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THE TWO-DIMENSIONAL BOOLE-TYPE TRANSFORM AND ITS ERGODICITY

ДВОВИМІРНЕ ПЕРЕТВОРЕННЯ БУЛЕВОГО ТИПУ ТА ЙОГО ЕРГОДИЧНІСТЬ

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Authors dedicate their work on commemoration of untimely passed away outstanding Ukrainian mathematician Professor A. M. Samoilenko

Based on the Schweiger's smooth fibered approach and the related Bernoulli shift transformation scheme the ergodicity of the two-dimensional Boole type transformations is proved. New multi-dimensional Boole type transformations, invariant with respect to the Lebesgue measure, and their ergodicity properties are also discussed.

Доведено ергодичність двовимірних перетворень булевого типу на основі гладких розшарувань Швайгера та застосування асоційованих зсувів Бернуллі. Розглянуто також властивість ергодичності багатовимірних перетворень булевого типу, інваріантних щодо міри Лебега.

1. Introduction. With its origins, going back several centuries, discrete analysis became now an increasingly central methodology for many mathematical problems related to discrete dynamical systems and algorithms, widely applied in modern science. Our theme, being related with studying topological and measure theoretical ergodicity aspects of the Boole type discrete dynamical systems [1 – 12], is of great interest in many branches of modern science and technology [11 – 25], especially in statistical mechanics, discrete mathematics, numerical analysis, chaos theory, statistics and probability theory as well as in electrical and electronic engineering. From this viewpoint this topic belongs to a much more general realm of mathematics, namely, to calculus, differential equations and differential geometry, because of the remarkable analogy of the subject especially to these branches of mathematics. Nonetheless, although the topic is discrete, our approach to treating topological and measure theoretical ergodicity and the related arithmetic properties of the generalized Boole type discrete dynamical systems will be completely analytical, resulting in the ergodicity proof of the two-dimensional Boole type transformation.

2. Ergodicity and Bernoulli type transformations. There is considered a class of mappings called [5, 12, 23, 26, 27] *smooth fibered multidimensional mappings* $\varphi\colon X \to X$, if the following conditions are satisfied:

(a) there is an invariant Lebesgue equivalent probability measure $\mu\colon \mathcal{B} \to \mathbb{R}_+$, for which there exist positive constants $c_1, c_2 \in \mathbb{R}_+$, such that $c_1\lambda(E) \le \mu(E) \le c_2\lambda(E)$ for every Borel set $E \subset X$;

(b) there is a family of finite or countable infinite digit sets $D_j,\ j = \overline{1,N}$;

(c) there is a mapping $k\colon X \to D$, where $D := D_1 \times D_2 \times \ldots \times D_N$, such that the sets $X_i := k^{-1}\{i\} = \{x \in X : k(x) = i\},\ i \in D$, are measurable and form a partition of the space X, that is sets $\sqcup_{i\in D} X_i = X$;

(d) the restrictions $\varphi|_{X_i}\colon X_i \to X,\ i \in D$, are injective and smooth maps.

It is easily observed that the mapping $\varphi\colon X \to X$ is equivalent to the Bernoulli shifts mapping $T_\varphi\colon D^\infty \to D^\infty$, where

$$T_\varphi\colon (k_1, k_2, k_3, \ldots, k_n, \ldots) \to (k_2, k_3, \ldots, k_n, \ldots) \tag{2.1}$$

with respect to the isomorphism $\psi\colon X \ni x \to (k_2, k_3, \ldots, k_n, \ldots) \in D^\infty$,

$$X(k_1, k_2, k_3, \ldots, k_n; x) \Longleftrightarrow (k_1, k_2, k_3, \ldots, k_n, \ldots), \tag{2.2}$$

determined for the rank-n cylinder sets $X_n(k_1, k_2, k_3, \ldots, k_n) \subset X,\ n \in \mathbb{N}:$

$$X_n(k_1, k_2, k_3, \ldots, k_n) := \cap_{j=\overline{1,n}} X_{k_j}. \tag{2.3}$$

The sequence $(k_1, k_2, k_3, \ldots, k_n) \in D^n$ is called *admissible* if there exists a point $x \in X$, such that $X_n(k_1, k_2, k_3, \ldots, k_n; x) \subset \cap_{j=\overline{1,n}} X_{k_j},\ n \in \mathbb{N}$.

In many concrete cases the ergodicity of a mapping $\varphi\colon X \to X$ can be stated more effectively using standard measure theoretical calculations. In particular, based on the construction above one can present a slightly alternative to [12, 28, 29] approach to proving ergodicity, making use of the following two classical measure theory [30, 31] lemmas.

Lemma 2.1 (Hahn – Caratheodory – Kolmogorov extension theorem)**.** *Let $\mathcal{A}$ be an algebra of subsets of X and $\mathcal{B}(\mathcal{A})$ denote the σ-algebra generated by $\mathcal{A}$. Suppose that a mapping $\mu\colon \mathcal{A} \to [0,1]$ satisfies the conditions:*

(a) $\mu(\varnothing) = 0$;

(b) *if $A_n \in \mathcal{A},\ n \in \mathbb{N}$, are pair wise disjoint and if $\sqcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$, then $\mu\big(\sqcup_{n\in\mathbb{N}} A_n\big) = \sum_{n\in\mathbb{N}} \mu(A_n)$.*

Then there is a unique probability measure $\mu\colon \mathcal{B}(\mathcal{A}) \to [0,1]$, which is an extension of the the mapping $\mu\colon \mathcal{A} \to [0,1]$.

Lemma 2.2. *Let $(X, \mathcal{B}, \mu)$ be a probability space and suppose that $\mathcal{A} \subset \mathcal{B}$ is an algebra that generates $\mathcal{B}$, that is $\mathcal{B} = \mathcal{B}(\mathcal{A})$. Suppose that there exists $C > 0$, such that for a fixed $B \in \mathcal{B}$ there holds the inequality*

$$\mu(B)\mu(I) \le C\mu(B \cap I) \tag{2.4}$$

for all $I \in \mathcal{A}$. Then the measure $\mu(B)\mu(B^c) = 0$, where $B^c := X \backslash B \in \mathcal{B}$ denotes the complement of the set $B \in \mathcal{B}$.

Owing to the special properties of *smooth fibered* multidimensional mappings $\varphi\colon X \to X$ being equivalent to the Bernoulli shifts [27, 28, 32] mapping (2.1), one can formulate the following important theorem.

