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THE THIRD ORDER DIFFERENTIAL EQUATION WITH THE TURNING POINT

ДИФЕРЕНЦІАЛЬНЕ РІВНЯННЯ ТРЕТЬОГО ПОРЯДКУ З ТОЧКОЮ ЗВОРОТУ

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We consider the third-order singular perturbed differential equation with the turning point. The asymptotics of the solution of this equation, which includes the turning point, is constructed.

Розглянуто сингулярно збурене диференціальне рівняння третього порядку з точкою звороту. Побудовано асимптотику розв'язку рівняння, що включає точку звороту.

1. Introduction. In present article we consider the equation

$$\mathbf{L}_\varepsilon U(x,\varepsilon) \equiv \varepsilon^5 U'''(x,\varepsilon) + \varepsilon^3 a(x) U''(x,\varepsilon) + \\ + \varepsilon^2 b(x) U'(x,\varepsilon) + c(x) U(x,\varepsilon) = h(x), \qquad (1)$$

where $\varepsilon \to +0,\ x \in I = [0; l]$. We study Eq. (1) when such conditions are satisfied:

$$a(x),\ b(x),\ c(x),\ h(x) \in C^\infty[I]. \qquad (2)$$

The purpose of this work consists in construction of uniform suitable asymptotic forms of the solution of Eq. (1) when singularly perturbed differential equation (SPDE) (1) contains the turning point $x = 0$, i.e., when

$$c(x) = x\tilde{c}(x) \le 0.$$

For the Liouville and Orr – Sommerfeld equations, similar problems were considered in [1 – 10]. A typical feature of these equations is that they contain derivatives of only even orders, which entails a significant simplification of the given problem. Obtaining the characteristic equation for Liouville and Orr – Sommerfeld equations was not difficult, and it has traditionally not been given special attention. Subsequently, the results of the work [4, 5, 7] were generalized to the systems of differential equations

$$\varepsilon^2 W''(x,\varepsilon) - AW(x,\varepsilon) = h(x). \qquad (3)$$

It is developped method of construction even asymptotic forms of system (3) under sufficiently broad conditions on spectrum of limit operator A (see [4, 7]).

The presence of even and odd orders in the SPDE with a turning point greatly complicates the construction of uniform suitable asymptotics. Therefore, there are much fewer scientific studies in this direction, compared to the studies of the Liouville and Orr – Sommerfeld equations.

One of the main works in this field is the work of Rudolf Langer [1], in which the equation

$$\omega'''(z,\lambda) + \lambda h_1(z,\lambda)\omega''(z,\lambda) + \lambda^2 h_2(z,\lambda)\omega'(z,\lambda) + \lambda^3 h_3(z,\lambda) = 0, \qquad (4)$$

were studied under $|\lambda| \to +\infty$. To construct the asymptotic form of the solution of this equation, Rudolf Langer reduced the order of the investigated equation by using one stable root of the characteristic equation.

The idea of reducing the order of the differential equation by means of the stable root of the characteristic equation is also used by the authors (see [2, 3, 5, 7 – 10]) in the study of the differential equation of the type (4).

However, this method is not promising for generalizing the results to systems of differential equations of a general type.

Significant results on SPDE systems of general type with turning points were obtained by Wolfgang Wasow [5]. The main idea of these studies is to split the SPDE system and study a second-order system for which the spectrum of the limit operator contains only two unstable elements

$$k_{1,2} = \pm i\sqrt{x\tilde{k}(x)}.$$

The purpose of this work consist in following: to generalise results obtained in [4 – 7] to general type SPDE with turning points, and then also to SPDE systems

$$\varepsilon W'(x,\varepsilon) - A(x,\varepsilon)W(x,\varepsilon) = h(x).$$

Let us show that it is possible to use the method constructed for Liuville and Orr – Sommerfeld equation and systems (3) (see [4, 5, 7]) also for the Eq. (1) and then for equations of general type.

First, we note the following: in the Eq. (1), the degrees of the small parameter are chosen in such a way that the asymptotics of this equation which contains the turning point is built on the integer degrees of the small parameter $\varepsilon > 0$. Therefore, to write the characteristic equation of the scalar equation type (1) is not obvious.

To write the characteristic equation, it is necessary to convert SPDE into SPDE system:

$$|A(x,\varepsilon) - \lambda E| = 0,$$

where $A(x,\varepsilon)$ — matrix corresponding to SPDE systems. After necessary transformations, we obtain characteristic equation

$$\lambda^3 + \varepsilon a(x)\lambda^2 + \varepsilon^3 b(x)\lambda + \varepsilon^4 c(x) = 0. \qquad (5)$$

In characteristic Eq. (5) we cannot get rid of low-order terms and go to additional characteristic equation, as this is done for Noayon-type SPDE with stable spectrum. Therefore we come to conclusion with complete characteristic Eq. (5). We write it in a form

$$\frac{\lambda^3(x,\varepsilon)}{\varepsilon^3} + a(x)\,\frac{\lambda^2(x,\varepsilon)}{\varepsilon^2} + \varepsilon b(x)\frac{\lambda(x,\varepsilon)}{\varepsilon} + \varepsilon c(x) = 0$$

and change $k(x,\varepsilon) = \dfrac{\lambda(x,\varepsilon)}{\varepsilon}$. We get equation

$$P(k(x,\varepsilon),\varepsilon) \equiv k^3 + a(x)k^2 + \varepsilon b(x)k + \varepsilon c(x) = 0 \tag{6}$$

which is more convenient to work with than with the characteristic Eq. (5).

