cases}$$

In accordance with Lemma 3.2, the fixed points of $\mathcal{N}$ are solutions to the inclusion problem (1.1). We shall show in five steps that the multi-valued operator $\mathcal{N}$ satisfies all assumptions of Mönch's fixed point theorem (Theorem 2.1) with $\overline{\mathcal{U}} = C(\mathbf{J}, \mathcal{K}).$

Step 1. $\mathcal{N}(x)$ is convex, for any $x \in C(\mathbf{J}, \mathcal{K}).$

For $f_1, f_2 \in \mathcal{N}(x),$ there exist $w_1, w_2 \in \mathcal{S}_{F,x}$ such that for each $t \in \mathbf{J},$ we have

$$f_i(t) = \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds\right) x_0 + \\ + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}\, w_i(\tau)\, d\tau\, du, \quad i = 1, 2.$$

Let $0 \le \mu \le 1$. Then, for $t \in \mathbf{J}$,

$$(\mu f_1 + (1-\mu) f_2)(t) = \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0 +$$

$$+\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}(\mu w_1 + (1-\mu)w_2)(\tau)\,d\tau\,du.$$

Since $\mathcal{S}_{F,x}$ is convex (because F has convex values), then $\mu f_1 + (1-\mu) f_2 \in \mathcal{N}(x)$.

Step 2. For each compact $G \in \overline{\mathcal{U}}$, $\mathcal{N}(G)$ is relatively compact.

Let $\{f_n\}$ be any sequence of $\mathcal{N}(G)$, and let $G \in \overline{\mathcal{U}}$ be a compact set. By using the Arzelà–Ascoli criterion of noncompactness in $C(\mathbf{J},\mathcal{K})$, we prove that $\{f_n\}$ has a convergent subsequence. Since $f_n \in \mathcal{N}(G)$, there exist $x_n \in G$ and $w_n \in \mathcal{S}_{F,x_n}$, such that

$$f_n(t) = \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0 +$$

$$+\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w_n(\tau)\,d\tau\,du$$

for $n \ge 1$. As a result of the Kuratowski noncompactness measure and Theorem 2.2, we have

$$\kappa(\{f_n(t)\}) \le 2\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \kappa\left(\left\{\exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) \le\right.\right.\right.$$

$$\left.\left.\left. \le \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w_n(\tau):\ n \ge 1\right\}\right)d\tau\,du\right\}. \tag{3.5}$$

The set $\{w_n(\tau) : n \ge 1\}$, on the other hand, is compact since G is compact. Hence,

$$\kappa(\{w_n(\tau) : n \ge 1\}) = 0$$

for a.e. $\tau \in \mathbf{J}$. Therefore, $\kappa(\{f_n(t)\}) = 0$ which implies that $\{f_n(t) : n \ge 1\}$ is relatively compact in $\mathcal{K}$ for each $t \in \mathbf{J}$. Furthermore, for each $t_1, t_2 \in \mathbf{J}$, $t_1 < t_2$, one obtain that:

$$|f_n(t_2) - f_n(t_1)| \le \left|\exp\left(-\int_0^{t_2} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0 - \exp\left(-\int_0^{t_1} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0\right| +$$

$$+\frac{1}{\Gamma(\alpha-1)}\left|\int_0^{t_2}\int_0^u \left[\exp\left(-\int_0^{t_2} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) - \right.\right.$$

$$\left.\left. - \exp\left(-\int_0^{t_1} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\right]\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}\,w_n(\tau)\,d\tau\,du\right| +$$

$$+\frac{1}{\Gamma(\alpha-1)}\left|\int\limits_{t_1}^{t_2}\int\limits_0^u \exp\left(-\int\limits_0^{t_1}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}\,w_n(\tau)\,d\tau\,du\right|.$$