Theorem 2.1. *Let the cylinder sets of a smooth fibered multidimensional mapping $\varphi X \to X$ satisfy the conditions of Lemma 2.2 with respect to the a Lebesgue measure λ on X, absolute continuous to the invariant measure μ on X. Then the mapping $\varphi\colon X \to X$ is ergodic with respect to its invariant probability measure μ on X.*

Example 2.1. A simplest example is given by the doubling mapping

$$\varphi\colon [0,1) \ni x \to \{2x\} \in [0,1), \tag{2.5}$$

where $k\colon [0,1) \ni x \to \lfloor 2x \rfloor \in \{0,1\} := D$.

It is ergodic [28, 29] with respect to the finite Lebesgue measure $d\lambda(x) = dx,\ x \in [0,1)$, and allows the generating partition $\xi = \{X_0 = [0,1/2), X_1 = [1/2,1)\},\ X_0 \sqcup X_1 = [0,1) = X$.

Concerning the ergodicity of the doubling mapping (2.5), it can be easily stated if we represent any number $x \in [0,1)$ as a binary expansion

$$x := (\cdot x_0 x_1 x_2 \ldots x_n \ldots) = \sum_{j \in \mathbb{Z}_+} x_j 2^{-(j+1)}, \tag{2.6}$$

where $x \in \{0,1\} = D$. Denote for convenience the set of all such expansions by $Y = \{(\cdot x_0 x_1 x_2 \ldots x_n \ldots)\colon x_j \in \{0,1\}\} \simeq \{0,1\}^{\mathbb{Z}_+}$. It is easy to observe that the mapping (2.5) is equivalent to the left Bernoulli type shift

$$T_\varphi(\cdot x_0 x_1 x_2 \ldots x_n \ldots) = (\cdot x_1 x_2 \ldots x_n \ldots) \tag{2.7}$$

for any element $(\cdot x_0 x_1 x_2 \ldots x_n \ldots) \in Y$. Now one can introduce so called dyadic intervals or cylinder sets to be the sets

$$I(k_0, k_1, \ldots, k_{n-1}) = \big\{x \in [0,1)\colon x_j = k_j,\ j = \overline{1, n-1}\,\big\}, \tag{2.8}$$

where, for instance, $I(0) = [0,1/2),\ I(1) = [1/2,1),\ I(0,0) = [0,1/4),\ I(0,1) = [1.4,1/2)$ etc. If $\mathcal{A}$ denotes the algebra of finite union of such cylinders, it is seen that it generates the usual Borel σ-algebra $\mathcal{B}$ of the integral $[0,1)$. Moreover, if one takes two separate points $x \neq y \in [0,1)$, their expansions are different at some place $n \in \mathbb{Z}_+$ of their 2-expansions, thus meaning that these numbers belong to different disjoint cylinders. Define now the following inverse to (2.5) mappings $\sigma_0\colon [0,1) \to [0,1/2)$, and $\sigma_1\colon [0,1) \to [1/2,1)$, where

$$\begin{aligned} \sigma_0(x) &= \Big\{ x/2, \quad \text{if} \quad x \in [0,1/2), \\ \sigma_1(x) &= \Big\{ (1+x)/2, \quad \text{if} \quad x \in [1/2,1), \end{aligned} \tag{2.9}$$

where $\varphi \circ \sigma_j(x) = x,\ j = \overline{0,1}$, for any $x \in [0,1)$ and whose actions on elements of the set Y are the corresponding right shifts:

$$\begin{aligned} \sigma_0(\ldots x_0 x_1 x_2 \ldots x_n \ldots) &= (\ldots 0 x_0 x_1 x_2 \ldots x_n \ldots), \\ \sigma_1(\ldots x_0 x_1 x_2 \ldots x_n \ldots) &= (\ldots 1 x_0 x_1 x_2 \ldots x_n \ldots). \end{aligned} \tag{2.10}$$

Based on the definition of cylinder sets (2.8) and actions (2.10), one can observe that

$$I_n := I(k_0, k_1, \ldots, k_n) = \sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}([0,1)), \tag{2.11a}$$

whose Lebesgue measure is easily calculated, being

$$\lambda(I_n) = 2^{-(n+1)} \sum_{j \in \mathbb{Z}_+} 2^{-j} = 2^{-n} \tag{2.12}$$

for any $n \in \mathbb{N}$.

We are now in a position to apply Lemmas 2.1 and 2.2. Let a measurable set $B \subset [0,1)$ is invariant: $B = \varphi^{-1}B = \varphi^{-n}B,\ n \in \mathbb{N}$, and calculate the Lebesgue measure

$$\lambda(B \cap I_n) = \int\limits_{[0,1)} \chi_{B \cap I_n}(x)\, dx = \int\limits_{[0,1)} \chi_B(z)\chi_{I_n}(x)\, dx = \int\limits_{I_n} \chi_B(x) dx =$$

$$= \int\limits_{[0,1)} \chi_B(\sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}(x)) d(\sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}(x)) =$$

$$= \int\limits_{[0,1)} \chi_{\tilde{\varphi}^{-n}B}(\sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}(x)) d(\sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}(x)) =$$

$$= \int\limits_{[0,1)} \chi_B(\tilde{\varphi}^n \circ \sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}(x)) d(\sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}(x)) =$$

$$= \int\limits_{[0,1)} \chi_B(x) d(\sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}(x)) =$$

$$= \int\limits_{[0,1)} \chi_B(x) \sigma'_{k_0} \sigma'_{k_1} \sigma'_{k_2} \ldots \sigma'_{k_n}(x) dx = 2^{-n}\lambda(B) = \lambda(I_n)\lambda(B), \tag{2.13}$$

that is $\lambda(I_n)\lambda(B) = \lambda(B \cap I_n) \le C\lambda(B \cap I_n)$, where $C = 1$. Thus, either the Lebesgue measure $\lambda(B) = 1$ or $\lambda(B) = 0$, meaning the ergodicity of the doubling mapping (2.5).

Example 2.2. A very interesting example is given by the classical continued fraction expansion via the Gauss ergodic mapping

$$\varphi\colon [0,1) \ni x \to \{1/x\} \in [0,1). \tag{2.14}$$

whose fibering is defined by the mapping $k\colon [0,1) \ni x \to \lfloor 1/x \rfloor \in \mathbb{N} := D$, generating partition is given by sets $X_i = (1/(i+1), 1/i],\ i \in \mathbb{N},\ X = \sqcup_{i \in \mathbb{N}} X_i$.

The invariant measure is the well known Gauss measure $d\mu(x) = d\lambda(x)/[(1+x)\ln 2]$, where $d\lambda(x) := dx,\ x \in [0,1)$. The ergodicity of the mapping (2.14) can be easily stated, reducing it [29] via the continuum fraction expansion to a Bernoulli shifting and applying Lemmas 2.1 and 2.2.