Let the roots of characteristic Eq. (6) satisfy conditions

$$k_1(x) < 0, \quad k_{2,3}(x,\varepsilon) = \pm i\sqrt{\varepsilon x\tilde{k}(x)}, \quad \tilde{k}(x) > 0 \quad \text{for all} \quad x \in I. \tag{7}$$

In this case point $x = 0$ is a turning point for Eq. (1) and simplified equation corresponding to the SPDE (1), i.e., equation

$$c(x)\omega(x) = h(x), \tag{8}$$

in the general case, has an essential discontinuity at the point $x = 0.$

Under the conditions (7), the characteristic Eq. (6) can be written as

$$P(k(x,\varepsilon),\varepsilon) \equiv \left[k^2(x,\varepsilon) + \varepsilon x\tilde{k}(x)\right][k(x,\varepsilon) - k_1(x)] = 0.$$

2. Extension of the perturbation problem. By analogy with [2–4, 7], to separate and save all essentially singular functions (ESF) in the SPDE solution (1), we introduce an additional vector variable $t = \{t_1, t_2\}$ according to the rule

$$t_1 = \varepsilon^{-p_1}\varphi_1(x) \equiv \Phi_1(x,\varepsilon), \quad t_2 = \varepsilon^{-p_2}\varphi_2(x) \equiv \Phi_2(x,\varepsilon), \tag{9}$$

where factors p_i and regular functions $\varphi_i(x),\ i = 1, 2,$ are to be determined.

By introducing additional vector variable $t,$ according to the method of regularization [6], instead of the function $U(x,\varepsilon)$, we will study the extended function $\tilde{U}(x,t,\varepsilon).$ The extension leads to the fulfillment of equality

$$\tilde{U}(x,t,\varepsilon)|_{t=\Phi(x,\varepsilon)} \equiv U(x,\varepsilon),$$

where $\Phi(x,\varepsilon) = \{\Phi_i(x,\varepsilon),\ i = 1, 2\}.$ We define total derivative

$$\frac{d^s\tilde{U}(x,t,\varepsilon)}{dx^s} \equiv \frac{d^sU(x,\varepsilon)}{dx^s}, \quad s = \overline{1,3},$$

and replace their values in the problem (1). Then, to determine the extended function $\tilde{U}(x,t,\varepsilon)$, we obtain the extended equation

$$\tilde{L}_\varepsilon\tilde{U}(x,t,\varepsilon) = h(x). \tag{10}$$

Here, for $i = 1, 2,$

$$\tilde{L}_\varepsilon \equiv \sum_{i=1}^{3} \tilde{L}_{\varepsilon i} + Y_\varepsilon^\perp,$$

$$\tilde{L}_{\varepsilon i} \equiv \varepsilon^{5-3p_i} \varphi_i'^3(x) \frac{\partial^3}{\partial t_i^3} + \varepsilon^{5-2p_i} L_{0i} \frac{\partial^2}{\partial t_i^2} + \varepsilon^{5-p_i} L_{1i} \frac{\partial}{\partial t} +$$
$$+ a(x) \left[\varepsilon^{3-2p_i} \varphi_i'^2(x) \frac{\partial^2}{\partial t_i^2} + \varepsilon^{3-p_i} d \frac{\partial}{\partial t_i} \right] + b(x) \varepsilon^{2-p_i} \varphi_i'(x) \frac{\partial}{\partial t_i}, \quad (11)$$

$$\tilde{L}_{\varepsilon 3} \equiv \varepsilon^5 \frac{\partial^3}{\partial x^3} + \varepsilon^3 a(x) \frac{\partial^2}{\partial x^2} + \varepsilon^2 b(x) \frac{\partial}{\partial x} + c(x),$$
$$d_i = 2\varphi_i'(x) \frac{\partial}{\partial x} + \varphi_i''(x), \quad (12)$$

$$L_{0i} \equiv 3\varphi_i'^2(x) \frac{\partial}{\partial x} + 3\varphi_i'(x) \varphi_i''(x),$$

$$L_{1i} \equiv 3\varphi_i'(x) \frac{\partial^2}{\partial x^2} + 3\varphi_i''(x) \frac{\partial}{\partial x} + \varphi_i'''(x).$$

The operator $Y_\varepsilon^\perp$ contains a removable derivative of t_1 and t_2 and will subsequently play the role of an annihilator. Therefore, it makes no sense to write it down explicitly.