By applying the mean value theorem to the function

$$\exp\left(-\int\limits_0^{t}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)$$

on $(t_1,t_2),$ we obtain that

$$\left|\exp\left(-\int\limits_0^{t_2}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)-\exp\left(-\int_0^{t_1}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\right|=$$

$$=\left|\frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}\exp\left(-\int\limits_0^{\xi}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)(t_2-t_1)\right|\le\left|\frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}\right|(t_2-t_1)\quad\forall\xi\in(t_1,t_2).$$

Therefore, we get

$$|f_n(t_2)-f_n(t_1)|\le\left|\frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}\right||x_0|(t_2-t_1)+$$

$$+\frac{1}{\Gamma(\alpha-1)M_{k_0}}\left|\frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}\right|(t_2-t_1)\int\limits_0^{t_2}\int\limits_0^{u}(u-\tau)^{\alpha-2}|w_n(\tau)|\,d\tau\,du+$$

$$+\frac{1}{\Gamma(\alpha-1)M_{k_0}}\int\limits_{t_1}^{t_2}\int\limits_0^{u}(u-\tau)^{\alpha-2}|w_n(\tau)|\,d\tau\,du\le$$

$$\le\left|\frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}\right||x_0|(t_2-t_1)+\frac{1}{\Gamma(\alpha-1)M_{k_0}}\left|\frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}\right|\times$$

$$\times(t_2-t_1)\int\limits_0^{t_2}\int\limits_0^{u}(u-\tau)^{\alpha-2}\varphi(\tau)\,d\tau\,du+$$

$$+\frac{1}{\Gamma(\alpha-1)M_{k_0}}\int\limits_{t_1}^{t_2}\int\limits_0^{u}(u-\tau)^{\alpha-2}\varphi(\tau)\,d\tau\,du.$$

The right hand side of the preceding inequality reduces to zero as $t_1\to t_2$. Thus, $\{w_n(\tau)\colon n\ge1\}$ is equicontinuous. Hence, $\{w_n(\tau)\colon n\ge1\}$ is relatively compact in $C(\mathbf{J},\mathcal{K})$.

Step 3. The graph of $\mathcal{N}$ is closed.

Let $x_n\to x_*,$ $f_n\in\mathcal{N}(x_n),$ and $f_n\to f_*$. It must be to show that $f_*\in\mathcal{N}(x_*)$. Now, $f_n\in\mathcal{N}(x_n)$ means that there exists $w_n\in\mathcal{S}_{F,x_n}$ such that, for each $t\in\mathbf{J}$,

$$f_n(t)=\exp\left(-\int\limits_0^{t}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0+$$

$$+\frac{1}{\Gamma(\alpha-1)}\int\limits_0^t\int\limits_0^u \exp\left(-\int\limits_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}\,w_n(\tau)\,d\tau\,du,$$

Consider the continuous linear operator $\Theta\colon L^1(\mathbf{J},\mathbf{E})\to C(\mathbf{J},\mathbf{E})$,

$$\Theta(w)(t)\mapsto f_n(t)=\exp\left(-\int\limits_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0+$$

$$+\frac{1}{\Gamma(\alpha-1)}\int\limits_0^t\int\limits_0^u \exp\left(-\int\limits_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w_n(\tau)\,d\tau\,du.$$

It is obvious that $\|f_n-f_*\|\to 0$ as $n\to\infty$. As a result of Lemma 2.1, we can conclude that $\Theta\circ\mathcal{S}_F$ is a closed graph operator. Additionally, $f_n(t)\in\Theta(\mathcal{S}_{F,x_n})$. Since, $x_n\to x_*$, Lemma 2.1 gives

$$f_*(t)=\exp\left(-\int\limits_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0+$$

$$+\frac{1}{\Gamma(\alpha-1)}\int\limits_0^t\int\limits_0^u \exp\left(-\int\limits_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w(\tau)\,d\tau\,du,$$

for some $w\in\mathcal{S}_{F,x}$.

Step 4. G is relatively compact in $C(\mathbf{J},\mathcal{K})$.