Namely, take a number $x \in [0,1)$ and denote by $[x_0, x_1, \ldots, x_n, \ldots]$ its continuous fraction expansion:

$$x = \frac{1}{x_0 +} \frac{1}{x_2 + \ldots} \frac{1}{x_n + \ldots}, \tag{2.15}$$

where $x_i \in \mathbb{Z}_+$ for all indices $i \in \mathbb{Z}_+$. Observe here that the induced continuous fraction mapping acts by left shifting as $T_\varphi[x_0, x_1, \ldots, x_n, \ldots] = [x_1, \ldots, x_n, \ldots]$ for any expansion (2.15). This expansion $[x_0, x_1, \ldots, x_n, \ldots]$ can be reduced to n-th order having defined for every $t \in [0,1)$ the rational t-fraction

$$[x_0, x_1, \ldots, x_{n-1} + t] := \frac{P_n(x_0, x_1, \ldots, x_{n-1}; t)}{Q_n(x_0, x_1, \ldots, x_{n-1}; t)}, \tag{2.16}$$

where, by definition,

$$P_n(x_0, x_1, \ldots, x_{n-1}; t)$$

and

$$Q_n(x_0, x_1, \ldots, x_{n-1}; t)$$

are coprime polynomials in the variables $x_0, x_1, \ldots, x_{n-1} \in \mathbb{Z}_+$ and $t \in [0,1)$ for all $n \in \mathbb{N}$. If to define the n-th order polynomials $P_n = P_n(x_0, x_1, \ldots, x_{n-1}) := P_n(x_0, x_1, \ldots, x_{n-1}; 0)$ and $Q_n = Q_n(x_0, x_1, \ldots, x_{n-1}) := Q_n(x_0, x_1, \ldots, x_{n-1}; 0)$, one can easily observe that the following iterative expressions hold:

$$\begin{aligned} P_n(x_0, x_1, \ldots, x_{n-1}; t) &= P_n + tP_{n-1}, \\ Q_n(x_0, x_1, \ldots, x_{n-1}; t) &= Q_n + tQ_{n-1}, \\ P_n(x_0, x_1, \ldots, x_{n-1}) &= Q_{n-1}(x_1, \ldots, x_{n-1}), \\ P_{n+1}(x_0, x_1, \ldots, x_{n-1}, x_n; t) &= x_n P_n + P_{n-1} + tP_n, \\ Q_{n+1}(x_0, x_1, \ldots, x_{n-1}, x_n; t) &= x_n Q_n + Q_{n-1} + tQ_n, \end{aligned} \tag{2.17}$$

for any $t \in [0,1)$ and arbitrary $n \in \mathbb{N}$. From (2.17), putting a parameter $t = 0$, one also derives the following important iterative relationships for all $n \in \mathbb{N}$:

$$P_{n+1} = x_n P_n + P_{n-1}, \qquad Q_{n+1} = x_n Q_n + Q_{n-1} \tag{2.18}$$

with initial conditions $P_0 = 0,\ P_1 = 1$ and $Q_0 = 1,\ Q_1 = x_0 \in \mathbb{Z}_+$. In particular, the following invariant condition $Q_n\ P_{n-1} - P_n Q_{n-1} = (-1)^n$ and inequality $Q_{n-1} \leq Q_n$ easily follow from (2.18) for all $n \in \mathbb{N}$. Let now indices $k_0, k_1, \ldots, k_{n-1} \in \mathbb{N}$ for every $n \in \mathbb{N}$ and define the cylindrical intervals $I_n \subset [0,1)$ as the corresponding collection of rational t-fractions

$$I_n = I_n(k_0, k_1, \ldots, k_{n-1}) := \big\{[k_0, k_1, \ldots, k_{n-1} + t] : t \in [0,1)\big\}. \tag{2.19}$$

If to define, in addition, the inverse mappings $[0,1) \ni x \to \dfrac{k}{k+x} \in I_1(k) \subset [0,1),\ k \in \mathbb{N}$, one easily ensues that the composition

$$\sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}} : [0,1) \to I_n(k_0, k_1, \ldots, k_{n-1}) \subset [0,1) \tag{2.20}$$

for every $n \in \mathbb{N}$. Moreover, there holds the condition $\varphi^n \circ \sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}}(x) = x$ for every $x \in [0,1)$ and $n \in \mathbb{N}$. Take now any $t \in [0,1)$, mention that

$$\sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}}(t) = [k_0, k_1, \ldots, k_{n-1} + t] = \frac{P_n + tP_{n-1}}{Q_n + tQ_{n-1}}, \tag{2.21}$$

and estimate the Lebesgue measure of the interval (2.19):

$$\lambda(I_n) := \int\limits_{[0,1)} \chi_{I_n(t)dt} = \int\limits_{I_n} dt = \int\limits_{[0,1)} \left| J_{\sigma_{k_0}\circ\sigma_{k_1}\circ\ldots\circ\sigma_{k_{n-1}}}(t) \right| dt =$$

$$= \int\limits_{[0,1)} \left| \frac{d}{dt}\left(\frac{P_n + tP_{n-1}}{Q_n + tQ_{n-1}}\right) \right| dt = \int\limits_{[0,1)} \frac{dt}{|Q_n + tQ_{n-1}|^2} \in \left[\frac{1}{4Q_n^2}, \frac{1}{Q_n^2}\right], \tag{2.22}$$

where we took into account that $0 < Q_{n-1} \le Q_n$ for all $n \in \mathbb{N}$.

Now we are in a position to estimate the Lebesgue measure $\lambda(B \cap I_n)$ for the intersection $B \cap I_n$ of an invariant set $B = \varphi^{-1}B = \varphi^{-n}B \subset [0,1)$ and arbitrary cylindrical interval $I_n \subset [0,1)$, $n \in \mathbb{N}$:

$$\lambda(B \cap I_n) = \int\limits_{I_n} \chi_B(x)dx =$$

$$= \int\limits_{[0,1)} \chi_B\big(\sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}}(x)\big)dx =$$

$$= \int\limits_{[0,1)} \chi_{\varphi^{-n}B}\big(\sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}}(x)\big) \left| J_{\sigma_{k_0}\circ\sigma_{k_1}\circ\ldots\circ\sigma_{k_{n-1}}}(x) \right| dx =$$

$$= \int\limits_{[0,1)} \chi_B\big(\varphi^n \circ \sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}} x\big) \left| J_{\sigma_{k_0}\circ\sigma_{k_1}\circ\ldots\circ\sigma_{k_{n-1}}}(x) \right| dx =$$

$$= \int\limits_{[0,1)} \chi_B(x) \left| J_{\sigma_{k_0}\circ\sigma_{k_1}\circ\ldots\circ\sigma_{k_{n-1}}}(x) \right| dx = \int\limits_{[0,1)} \chi_B(x) \frac{dt}{|Q_n + xQ_{n-1}|^2} \ge$$

$$\ge \frac{1}{4Q_n^2}\lambda(B) \ge \frac{1}{4}\lambda(I_n)\lambda(B), \tag{2.23}$$

which fits completely the conditions of Lemma 2.2 with constant $C = 4$. Thus, as a consequence from the estimation (2.23) one deduces that either the measure $\lambda(B) = 1$ or $\lambda(B) = 0$, stating the ergodicity both of the Lebesgue measure $d\lambda(x)$, $x \in [0,1)$, and the invariant Gauss measure $d\mu(x) = \dfrac{dx}{(1+x)\ln 2}$, $x \in [0,1)$, on the unit interval $[0,1)$.