3. Spaces of nonresonance solutions. We consider sets (subspaces) of functions

$$Y_{r1} = \{\alpha_r(x) \exp t_1\}, \quad Y_{r2k} = \{V_{rk}(x) U_k(t_2) + Q_{rk}(x) U_k'(t_2)\},$$
$$Y_{r3} = \{f_r(x) \psi(t_2) + g_r(x) \psi'(t_2)\}, \quad Y_{r4} = \{\omega_r(x)\},$$

where $\alpha_r(x),\ V_{rk}(x),\ Q_{rk}(x),\ f_r(x),\ g_r(x),\ \omega_r(x) \in C^\infty[I]$.

The Airy – Dorodnitsyn functions $U_k(t_2)$ are linearly independent solutions of the differential equation

$$U''(t_2) - t_2 U(t_2) = 0,$$

which characteristics is described in [6, 7].

The Scorer function $\psi(t_2)$ solves the equation

$$\psi''(t_2) - t_2 \psi(t_2) = 1,$$

and its characteristics is described in [5, 7].

A new space is formed from these subspaces:

$$Y_r = Y_{r1} \bigoplus_{k=1}^{2} Y_{r2k} \bigoplus Y_{r3} \bigoplus Y_{r4}. \quad (13)$$

Element $W_r(x,t) \in Y_r$ is of the form of

$$W_r(x,t) = \alpha_r(x) \exp t_1 + \sum_{k=1}^{2} \big[V_{rk}(x) U_k(t_2) + Q_{rk}(x) U_k'(t_2) \big] +$$
$$+ f_r(x) \psi(t_2) + g_r(x) \psi(t_2) + \omega_r(x).$$

By analogy with [2 – 4, 7], we study the action of the extended operator $\tilde{L}_\varepsilon$ on stable element $\alpha_r(x)\exp t_2 \in Y_{r1}$. Taking into account (7), (9), (11), (12), we have

$$\tilde{L}_\varepsilon \alpha_r(x)\exp t_1 \equiv \Big(\tilde{L}_{\varepsilon 1} + \tilde{L}_{\varepsilon 3}\Big)\alpha_r(x)\exp t_1 \equiv$$
$$\equiv \Bigg[\tilde{P}\big(\varphi_1'(x),\varepsilon\big) + \varepsilon^{5-2p_1}L_{01} + \varepsilon^{5-p_1}L_{11} + \varepsilon^{3-p_1}a(x)d_1 +$$
$$+ \varepsilon^2 b(x)\frac{\partial}{\partial x} + \varepsilon^3 a(x)\frac{\partial^2}{\partial x^2} + \varepsilon^5\frac{\partial^3}{\partial x^3}\Bigg]\alpha_r(x)\exp t_1, \tag{14}$$

where

$$\tilde{P}\big(\varphi_1'(x),\varepsilon\big) \equiv \varepsilon^{5-3p_1}{\varphi_1'}^3(x) + \varepsilon^{3-2p_1}a(x){\varphi_1'}^2(x) + \varepsilon^{2-p_1}b(x)\varphi_1'(x) + c(x).$$

It is necessary to choose a regular function $\varphi_1(x)$ and expand p_1 so that $\tilde{P}(\varphi_1'(x),\varepsilon) \equiv 0$. For this, it is necessary to put $\varphi_1(x) = \int_0^x k_1(x)dx$ and $p_1 = 2$. We get

$$\tilde{P}\big(\varphi_1'(x),\varepsilon\big) \equiv \frac{P(k_1(x),\varepsilon)}{\varepsilon} \equiv 0.$$

Thus, the regular function $\varphi_1(x)$ and the factor $p_1 = 2$ are determined. The equality (14) have the form

$$\tilde{L}_\varepsilon \alpha_r(x)\exp t_1 \equiv \Bigg\{\frac{P(k_1(x),\varepsilon)}{\varepsilon} + \varepsilon[L_{01} + a(x)d_1] + \varepsilon^2 b(x)\frac{\partial}{\partial x} +$$
$$+ \varepsilon^3\Bigg[L_{11} + a(x)\frac{\partial^2}{\partial x^2}\Bigg] + \varepsilon^5\frac{\partial^3}{\partial x^3}\Bigg\}\alpha_r(x)\exp t_1 \equiv$$
$$\equiv R_{1\varepsilon}\alpha_r(x)\exp t_1. \tag{15}$$

The correctness of choosing the regular functions $\varphi_1(x)$ and the factor p_1 will be fully motivated below, when from the equality (15) we can determine the function $\alpha_r(x)$. Hence, we proceed to the study of the equation $L_{01}\alpha_r(x) = F_r^\alpha(x)$, where $F_r^\alpha(x)$ is a known sufficiently smooth function. Considering (11) we have

$$[L_{01} + a(x)d]\alpha_r(x) \equiv k_1^2(x)\alpha_r'(x) + 2k_1(x)k_1'(x)\alpha_r(x) = F_r^\alpha(x). \tag{16}$$

Solution of this differential equation is the function

$$\alpha_r(x) = \frac{1}{k_1^2(x)}\left[C_{1r} + \frac{1}{3}\int\limits_0^x F_r^\alpha(x)dx\right], \tag{17}$$

where C_{1r} is a free constant.

Therefore, the regular function $\varphi_1(x)$ and the factor p_1 are chosen correctly.