Assume that $G\subset\overline{\mathcal{U}}$, $G\subset\operatorname{conv}(\{0\}\cup\mathcal{N}(G))$, and $\overline{G}=\overline{C}$, for some countable set $C\subset\subset G$. By using a similar approach as in Step 2, one can obtain that $\mathcal{N}(G)$ is equicontinuous. In accordance to $G\subset\operatorname{conv}(\{0\}\cup\mathcal{N}(G))$, it follows that G is equicontinuous. In addition, since $C\subset G\subset\operatorname{conv}(\{0\}\cup\mathcal{N}(G))$ and C is countable, then we can find a countable set $\mathbf{P}=\{f_n: n\geq 1\}\subset\mathcal{N}(G)$ with $C\subset\operatorname{conv}(\{0\}\cup\mathbf{P})$. Thus, there exist $x_n\in G$ and $w_n\in\mathcal{S}_{F,x_n}$ such that

$$f_n(t)=\exp\left(-\int\limits_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0+$$

$$+\frac{1}{\Gamma(\alpha-1)}\int\limits_0^t\int\limits_0^u \exp\left(-\int\limits_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w_n(\tau)\,d\tau\,du.$$

In the light of Theorem 2.2 and the fact that $G\subset\overline{C}\subset\overline{\operatorname{conv}}(\{0\}\cup\mathbf{P})$, we get

$$\kappa(G(t))\leq\kappa\big(\overline{C}(t)\big)\leq\kappa(\mathbf{P}(t))=\kappa(\{f_n(t): n\geq 1\}).$$

By virtue of (3.5) and the fact that $w_n(\tau)\in G(\tau)$, we get

$$\kappa(G(t))\leq$$

$$\leq 2\left\{\frac{1}{\Gamma(\alpha-1)}\int\limits_0^t\int\limits_0^u \kappa\left(\left\{\exp\left(-\int\limits_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w_n(\tau): n\geq 1\right\}\right)d\tau\,du\right\}\leq$$

$$\leq 2\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}\kappa(G(\tau))\,d\tau\,du\right\} \leq$$

$$\leq 2\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}\vartheta(\tau,\kappa(G(\tau)))\,d\tau\,du\right\}.$$

Additionally, the function θ represented by $\theta(t) = \kappa(G(t))$ belongs to $C(\mathbf{J},[0,2\varrho])$. Hence, by (H3), $\theta \equiv 0$, that is $\kappa(G(t)) = 0$ for all $t \in \mathbf{J}$.

The Arzelà – Ascoli theorem states that G is relatively compact in $C(\mathbf{J},\mathcal{K})$.

Step 5. Let $f \in \mathcal{N}(x)$ with $x \in \overline{\mathcal{U}}$. Since $x(\tau) \leq \varrho$ and (H2), we have $\mathcal{N}(\overline{\mathcal{U}}) \subset \overline{\mathcal{U}}$, because if it is not true, there exists a function $x \in \overline{\mathcal{U}}$ but $\|\mathcal{N}(x)\| > \varrho$ and

$$f(t) = \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0 +$$
$$+\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w(\tau)\,d\tau\,du,$$

for some $w \in \mathcal{S}_{F,x}$. Alternatively, we have

$$\varrho < \|\mathcal{N}(x)\| \leq \left|\exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0\right| +$$
$$+\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \left|\exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\right|\frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}|w(\tau)|\,d\tau\,du \leq$$
$$\leq |x_0| + \frac{1}{\Gamma(\alpha-1)M_{k_0}}\int_0^t\int_0^u (u-\tau)^{\alpha-2}|w(\tau)|\,d\tau\,du =$$
$$= |x_0| + \frac{1}{\Gamma(\alpha-1)M_{k_0}}\int_0^t\int_\tau^t (u-\tau)^{\alpha-2}|w(\tau)|\,du\,d\tau =$$
$$= |x_0| + \frac{1}{\Gamma(\alpha)M_{k_0}}\int_0^t (t-\tau)^{\alpha-1}|w(\tau)|\,d\tau \leq$$
$$\leq |x_0| + \frac{t}{\Gamma(\alpha)M_{k_0}}\int_0^t \varphi(\tau)\,d\tau \leq |x_0| + \frac{b}{\Gamma(\alpha)M_{k_0}}\int_0^b \varphi(\tau)\,d\tau.$$

Dividing both sides by ϱ and taking the lower limit as $\varrho \to \infty$, we infer that $\dfrac{b}{\Gamma(\alpha)M_{k_0}}\ell \geq 1$ which contradicts (3.4). Hence $\mathcal{N}(\overline{\mathcal{U}}) \subset \overline{\mathcal{U}}$.