3. One-dimensional Boole type mappings and invariant ergodic measures. The classical one dimensional Boole [4] mapping is defined as

$$\varphi : \mathbb{R}\backslash\{0\} \ni x \to x - 1/x \in \mathbb{R}. \tag{3.1}$$

As it was shown by Adler and Weiss in [33], the Boole mapping (3.1) is ergodic with respect to the invariant σ-finite Lebesgue measure $d\lambda(x) := dx$, $x \in \mathbb{R} := X$. Their proof of the ergodicity was strongly based on the measure theoretic reducing the mapping (3.1) to the corresponding *induced* [28, 29, 32] transformation $\varphi_A : [-1,1] \in [-1,1] \subset \mathbb{R}$ and proving its ergodicity. The φ-invariance of the Lebesgue measure $d\lambda(x) := dx$, $x \in \mathbb{R}$, is easily checked making use of the

Perron – Frobenius condition: for preimages $u_\pm := u_\pm(x) \in \mathbb{R},\ x \in \mathbb{R}$, where $\varphi(u_\pm(x)) = x$, $u_+ + u_- = x,\ u_- u_+ = -1$, one verifies straightforwardly that the preimage measure

$$\sum_\pm du_\pm(x) = \sum_\pm \left|\frac{du_\pm}{dx}\right| dx = \sum_\pm \frac{dx}{|J_\varphi(u_\pm)|} = \sum_\pm \frac{dx}{(1+u_\pm^{-2})} =$$
$$= \sum_\pm \frac{u_\pm^2 dx}{(1+u_\pm^2)} = \frac{(u_+^2+2+u_-^2)dx}{1+(u_+u_-)^2+u_+^2+u_-^2} = \frac{\big(u_+^2+2+u_-^2\big)dx}{2+u_+^2+u_-^2} = dx \tag{3.2}$$

coinciding exactly with the Lebesgue measure on the axis $\mathbb{R}$.

Below we present a modified proof of the ergodicity of the Boole transformation (3.1). Namely, the discussed above approach, when applied to the Boole transformation (3.1) appeared to be successful and allowed to obtain a new proof of the Adler – Weiss [33] result about the ergodicity of this Boole transformation.

Theorem 3.1. *The one-dimensional Boole transformation (3.1) is ergodic with respect to the invariant Lebesgue measure λ on $\mathbb{R}$.*

Proof. As it was mentioned above, it can be reduced to Theorem 2.1, taking into account its mentioned above relation (3.4) to the doubling mapping $T_\varphi\colon [0,1) \ni s \to \{2s\} \in [0,1)$. As it was also shown in [12, 23, 34], the Boole transformation (3.1) can be related to the doubling mapping $T_\varphi\colon [0,1) \ni s \to \{2s\} \in [0,1)$ via the commutative diagram

$$\begin{array}{ccccc} [0,1) & \overset{\cot(\pi\circ)}{\rightarrow} & \mathbb{R} & \overset{\varphi}{\rightarrow} & \mathbb{R} \\ \downarrow\uparrow Id & & & & \downarrow \pi^{-1}\cot^{-1}, \\ [0,1) & \overset{T_\varphi}{\rightarrow} & [0,1) & \overset{\alpha^{-1}(\pi\circ)}{\rightarrow} & [0,1) \end{array} \tag{3.3}$$

where $\alpha^{-1}\colon [0,1) \to [0,1)$ is a diffeomorphism, defined by means of the expression $\alpha(s) = \pi^{-1}\operatorname{arccot}(\pi s/2),\ s \in [0,1)$, related with the mapping (3.1) as

$$\varphi = \cot\pi \circ \alpha^{-1} \circ T_\varphi\big(\pi^{-1} \circ \cot^{-1}\big). \tag{3.4}$$

Let now $\tilde\varphi\colon [0,1) \to [0,1),\ \tilde\varphi := \alpha^{-1} \circ T_\varphi$, be the equivalent to (3.4) mapping, where $\tilde\varphi(s) = \pi^{-1}\cot^{-1}(\varphi(\cot(\pi s)))$ for any $s \in [0,1)$. As every number $a \in [0,1)$ has the binary expansion

$$a := (\cdot k_0 k_1 k_2 \ldots k_n \ldots) = \sum_{j\in\mathbb{Z}_+} k_j 2^{-(j+1)}, \tag{3.5}$$

one can define the so called proper [12, 34] cylindrical sets $I_n := I_n(k_0, k_{1,}, \ldots, k_n) \subset [0,1)$, $n \in \mathbb{Z}_+$, as

$$I_n = \big\{(\sigma_{k_{n-1}} \circ \alpha) \circ (\sigma_{k_{n-1}} \circ \alpha) \circ \ldots \circ (\sigma_{k_0} \circ \alpha)(t) : t \in [0,1)\big\}, \tag{3.6}$$

where $\sigma_0(s) = s/2$, if $s \in [0,1/2)$, and $\sigma_1(s) = (1+s)/2$, if $s \in [1/2,1)$. Remark also that $\tilde\varphi \circ \big(\sigma_{k_j} \circ \alpha\big)(s) = s$ for every $s \in [0,1),\ k_j \in \{0,1\},\ j = \overline{0, n-1}$. The Lebesgue measure of the interval (3.6) can be easily estimated as follows. We have, by definition,

$$\lambda(I_n) = \int_{I_n} dx = \int_{\mathbb{R}} \chi_{I_n}(x)dx\Bigg|_{x=\cot(\pi t)} = \int_{[0,1)} \Big|J_{(\sigma_{k_n}\circ\alpha)\circ(\sigma_{k_{n-1}}\circ\alpha)\circ\ldots\circ(\sigma_{k_0}\circ\alpha))}(t)\Big| dt =$$
$$= \int_{[0,1)} \Big(\sigma'_{k_{n-1}}\alpha'(t_{n-1})\Big)\Big(\sigma'_{k_{n-1}}\alpha'(t_{n-1})\Big)\ldots\big(\sigma'_{k_0}\alpha\prime(t)\big)dt, \tag{3.7}$$