Next, we proceed to the main study, i.e., to study the action of the extended operator $\tilde{L}_\varepsilon$ on elements from the subspaces $Y_{r2k},\ k=1,2$. With a fixed $k=1,2$, we have

$$\begin{aligned}&\tilde{L}_\varepsilon[V_{rk}(x)U_k(t_2)+Q_{rk}(x)U'_k(t_2)]\equiv\\ &\quad\equiv\left(\tilde{L}_{\varepsilon 2}+\tilde{L}_{\varepsilon 3}\right)[V_{rk}(x)U_k(t_2)+Q_{rk}(x)U'_k(t_2)]\equiv\\ &\quad\equiv A_k(x,\varepsilon)U_k(t_2)+B_{rk}(x,\varepsilon)U'_k(t_2).\end{aligned}\tag{18}$$

Here

$$\begin{aligned}A_{rk}(x,\varepsilon)\equiv{}&\varepsilon^{5-3p_2}{\varphi'_2}^3(x)\left[-V_{rk}(x)+\varepsilon^{-2p_2}\varphi_2^2(x)Q_{rk}(x)\right]-\\ &-\varepsilon^{5-3p_2}\varphi_2(x)L_{02}V_{rk}(x)-\varepsilon^{5-2p_2}[L_{02}+\varphi_2(x)L_{12}]Q_{rk}(x)-\\ &-a(x)\Big[\varepsilon^{3-3p_2}{\varphi'_2}^2(x)\varphi_2(x)V_{rk}(x)+\\ &+\varepsilon^{3-2p_2}{\varphi'_2}^2(x)Q_{rk}(x)+\varepsilon^{3-2p_2}\varphi_2(x)dQ_{rk}(x)\Big]-\\ &-b(x)\varepsilon^{2-2p_2}\varphi'_2(x)\varphi_2(x)Q_{rk}(x)+\tilde{L}_{\varepsilon 3}V_{rk}(x),\end{aligned}\tag{19}$$

$$\begin{aligned}B_{rk}(x,\varepsilon)\equiv{}&\varepsilon^{5-3p_2}{\varphi'_2}^3(x)\left[-\varepsilon^{-p_2}\varphi_2(x)V_{rk}(x)-2Q_{rk}(x)\right]-\\ &-\varepsilon^{5-3p_2}\varphi_2(x)L_{02}Q_{rk}(x)+\varepsilon^{5-p_2}L_{12}V_{rk}(x)+\\ &+a(x)\Big[-\varepsilon^{3-3p_2}{\varphi'_2}^2(x)\varphi_2(x)Q_{rk}(x)+\varepsilon^{3-p_2}dV_{rk}(x)\Big]+\\ &+\varepsilon^{2-p_2}b(x)\varphi'_2(x)V_{rk}(x)+\tilde{L}_{\varepsilon 3}Q_{rk}(x).\end{aligned}\tag{20}$$

In the Eq. (18), we need to obtain an analogue of what was obtained in the Eq. (14). The first step of such an analogy is the separation of the main part in (19) and (20), in addition, the elements from the subspaces $Y_{r2k},\ k=1,2$, must belong to the kernel of the main part of equality (18).

Further studies show that the main part in the equality (19) is

$$\begin{aligned}&\Big[\varepsilon^{5-5p_2}{\varphi'_2}^3(x)\varphi_2^2(x)-\varepsilon^{2-2p_2}b(x)\varphi'_2(x)\varphi_2(x)\Big]Q_{rk}(x)-\\ &\quad-\Big[\varepsilon^{3-3p_2}a(x){\varphi'_2}^2(x)\varphi_2(x)-c(x)\Big]V_{rk}(x)\equiv\\ &\equiv\Big[{\varphi'_2}^2(x)\varphi_2(x)-\varepsilon^{3p_2-3}x\tilde{k}(x)\Big]\times\\ &\quad\times\left[\varepsilon^{5-5p_2}\varphi'_2(x)\varphi_2(x)Q_{rk}(x)-\varepsilon^{3-3p_2}a(x)V_{rk}(x)\right].\end{aligned}$$

To define the factor p_2 and the regular function $\varphi_2(x)$ we set equal to zero the expression in the first square brackets. Then we define the factor $p_2=1$ and obtain a relatively regular function necessary for the solution:

$${\varphi'_2}^2(x)\varphi_2(x)=x\tilde{k}(x),\quad \varphi_2(0)=0.\tag{21}$$

Solution of the problem (21) is the function

$$\varphi_2(x)=\left(\frac{3}{2}\int\limits_0^x\sqrt{x\tilde{k}(x)}dx\right)^{\frac{2}{3}}.\tag{22}$$

We define the factor $p_2 = 1$ and the regular function $\varphi_2(x)$ from (22) and write the equality (19) in the form

$$A_{rk}(x,\varepsilon) \equiv -\varepsilon a(x)\Big[{\varphi_2'}^2(x)Q_{rk}(x) + \varphi_2(x)d_2Q_{rk}(x)\Big] - \\ - \varepsilon^2\Big[{\varphi_2'}^3(x)V_{rk}(x) + \varphi_2(x)L_{02}V_{rk}(x) - b(x)V_{rk}'(x)\Big] - \\ - \varepsilon^3[L_{02} + \varphi_2 L_{12}]Q_{rk}(x) + \varepsilon^3 a(x)V_{rk}'' + \varepsilon^5 V_{rk}'''(x).$$