As a result of Steps $1-5$ and Theorem 2.1, we may deduce that $\mathcal{N}$ has a fixed point $x \in C(\mathbf{J}, \mathcal{K})$ that is a solution to the inclusion problem (1.1).

4. Example. Consider the fractional differential inclusion

$$\begin{cases} {}^{PC}_{\ 0}\mathcal{D}_t^{\frac{1}{2}} x(t) \in F(t, x(t)), \quad \text{a.e. on} \quad [0,1], \\ x(0) = 0, \end{cases} \tag{4.1}$$

where $\alpha = \dfrac{1}{2}$, $b = 1$, $x_0 = 0$, and $F\colon [0,1] \times \mathbb{R} \to \mathfrak{P}(\mathbb{R})$ is a multi-valued map given by

$$x \mapsto F(t,x) = \left(e^{-|x|} + \sin t, 3 + \frac{|x|}{1+x^2} + 5t^3\right).$$

For $f \in F$, one has

$$|f| = \max\left(e^{-|x|} + \sin t, 3 + \frac{|x|}{1+x^2} + 5t^3\right) \le 9, \quad x \in \mathbb{R}.$$

Thus

$$\|F(t,x)\|_{\mathfrak{P}} = \{|f|\colon f \in F(t,x)\} = \max\left(e^{-|x|} + \sin t, 3 + \frac{|x|}{1+x^2} + 5t^3\right) \le 9 = \varphi(t),$$

for $t \in [0,1]$, $x \in \mathbb{R}$. Clearly, the value of F is compact and convex valued, and it is upper semi-continuous.

Furthermore, for $(t,x) \in [0,1] \times \in \mathbb{R}$ with $|x| \le \varrho$, one has

$$\liminf_{\varrho \to \infty} \frac{\displaystyle\int_0^1 \varphi(t)\,dt}{\varrho} = 0 = \ell.$$

Therefore, for a suitable M_{k_0}, the condition (3.4) implies that

$$\frac{\Gamma(1/2) M_{k_0}}{b} = M_{k_0}\sqrt{\pi} > 0.$$

Finally, we assume that there exists a Carathèodory function $\vartheta\colon [0,1] \times [0, 2\varrho] \to \mathbb{R}_+$ such that

$$\kappa(F(t,G)) \le \vartheta(t, \kappa(G)),$$

a.e. $t \in [0,1]$ and each $G \subset \mathcal{K} = \{x \in \mathbb{R}\colon |x| \le \varrho\}$, and the unique solution $\theta \in C([0,1],[0,2\varrho])$ of the inequality

$$\theta(t) \le 2\left\{\frac{1}{\Gamma(\alpha-1)} \int\limits_0^t \int\limits_0^u \exp\left(-\int\limits_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} \vartheta(\tau, \kappa(G(\tau)))\,d\tau\,du\right\}, \quad t \in \mathbf{J},$$

is $\theta \equiv 0$.

Hence all the assumptions of Theorem 3.1 hold true and we infer that the inclusion problem (4.1) possesses at least one solution on $[0,1]$.

5. Conclusions. We expand our analysis of fractional differential inclusions in Banach space to the case of hybrid Caputo-proportional fractional derivatives. The existence theorem of the solutions for the suggested inclusion problem is based on the set-valued version of the Monch fixed point theorem combined with the Kuratowski measure of noncompactness. An example is provided to assist in comprehending the theoretical finding. In the future, we intend to use the HCPFDs to investigate the existence, controllability, and stability of nonlinear fractional differential equations and apply them to novel models.

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