where the derivatives $\sigma'_{k_j} = 1/2,\ \alpha'(t_j) = 2/\big[1 + 3\sin^2(\pi t_j)\big],\ t_j := \sigma_{k_j} \circ \alpha \circ \ldots \circ \sigma_{k_0} \circ \alpha(t)$, $j = \overline{0, n-1},\ t \in [0,1)$. Taking into account that

$$\begin{gathered} \lambda(\sigma_0 \circ \alpha)([0,1)]) = \frac{1}{2} = \lambda(\sigma_1 \circ \alpha([0,1]), \\ \sigma_{k_j} \circ \alpha \circ \ldots \circ \sigma_{k_0} \circ \alpha([0,1)) \subset [1/2^{j+1}, 1/2^j) \end{gathered} \tag{3.8}$$

for any $j = \overline{0, n-1}$, owing to the classical average value theorem, applied to (3.8), one easily obtains that

$$\alpha'(\bar{t}_j)) = 2^{-j}/\big[3\sin^2(\pi\bar{t}_j) + 1\big], \tag{3.9a}$$

where numbers $\bar{t}_j \in \big(1/2^{j+1}, 1/2^j\big) \subset [0,1)$ for all $j \in \overline{0,n}$. Now, owing to the evident inequalities $2t \le \sin(\pi t) \le \pi t$ for all $t \in [0, 1/2)$, one derives the following two-side estimations:

$$\frac{2^{-j}}{\exp 3(\pi 2^{-j})^2} \le \frac{2^{-j}}{3\sin^2(\pi 2^{-j}) + 1} \le \frac{2^{-j}}{3\sin^2(\pi t_j) + 1} \le \frac{2^{-j}}{3\sin^2\big(\pi 2^{-(j+1)}\big) + 1} \le \frac{2^{-j}}{3(2^{-(j+1)})^2 + 1} \tag{3.10}$$

for any $j = \overline{0, n}$. Thus, from expressions (3.7) and (3.10), one ensues right away the needed Renyi type [12, 28, 29, 31, 32] estimations

$$\begin{gathered} \frac{\exp(-3\pi^2)}{2^{n(n+1)/2}} \le \frac{\exp\big[\pi^2(-4 + 4^{-n})\big]}{2^{n(n+1)/2}} = \prod_{j=0}^{n} \frac{2^{-j}}{\exp[3(\pi 2^{-j})]^2} \le \lambda(I_n) \le \\ \le \prod_{j=0}^{n} \frac{2^{-j}}{3(2^{-(j+1)})^2 + 1} \le \frac{2^{-n(n+1)/2}}{\sum_{j=0}^{n} 3(2^{-(j+1)})^2 + 1} = \frac{2^{-n(n+1)/2}}{2 - 4^{-(n+1)}} \le \frac{4/7}{2^{n(n+1)/2}} \end{gathered} \tag{3.11}$$

for all $n \in \mathbb{Z}_+$. In particular, from (3.11) one ensues that $\lim_{n\to\infty} \lambda(I_n) = 0$, meaning that the family of such cylindrical sets generates [23, 30, 34] the Borel σ-algebra $\mathcal{B}$ on the interval $[0,1)$. Thus, we arrived at a position allowing to apply Lemmas 2.1 and 2.2. So, let a measurable set $B \subset [0,1)$ is invariant: $B = \varphi^{-1}B = \varphi^{-n}B,\ n \in \mathbb{N}$, and calculate the following Lebesgue measure:

$$\begin{aligned} \lambda(B \cap I_n) &= \int\limits_{[0,1)} \chi_{B\cap I_n}(t)dt = \int\limits_{[0,1)} \chi_B(t)\chi_{I_n}(t)dt = \int\limits_{I_n} \chi_B(t)dt = \\ &= \int\limits_{[0,1)} \chi_B\big((\sigma_{k_n} \circ \alpha) \circ (\sigma_{k_{n-1}} \circ \alpha) \circ \ldots \circ (\sigma_{k_0} \circ \alpha)(t)\big) \times \\ &\qquad \times d((\sigma_{k_n} \circ \alpha) \circ (\sigma_{k_{n-1}} \circ \alpha) \circ \ldots \circ (\sigma_{k_0} \circ \alpha)(x)) = \\ &= \int\limits_{[0,1)} \chi_{\varphi^{-n}B}((\sigma_{k_n} \circ \alpha) \circ (\sigma_{k_{n-1}} \circ \alpha) \circ \ldots \circ (\sigma_{k_0} \circ \alpha)(x)) \times \\ &\qquad \times d((\sigma_{k_n} \circ \alpha) \circ (\sigma_{k_{n-1}} \circ \alpha) \circ \ldots \circ (\sigma_{k_0} \circ \alpha)(x)) = \end{aligned}$$

$$= \int\limits_{[0,1)} \chi_B\big(\tilde{\varphi}^n \circ (\sigma_{k_n} \circ \alpha) \circ (\sigma_{k_{n-1}} \circ \alpha) \circ \ldots \circ (\sigma_{k_0} \circ \alpha)(t)\big) \times$$

$$\times\, d((\sigma_{k_n} \circ \alpha) \circ (\sigma_{k_{n-1}} \circ \alpha) \circ \ldots \circ (\sigma_{k_0} \circ \alpha)(t)). \tag{3.12}$$

Since the composition $\varphi \circ (\sigma_{k_j} \circ \alpha) = Id$ for any $j = \overline{0, n}$, from (3.12) one ensues that

$$\lambda(B \cap I_n) = \int\limits_{[0,1)} \chi_B(x) d\big((\sigma_{k_n} \circ \alpha) \circ (\sigma_{k_{n-1}} \circ \alpha) \circ \ldots \circ (\sigma_{k_0} \circ \alpha)(x)\big) =$$

$$= \int\limits_{[0,1)} \chi_B(x) \sigma'_{k_n} \alpha' \sigma'_{k_{n-1}} \alpha' \sigma'_{k_{n-2}} \alpha' \ldots \sigma'_{k_0} \alpha'(x) dx \geq$$

$$\geq \frac{\exp(-3\pi^2)}{2^{n(n+1)/2} \lambda(I_n)} \lambda(I_n) \lambda(B) \geq \frac{7}{4 \exp(3\pi^2)}\, \lambda(I_n) \lambda(B),$$

that is the Lebesgue measure $\lambda(I_n)\lambda(B) \leq C\lambda(B \cap I_n)$ for all $n \in \mathbb{Z}_+$, where the constant $C = 4\exp\big(3\pi^2\big)/7$. Thus, owing to Lemma 2.2, either the Lebesgue measure $\lambda(B) = 1$ or $\lambda(B) = 0$, meaning simultaneously the ergodicity of the Boole mapping (3.1) with respect to the same invariant Lebesgue measure λ on $\mathbb{R}$, proving the next theorem.