To find functions $Q_{rk}(x)$, we obtain differential equations

$$DQ_{rk}(x) \equiv 2\varphi_2(x)\varphi_2'(x)Q_{rk}'(x) + \Big[\varphi_2(x)\varphi_2''(x) + {\varphi_2'}^2(x)\Big]Q_{rk}(x) = \frac{1}{k_1(x)}\,F_{rk}^Q(x). \tag{23}$$

The point $x = 0$ is a regular point corresponding to differential Eqs. (23). Therefore, there are quite smooth solutions of the Eqs. (23), which satisfies the conditions $|Q_{rk}(0)| < \infty$.

Under the condition of defined $p_2 = 1$ and a regular function $\varphi_2(x)$, there exist equalities

$$c(x) - a(x){\varphi_2'}^2(x)\varphi_2(x) \equiv b(x) - {\varphi_2'}^2(x)\varphi_2(x) \equiv 0.$$

Then the form of equality (20) is simplified, i.e.,

$$B_{rk}(x) \equiv \varepsilon^2\big[a(x)d_2V_{rk}(x) - 2Q_{rk}(x) - \varphi_2(x)L_{02}Q_{rk}(x) + b(x)Q_{rk}'(x)\big] + \\ + \varepsilon^3 a(x)Q_{rk}''(x) + \varepsilon^4 L_{12}V_{rk}(x) + \varepsilon^5 Q_{rk}'''(x). \tag{24}$$

It can be seen from the obtained equalities (24) that the functions $V_{rk}(x)$ will be defined as solutions of differential equations

$$d_2V_{rk}(x) = \frac{1}{k_1(x)}\,F_{rk}^V(x). \tag{25}$$

Since the coefficients obtained in Eqs. (25) are not equal to zero at any point of the segment $I = [0; l]$, we obtain sufficiently smooth solutions of inhomogeneous differential Eqs. (25) in the form

$$V_{rk}(x) = \frac{1}{\sqrt{\varphi_2'(x)}}\left[C_{r(k+1)} + \int\limits_0^x \frac{\sqrt{\varphi_2'(x)}}{k_1(x)}F_{rk}^V(x)dx\right], \quad k = 1, 2, \tag{26}$$

where $C_{r(k+1)}$ is an arbitrary constant.

Lemma 1.

1. *As a result of the action of the extended operator $\tilde{L}_\varepsilon$ on the elements of the subspace Y_{r1}, we uniquely determine the factor $p_1 = 2$, the regular function $\varphi_1(x)$ and the obtained differential Eqs. (16), which are used to construct the first linear independent solution of the SPDE (1).*

2. *To define the extended operator $\tilde{L}_\varepsilon$ on elements of the spaces of nonresonance solutions (SNS) Y_{r2k}, $k = 1, 2$:*

(a) *regardless of the index k, we determine the factor $p_2 = 1$ and the regular function $\varphi_2(x)$ (see (22)), i.e., we uniquely determine the variable t_2;*

(b) *in the process of construction, the asymptotic forms of the solution of the extended Eq. (11) — the functions* $Q_{rk}(x)$ *— are uniquely determined as solutions of the differential Eqs. (23) under the conditions* $|Q_{rk}(0)| < \infty$*;*

(c) *the functions* $V_{rk}(x)$ *contain a free constant (see (26)). This fact allows us to construct two more linear independent solutions of the SPDE (1) corresponding to two unstable elements* $k_{2,3}(x) = \pm i\sqrt{x\tilde{k}(x)}$.

4. Formalism of construction of solution of extented problem. The main stages of constructing linear independent SPDE solutions (9) were motivated above. Construction of the partial solution of the inhomogeneous differential Eq. (9) does not affect the definition of regular functions $\varphi_i(x)$ and factors p_i. Thus, in this section, the formalism of constructing the fundamental system of solutions (FSS) of the uniform partial solution of the inhomogeneous extended Eq. (11) will be directly considered. By analogy with the previous one, we investigate the effect of the extended operator $\tilde{L}_\varepsilon$ on the element of the subspace Y_{r3}. By combining all the obtained results, the action of the extended operator $\tilde{L}_\varepsilon$ on the SNF element (13) can be written in the form of the following equality:

$$\tilde{L}_\varepsilon W_r(x,t) \equiv \tilde{L}_\varepsilon[\alpha_r(x)\exp t_1 + y_r(x,t_2)] \equiv R_1\varepsilon\alpha_r(x)\exp T_1 + R_\varepsilon y_r(x,t). \tag{27}$$