It is worth to mention here the well known [1, 11, 28, 32, 35, 36] doubling mapping (2.5) is isomorphic to the following one-dimensional Boole type transformation:

$$\varphi \colon \mathbb{R} \ni x \to (x - 1/x)/2 \in \mathbb{R}, \tag{3.13}$$

which is invariant with respect to the probability measure $d\mu(x) = dx/[\pi(1 + x^2)],\ x \in \mathbb{R}$. The Boole mapping (3.1) was generalized as

$$\mathbb{R}\backslash\{b_j : j = \overline{1, N}\} \ni x \to \varphi(x) := cx + a - \sum_{j=1}^{N} \frac{\beta_j}{x - b_j} \in \mathbb{R}, \tag{3.14}$$

where a and $b_j \in \mathbb{R},\ j = \overline{1, N}$, are some real values, $\alpha, \beta_j \in \mathbb{R}_+,\ j = \overline{1, N}$, and was analyzed in [1, 2, 9, 37, 38]. In the case $c = 1,\ a = 0,$ a similar ergodicity result was proved in [2, 39–41] making use of a specially devised inner function method. The related spectral aspects of the mapping (3.14) were in part studied also in [1, 2]. In spite of these results the case $\alpha \neq 1$ still persists to be challenging as the only relating result [1, 2] concerns the following special case of (3.14):

$$\mathbb{R} \ni x \to \varphi(x) := cx + a - \frac{\beta}{x - b} \in \mathbb{R} \tag{3.15}$$

for $0 < c < 1$, and arbitrary $a, b \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$. The related to mappings (3.15) invariant measures and ergodicity were analyzed in [9, 35–37], owing to their equivalence

$$[0, 1) \ni s \colon \to T_\varphi(s) = 2s \bmod 1 \in [0, 1), \tag{3.16}$$

following from the commutative diagram

$$\begin{array}{ccc} [0,1) & \overset{T_\varphi}{\to} & [0,1) \\ f\downarrow & & \downarrow f \\ \mathbb{R} & \overset{\varphi}{\to} & \mathbb{R} \end{array}\ , \tag{3.17}$$

for which the condition $f \circ T_\varphi = \varphi \circ f$, where $f(s) := (2\beta)^{1/2} \cot \pi s + 2a, s \in [0, 1)$, holds. It is also important to mention here that in the framework of the theory of inner functions in [1, 39 – 41] there was stated that there exists an invariant measure $d\mu(x),\ x \in \mathbb{R}$, on the axis $\mathbb{R}$, such that the generalized Boole type transformation (3.14) for any $N > 1,\ c = 1$, and $a = 0$ is ergodic.

If $\alpha = 1$ and $a \neq 0$, the transformation (3.14) appears to be not ergodic, being totally dissipative, that is the wandering set $\mathcal{D}(\varphi) := \cup \mathcal{W}(\varphi) = \mathbb{R}$, where $\mathcal{W}(\varphi) \subset \mathbb{R}$ are such subsets that all sets $\varphi^{-n}(\mathcal{W}),\ n \in \mathbb{Z}_+$, are disjoint. A similar to this statement can be also formulated [1] for the generalized Boole type transformation

$$\mathbb{R} \ni x \to \varphi(x) := \alpha x + a + \int\limits_{\mathbb{R}} \frac{d\nu(s)}{s - x} \in \mathbb{R}, \tag{3.18}$$

where $a \in \mathbb{R},\ c \in \mathbb{R}_+$, a measure $d\nu(s),\ s \in \mathbb{R}$, on $\mathbb{R}$ (not necessary absolutely continuous with respect to the Lebesgue measure) has the compact support $\operatorname{supp} \nu \subset \mathbb{R}$ and satisfies the following natural conditions:

$$\int\limits_{\mathbb{R}} \frac{d\nu(s)}{1 + s^2} = a, \qquad \int\limits_{\mathbb{R}} d\nu(s) < \infty, \tag{3.19}$$

ensuring the boundedness of its topological characteristics.

4. Two-dimensional Boole type transformations and their ergodicity. Multi-dimensional endomorphism of measurable spaces are of great interest [12, 32] in mathematics from many points of view including number-theoretical aspects, numerical theory, dynamical systems theory and diverse physical applications. It is worth to mention here the works [12, 42, 43], where author reviewed a lot of very interesting measure preserving and ergodic multi-dimensional mappings. Recently enough in works [9, 35, 37, 38] there was also proposed a set of new multi-dimensional Boole type transformations $\varphi_\sigma : \mathbb{R}^n \to \mathbb{R}^n$, where

$$\varphi_\sigma(x_1, x_2, \ldots, x_n) := (x_1 - 1/x_{\sigma(1)}, x_2 \pm 1/x_{\sigma(2)}, \ldots, x_n \pm 1/x_{\sigma(n)}) \tag{4.1}$$

for any $n \in \mathbb{N}$ and arbitrary permutations $\sigma \in S_n$ (the signs "$\pm$" are chosen from the nondegeneracy condition $J_\varphi(x) \neq 0,\ x \in \mathbb{R}^n \backslash \{0\}$).

For the case $n = 2,\ (x, y) \in \mathbb{R}^2 \backslash \{0, 0\}$, one obtains the following nontrivial two-dimensional Boole type mapping:

$$\varphi(x, y) := (x - 1/y, y + 1/x), \tag{4.2}$$

and for the case $n = 3,\ (x, y, z) \in \mathbb{R}^3 \backslash \{0, 0, 0\}$, one obtains the following nontrivial three-dimensional Boole type mapping:

$$\begin{aligned} \varphi_+(x, y, z) &:= (x - 1/y, y + 1/z, z + 1/x), \\ \varphi_-(x, y, z) &:= (x - 1/y, y - 1/z, z - 1/x). \end{aligned} \tag{4.3}$$

We have observed that the infinitesimal Lebesgue measure $d\lambda(x, y) := dxdy,\ (x, y) \in \mathbb{R}^2$, on the plane $\mathbb{R}^2$ is invariant subject to the mapping (4.2), that can be easily checked making use of the

Perron – Frobenius condition: for the corresponding preimages $(u_\pm, v_\pm) := (u_\pm(x,y), v_\pm(x,y)) \in \in \mathbb{R}^2$, where $u_+u_- = xy^{-1}$, $v_+v_- = -yx^{-1}$, $u_+ + u_- = 2y^{-1} + x$, $v_+ + v_- = y - 2x^{-1}$, $\varphi(u_\pm, v_\pm) = (x,y) \in \mathbb{R}^2$, one verifies that the measure

$$\sum_{\pm} du_\pm dv_\pm(x,y) = \sum_{\pm} \left|J_{(u_\pm,v_\pm)}(x,y)\right| dxdy =$$

$$= \sum_{\pm} \frac{dxdy}{|J_\varphi(u_\pm, v_\pm)|} = \sum_{\pm} \frac{dxdy}{(1 + (u_\pm v_\pm)^{-2})} =$$

$$= \sum_{\pm} \frac{(u_\pm v_\pm)^2 dx}{(1 + (u_\pm v_\pm)^2)} = \frac{\left[2(u_+v_+u_-v_-)^2 + (u_-v_-)^2 + (u_+v_+)^2\right] dxdy}{[1 + (u_-v_-)^2 + (u_+v_+)^2 + (u_+v_+u_-v_-)^2]} =$$

$$= \frac{\left[(u_-v_-)^2 + (u_+v_+)^2 + 2\right] dxdy}{[2 + (u_-v_-)^2 + (u_+v_+)^2]} = dxdy, \tag{4.4}$$

coinciding exactly with the Lebesgue measure $d\lambda(x,y) := dxdy,\ (x,y) \in \mathbb{R}^2$.

Concerning the ergodicity of the Lebesgue measure preserving mapping (4.2), the approach based on Theorem 2.1 subject to *smooth fibered* multidimensional mappings failed to be effective. Taking into account that the ergodicity result of [33] subject to the one-dimensional Boole mapping (3.1) was strongly based on the induced Kakutani transformation technique, one can expect that it can be also employed for the two-dimensional Boole mapping case (4.1).

Proceed now to a notion of the induced transformation [28, 29, 32] for a measure preserving mapping $\varphi\colon X \to X$, which was effectively used by Adler and Weiss [33], when proving the ergodicity of the one-dimensional Boole transformation (3.1), being, in part, closely related to the classical Poincaré recurrence theorem [15, 28]. Namely, let $(X; \mathcal{B}, \mu, \varphi)$ be a measure preserving discrete system and let $A \subset X$ be a measurable set with $\mu(A) > 0$, for which there holds such a covering condition:

$$\cup_{n\in\mathbb{N}} \varphi^{-n} A = X \tag{4.5}$$

modulo a zero measure set.

Remark 4.1. It is worth to mention here [15, 28, 29, 32] that if a measure preserving system $(X; \mathcal{B}, \mu, \varphi)$, $\mu(X) = 1$, satisfies for arbitrarily chosen measurable $A \subset X$, $\mu(A) > 0$, the covering condition (4.5), then the mapping $\varphi\colon X \to X$ is ergodic. Really, if the measure preserving mapping $\varphi\colon X \to X$ is ergodic, then for an arbitrary measurable set $B \subset X$, satisfying the condition $\mu(B \Delta T^{-1} B) = 0$, there holds either $\mu(B) = 1$, or $\mu(B) = 0$. Now let $A \subset X$ be a measurable set with $\mu(A) > 0$ and construct the set $B := \cup_{n\in\mathbb{N}} \varphi^{-n} A$. Since $\varphi^{-1} B \subset B$, one obtains that $\mu\big(\varphi^{-1} B\big) = \mu(B)$, giving rise to the equality $\mu\big(B \Delta T^{-1} B\big) = 0$, that is either $\mu(B) = 1$, or $\mu(B) = 0$. Moreover, as $\varphi^{-1} A \subset B$ one ensues $\mu(B) \geq \mu(A)$ or $\mu(B) = 1$. The latter, evidently, means that $B = \cup_{n\in\mathbb{N}} \varphi^{-n} A = X$ modulo a zero measure set.

Now, owing to the condition (4.5), the first return time $\tau_A \in \mathbb{N}$ can be defined by the condition

$$\tau_A(x) := \inf_{n\in\mathbb{N}} \{n : \varphi^n(x) \in A,\ x \in A\}, \tag{4.6}$$

exists almost everywhere and is finite.

Definition 4.1. *Let a measure preserving system $(X; \mathcal{B}, \mu, \varphi)$ satisfies the condition (4.5). Then a mapping $\varphi_A : A \to A$ defined as*

$$\varphi_A(x) := \varphi^{\tau_A(x)}(x) \tag{4.7}$$

for almost all $x \in A$ *is called the transformation induced by the measure preserving mapping* $\varphi\colon X \to X$ *on the set* $A \subset X$.

The constructed above induced mapping is characterized by the following [28, 29, 32] important theorems.

Theorem 4.1 (M. Kac's theorem)**.** *Let a mapping* $\varphi\colon X \to X$ *is ergodic and a measurable set* $A \subset X$ *is chosen such* $0 < \mu(A) < \infty$. *Then the average returning time is proportional to the measure* $\mu(A)$, *that is*

$$\int\limits_A \tau_A(x)d\mu(x) = \mu(A). \tag{4.8}$$

Theorem 4.2. *The induced transformation (4.7) is a measure preserving mapping on the space* $(A, \mathcal{B}|_A, \mu_A = \mu(A)^{-1}\mu|_A, \varphi_A)$, *where* $\mathcal{B}|_A := \{B \cap A\colon B \in \mathcal{B}\}$, $0 < \mu(A) < \infty$. *Moreover, if the mapping* $\varphi\colon X \to X$ *is ergodic, with respect to the measure* μ, *the induced transformation* $\varphi_A\colon A \to A$ *is ergodic with respect to the measure* $\mu_A := \mu/\mu(A)$ *induced on the set* A.

As it was already mentioned before, namely this theorem was used in the work [33] for proving the ergodicity of the Boole mapping (3.1). As it was demonstrated above, there exists also an effective second essentially analytical approach to proving the ergodicity, and it would be useful to present also two proofs, if any, of the ergodicity property of the two-dimensional Boole type mapping (4.2).