Here, the operator $R_{1\varepsilon}$ corresponds to the equality (15), but the operator R_ε in its action on the element $y_r(x,t_2) \in Y_{r21} \bigoplus Y_{r22} \bigoplus Y_{r3} \bigoplus Y_{r4} = \tilde{Y}_r$ can be presented in the form of equalities

$$R_\varepsilon y_r(x,t_2) \equiv \big[R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \varepsilon^3 R_3 + \varepsilon^4 R_4 + \varepsilon^5 R_5\big] y_r(x,t_2), \tag{28}$$

$$\begin{aligned} R_0 y_r(x,t_2) \equiv{} & \Big[{\varphi'_2}^2(x)\varphi_2(x) - x\tilde{k}(x)\Big]\times \\ & \times\Bigg\{\big[\varphi'_2(x)\varphi_2(x) + k_1(x)\big]\left[\sum_{k=1}^{2} Q_{rk}(x)U'_k(t_2) + g_r(x)\psi'(t_2)\right] - \\ & - \big[\varepsilon\varphi'_2(x) + k_1(x)\big]\left[V_{rk}(x)U_k(t_2) + f_r(x)\psi(t_2)\right]\Bigg\} + c(x)\omega_r(x), \end{aligned} \tag{29}$$

$$R_1 y_r(x,t_2) \equiv a(x)D\left[\sum_{k=1}^{2} Q_{rk}(x)U_k(t_2) + g_r(x)\psi(t_2)\right] + a(x)\varphi'_2(x)f_r(x), \tag{30}$$

$$\begin{aligned} R_2 y_r(x,t) \equiv{} & \sum_{k=1}^{2}\left[a(x)dV_{rk}(x) - \left[2 + \varphi_2(x)L_{02} - b(x)\frac{\partial}{\partial x}\right]Q_{rk}(x)U'_k(t_2)\right] + \\ & + \left[a(x)df_r(x) - \left[2\varphi_2(x)L_{02} - b(x)\frac{\partial}{\partial x}\right]g_r\right]\psi'(t_2) - \\ & - \left[{\varphi'_2}^3(x) + \varphi_2(x)L_{02} + b(x)\frac{\partial}{\partial x}\right]\left[\sum_{k=1}^{2} V_{rk}(x)U_k(t_2) + f_r(x)\psi(t_2)\right] - \\ & - \Big[{\varphi'_2}^3(x) - a(x)d\Big]g_r(x) + b(x)\omega'_r(x), \end{aligned} \tag{31}$$

$$R_3y_r(x,t_2) \equiv -\Bigg\{[L_{02}+\varphi_2(x)L_{12}]\left[\sum_{k=1}^{2} Q_{rk}(x)U_k(t_2)+g_r(x)\psi(t_2)\right]+$$

$$+a(x)\left[\sum_{k=1}^{2} V''_{rk}(x)U_k(t_2)+\sum_{k=1}^{2} Q''_{rk}(x)U'_k(t_2)+\right.$$

$$\left.\left.+f_r(x)\psi(t_2)+g''_r(x)\psi'(t_2)\right]\right\}+a(x)\omega''_r(x), \tag{32}$$

$$R_4y_r(x,t_2) \equiv L_{12}\left[\sum_{k=1}^{2} V_{rk}(x)U'_k(t_2)+f_r(x)\psi(t_2)\right]L_{12}g_r(x), \tag{33}$$

$$R_5y_r(x,t_2) \equiv \sum_{k=1}^{2}\big[V'''_{rk}(x)U_k(t_2)+Q'''_{rk}(x)U'_k(t_2)\big]+f'''_r(x)\psi(t_2)+g'''_r(x)\psi'(t_2)+\omega'''_r(x). \tag{34}$$

Lemma 2. *The following conclusions can be drawn from the obtained identities (27) – (34):*
1. *Spaces $Y_{r1},\ Y_{r2k},\ Y_{r3},\ Y_{r4}$ are invariant for extended operator $\tilde{L}_\varepsilon$.*
2. *Operator $\tilde{P}(k_1(x),\varepsilon)$ and R_0 are main forming extended operators $\tilde{L}_\varepsilon$ in subspaces Y_{r1} and $\tilde{Y}_\varepsilon = Y_{r21}\bigoplus Y_{r22}\bigoplus Y_{r3}\bigoplus Y_{r4}$ respectively.*
3. *Extented problem (11) is regularly perturbed in subspaces Y_{r1} and $\tilde{Y}_\varepsilon$, hence also in space Y_r.*

On the basis of the obtained conclusions, the asymptotic forms of the extended problem (11) are constructed in the form of a series

$$\tilde{U}(x,t,\varepsilon)=\sum_{r=-1}^{+\infty}\varepsilon^r W_r(x,t), \quad W_r(x,t)\in Y_r. \tag{35}$$

We substitute the formal series (35) into the extended Eq. (10) and equate the terms by the same powers of the small parameter $\varepsilon > 0$. Then, to determine the coefficients of the series (35), we obtain two independent sequences of the systems of equations.

The first of them, corresponding to the stable root $k_1(x)$, has the form

$$L_{01}\alpha_r(x) = -a(x)d\alpha_{r-1}(x)-\left[L_{11}+b(x)\frac{\partial}{\partial x}\right]\alpha_{r-3}(x)-a(x)\alpha''_{r-4}(x)-\alpha'''_{r-6}(x), \quad r\geq 0. \tag{36}$$

The second one is represented by equations

$$R_0y_{-1}(x,t_2)=0, \quad R_0y_0(x,t_2)=h(x)-R_1y_{-1}(x,t_2), \tag{37}$$

$$R_0y_r(x,t_2) = -\sum_{i=1}^{5} R_iy_{r-i}(x,t_2), \quad r\geq 1. \tag{38}$$

The obtained independent sequences of Eqs. (36), (37), (38) show the following. We split the solution of the extended Eq. (10), but therefore also the SPDE (1)), thereby a linearly independent solution of the Eq. (1) corresponding to the stable root characteristic $k_1(x)$ of the Eq. (6) can be constructed independently of the constructions of the other two linearly independent solutions of the equation (1).

Let us note one more detail that is characteristic of SPDEs with an untidy spectrum. Only the $k_{2,3}(x)$ roots of the characteristic equation (6) participate in the construction of the partial solution of the inhomogeneous SPDE (1) with a turning point.

5. Construction of asymptotic solution of extended problem. We need to show that from the sequence of Eqs. (36) – (38) it is possible to determine sufficiently smooth factors of the functions $W_r(x,t)$. By gradually solving the sequence of differential Eqs. (36), we determine all functions $\alpha_r(x),\ r \geq 0,$ by (17). Therefore, we define one linearly independent solution of the extended Eq. (11), but correspondingly also the SPDE (1).

By using equality (29), for $i = 1, 2,$ we define

$$\mathrm{Ker}\, R_0 = \big\{V_{ri}(x)U_i(t_2) + Q_{ri}(x)U_i'(t_2),\ f_r(x)\psi(t_2) + g_r(x)\psi'(t_2)\big\},$$

where $f_r(x),\ g_r(x)$ are arbitrary sufficiently smooth functions at $x \in I.$

Then general solution of Eq. (37) in SNS Y_{-1} is the function

$$y_{-1}(x,t_2) = \sum_{i=1}^{2}\big[V_{(-1)i}(x)U_i(t_2) + Q_{(-1)i}(x)U_i'(t_2)\big] + f_{-1}(x)\psi(t_2) + g_{-1}(x)\psi'(t_2) \equiv Z_{-1}(x,t_2).$$

We calculate the shift in the right-hand side of the Eq. (37). Using the arbitrariness of the functions $Q_{(-1)i}(x)$ and $f_{-1}(x)$ requires that the right-hand side of this equation does not contain elements of the kernel of the operator R_0. To fulfill these conditions, the functions $Q_{(-1)i}(x)$ and $f_{-1}(x)$ must satisfy the equations

$$DQ_{(-1)i}(x) = 0, \quad Dg_{-1}(x) = 0. \tag{39}$$

The bounded solutions of the equations $x \in I,$ (39) will be identical zeros. Furthermore, we have an initial condition

$$f_{-1}(0) = -\frac{h(0)}{k_1(0)\varphi_2'(0)} = f_{-1}^0.$$

When these conditions are fulfilled in SNS Y_0, there exists a solution of the Eq. (37)

$$y_0(x,t_2) = Z_0(x,t_2) + \frac{h(x) - a(x)\varphi_2'(x)f_{-1}(x)}{c(x)} \equiv Z_0(x,t_2) + \omega_0(x).$$

When solving the iterative Eqs. (38) in SNS Y_1 at $r = 1$, we determine the functions $Q_{0i}(x) \equiv g_0(x) \equiv 0$ and obtain initial condition $f_0(0) = 0$ and differential equations

$$dV_{(-1)i}(x) = 0, \quad df_{-1}(x) = 0.$$

Using the initial condition $f_{-1}(0) = f_{-1}^0$, we uniquely define the function $f_0(x)$, but the functions $V_{(-1)i}(x)$ are defined with accuracy up to an arbitrary constant $C_{r(i+1)},\ i = 1, 2$ (see (26)).

Continuing to solve the iterative Eq. (22) for $r \geq 2,$ we uniquely determine the functions $Q_{ri}(x)$ and $f_r(x)$, but $V_{ri}(x)$ — up to an arbitrary constant $C_{r(i+1)}$.

The main term of the asymptotic form of the solution of the extended Eq. (11) has the form

$$U_0(x,t,\varepsilon)=\varepsilon^{-1}f_{-1}(x)\psi(t_2)+\sum_{i=1}^{2}V_{0i}(x)U_i(t_2)+\alpha_0(x)\exp t_1+\omega_0(x). \tag{40}$$

Let us restrict equality (40) at $t=\Phi(x,\varepsilon.)$ We obtain the main member of an asymptotic form of the solution for the inhomogeneous SPDE (1).

6. Solution structure of fundamental system. For extended uniform Eq. (10) we have three solutions. The first solution, corresponding to the stable root $k_1(x)$, is

$$\tilde{U}_1(x,t,\varepsilon)\equiv\tilde{U}_1(x,t_1,\varepsilon)=\sum_{r=0}^{+\infty}\varepsilon^r\alpha_r(x)\exp t_1, \tag{41}$$

where $\alpha_r(x)$ are solutions of differential Eqs. (16), moreover, $\alpha_r(x)=C_{1r}k_1^{-2}(x),\ r=0,1$ (see (17)).

Other two solutions, corresponding to unstable elements $k_{2,3}(x,\varepsilon)=\pm i\sqrt{\varepsilon x\tilde{k}(x)}$, are presented in the form

$$\tilde{U}_{s+1}(x,t,\varepsilon)\equiv\tilde{U}_{s+1}(x,t_2,\varepsilon)\equiv\sum_{r=0}^{+\infty}\varepsilon^r\big[V_{rs}(x)U_s(t_2)+Q_{rs}(x)U_s'(t_2)\big], \tag{42}$$

where $Q_{rs}(x)\equiv 0,\ r=-1,0,\ s=1,2.$

The particular solutions (41) and (42) satisfy the conditions

$$\tilde{U}_1(0,0,\varepsilon)=1,\quad \tilde{U}_2(0,0,\varepsilon)=1,\quad \tilde{U}_3(0,0,\varepsilon)=0.$$

In this case Wronskian $W\big(\tilde{U}_i(0,0,\varepsilon)\big)=O\big(\varepsilon^{-2}\big)\neq 0$ under $\varepsilon>0$.

We make the restriction in equalities (41), (42) at $t=\Phi(x,\varepsilon)$. Then we obtain three linearly independent solutions of the SPDE (1), which can be represented in the form of equalities

$$\tilde{Y}_i(x,t,\varepsilon)\equiv\tilde{Y}_{im}(x,t,\varepsilon)+\varepsilon^m\tilde{\xi}_{im}(x,t,\varepsilon),\quad i=\overline{1,3}, \tag{43}$$

where $\tilde{Y}_{im}(x,t,\varepsilon)=\sum_{r=0}^{m}\varepsilon^rW_{ri}(x,t)$.

By restriction in (43), we obtain

$$Y_i(x,\Phi(x,\varepsilon),\varepsilon)\equiv Y_{im}(x,\Phi(x,\varepsilon),\varepsilon)+\varepsilon^m\xi_{im}(x,\Phi(x,\varepsilon),\varepsilon),\quad i=\overline{1,3}.$$

By the method described in [2 – 7], it is possible to show that at sufficiently small values of parameter $\varepsilon>0$ there exist estimations

$$\|\xi_{im}(x,\Phi(x,\varepsilon),\varepsilon)\|\leq K,\quad i=\overline{1,3}, \tag{44}$$

where constant K does not depend on $x\in I$ and small parameter $\varepsilon>0$.

General solution of the SPDE (1) is

$$Y(x,\Phi(x,\varepsilon),\varepsilon)=\sum_{i=1}^{3}\gamma_iY_i(x,\Phi(x,\varepsilon),\varepsilon)+Y_{\text{part.}}(x,\Phi(x,\varepsilon),\varepsilon),$$

where γ_i is arbitrary constant, but particular solution of the SPDE (1) is

$$Y_{\text{part.}}(x, \Phi(x,\varepsilon), \varepsilon) = \frac{f_{-1}(x)}{\varepsilon}\, \psi\left(\frac{\varphi_2(x)}{\varepsilon}\right) + \\ + \sum_{r=0}^{+\infty} \varepsilon^r \left[f_r(x)\psi\left(\frac{\varphi_2(x)}{\varepsilon}\right) + g_r(x)\, \frac{d\psi\left(\frac{\varphi_2(x)}{\varepsilon}\right)}{d\left(\frac{\varphi_2(x)}{\varepsilon}\right)} + \omega_r(x)\right].$$

Theorem 6.1. *Let the following assumptions be satisfied:*

1. *Condition (2) is satisfied.*

2. *Roots of characteristic Eq. (6) satisfy conditions (7). Then for sufficiently small values of parameter* $\varepsilon > 0$*:*

(a) *by introducing an additional vector variable* $t = \{t_1, t_2\}$ *in corresponding with the form (9), by the above-described SPDE algorithm (1), it can be put in corresponding with the extended equation (11);*

(b) *in SNS (13) there exist three linear independent solutions* $\tilde{Y}_i(x, t, \varepsilon)$, $i = \overline{1,3}$, *of uniform extended equation (11), presented as asymptotic series (41) and (42).*

(c) *restrictions of these series at* $t = \Phi(x, \varepsilon)$ *are asymptotic series for linearly independent solutions of the SPDE (1);*

(d) *for linearly independent solutions* $Y_i(x, \Phi(x,\varepsilon), \varepsilon)$, *there exist the estimations (44);*

(e) *for any compact subset of* I *that does not contain turning point* $x = 0$, *there exists the limit relation*

$$\lim_{\varepsilon\to 0} Y_{\text{part.}}(x, \Phi(x,\varepsilon), \varepsilon) = \frac{h(x)}{k_1(x)x\tilde{k}(x)} \equiv \omega_0(x),$$

where $\omega_0(x)$ *is the solution of simplified equation (8).*

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