Concerning the approach based on Theorem 4.2, its main technical ingredients are strongly related to construction of a special generating partition of the measured space X, suggested by Kakutani and Rokhlin [44, 45] for the corresponding induced mapping $\varphi_A\colon A \to A$ introduced above. In particular, let a mapping $\varphi\colon X \to X$ be ergodic and consider for a measurable set $A \subset X$, satisfying the condition $0 < \mu(A) < \infty$, its induced mapping $\varphi_A\colon A \to A$. As the condition (4.5) *a priori* [15, 28, 29, 32, 33] holds, one can construct the following disjoint measurable *first return iteration* subsets:

$$X_n := \{x \in X : \varphi^n(x) \in A,\ \varphi^j(x) \notin A,\ j = \overline{1, n-1}\}, \tag{4.9}$$

where $\sqcup_{n\in\mathbb{N}} X_n = X$, $X_n \cap X_m = \varnothing$, $m \neq n \in \mathbb{N}$, and for which the iteration expression

$$X_{n+1} = \varphi^{-1}X_n \cap \varphi^{-1}A^c \tag{4.10}$$

is satisfied. Based on the sets (4.9) one constructs for all $n \in \mathbb{N}$ the sets

$$A_n := X_n \cap A, \qquad B_n := X_n \cap A^c, \tag{4.11}$$

satisfying the following important disjoint sum properties:

$$\varphi^{-1}B_n = B_{n+1} \sqcup A_{n+1}, \qquad \sqcup_{n\in\mathbb{N}} A_n = A, \qquad \sqcup_{n\in\mathbb{N}} B_n = A^c. \tag{4.12}$$

Consider now any measurable subset $E \subset A$ and mention that

$$\varphi_A^{-1}E = \sqcup_{n\in\mathbb{N}}(\varphi^{-n}E \cap A_n), \tag{4.13}$$

giving rise to the equality

$$\mu(\varphi_A^{-1}E) = \mu(\sqcup_{n\in\mathbb{N}}(\varphi^{-n}E \cap A_n)) = \sum_{n\in\mathbb{N}} \mu(\varphi^{-n}E \cap A_n). \tag{4.14}$$

Based now on the representation (4.12) and the measure invariance, one can easily calculate the equalities

$$
\begin{aligned}
\mu(E) = \mu\big(\varphi^{-1}E\big) = \mu\big(\varphi^{-1}E \cap (B_1 \sqcup A_1)\big) = \\
= \mu\big(\varphi^{-1}E \cap B_1\big) + \mu\big(\varphi^{-1}E \cap A_1\big), \\
\mu(B_n) = \mu\big(\varphi^{-1}B_n\big) = \mu(B_{n+1} \sqcup A_{n+2}) = \mu(B_{n+1}) + \mu(A_{n+2}), \\
\mu\big(\varphi^{-1}E \cap B_1\big) = \mu\big(\varphi^{-1}\big(\varphi^{-1}E \cap B_1\big)\big) = \mu\big(\varphi^{-2}E \cap \varphi^{-1}B_1\big) = \\
= \mu\big(\varphi^{-2}E \cap (B_2 \sqcup A_2)\big) = \mu\big(\varphi^{-2}E \cap B_2\big) + \mu\big(\varphi^{-2}E \cap B_2\big), \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\mu\big(\varphi^{-n}E \cap B_n\big) = \mu\Big(\varphi^{-(n+1)}E \cap B_{n+1}\Big) + \mu\Big(\varphi^{-(n+1)}E \cap A_{n+1}\Big),
\end{aligned}
\tag{4.15}
$$

which hold for all $n \in \mathbb{N}$. As a simple consequence from the equalities (4.15) one derives also such equalities

$$
\mu\big(\varphi^{-n}E \cap B_n\big) = \sum_{k=n+1}^{\infty} \mu\big(\varphi^{-n}E \cap A_n\big), \qquad \mu(B_n) = \sum_{k=n+1}^{\infty} \mu(A_n), \tag{4.16}
$$

reducing to the next two expressions;

$$
\begin{gathered}
\mu\big(\varphi^{-n}E \cap B_n\big) + \sum_{k=\overline{1,n}} \mu\big(\varphi^{-n}E \cap A_n\big) = \sum_{n\in\mathbb{N}} \mu\big(\varphi^{-n}E \cap A_n\big) := \eta_A, \\
\mu(A) = \sum_{k=1}^{\infty} \mu(A_n), \qquad \mu(B_1) = \sum_{k=2}^{\infty} \mu(A_n),
\end{gathered}
\tag{4.17}
$$

simply meaning the invariance of the positive quantity $\eta_A \in \mathbb{R}_+$ with respect to $n \in \mathbb{N}$ and the boundedness of the measure $\mu(B_1) \le \mu(A)$, since the measure $\mu(A) = \mu\big(\sqcup_{n\in\mathbb{N}} A_n\big) < \infty$ is, by assumption, bounded. Taking into account the first equality of (4.15) one obtains right away that $\eta_A = \mu(A) > 0$, that is

$$
\mu\big(\varphi^{-n}E \cap B_n\big) + \sum_{k=\overline{1,n}} \mu\big(\varphi^{-n}E \cap A_n\big) = \mu(E). \tag{4.18}
$$

Recalling now the equality (4.14), we obtain from (4.18) and (4.17) that

$$
\begin{aligned}
\big|\mu\big(\varphi_A^{-1}E\big) - \mu(E)\big| = \lim_{n\to\infty}\left(\sum_{k=n+1}^{\infty} \mu\Big(\varphi^{-k}E \cap A_k\Big)\right) + \mu\big(\varphi^{-n}E \cap B_n\big) \le \\
\le 2\lim_{n\to\infty}\left(\sum_{k=n+1}^{\infty} \mu(A_k)\right) = 0
\end{aligned}
\tag{4.19}
$$

owing to the convergence condition (4.10) for the measure $\mu(A) < \infty$, thus stating that $\mu\big(\varphi_A^{-1}E\big) = \mu(E)$ for any measurable set $E \subset A$. The latter means that the suitably induced on the set $A \subset X$ measure $\mu_A = \mu/\mu(A)$ is also invariant with respect to the induced mapping $\varphi_A : A \to A$.

Assume now that the induced mapping $\varphi_A : A \to A$ is ergodic and take a set $D \subset X$, $\mu(D \cap A) > 0$, since either $\mu(D \cap A) > 0$, or $\mu\big(D \cap A^c\big) > 0$, and which is invariant with respect to the mapping $\varphi : X \to X$, that is $\varphi^{-1} D = D$. From the expansion (4.13) one obtains that

$$\varphi_A^{-1}(D \cap A) = \sqcup_{n\in\mathbb{N}}\big(\varphi^{-n}(D \cap A) \cap A_n\big) = \sqcup_{n\in\mathbb{N}}\big(D \cap \varphi^{-n}A \cap A_n\big) = \\ = D \cap \big(\sqcup_{n\in\mathbb{N}}\big(\varphi^{-n}A \cap A_n\big)\big) = D \cap \varphi_A^{-1}A = D \cap A, \tag{4.20}$$

since the initial assumption $\cup_{n\in\mathbb{N}}\varphi^{-n}A = X$ assures that $\varphi_A^{-1}A = A$ modulo a zero measure set. As the induced mapping is assumed to be ergodic, from (4.20) and the condition $\mu(D \cap A) > 0$ one derives right away that $D \cap A = A$. Thus, based once more on the initial assumption $\cup_{n\in\mathbb{N}}\varphi^{-n}A = X$ one simply obtains that

$$X = \cup_{n\in\mathbb{N}}\varphi^{-n}(D \cap A) = \cup_{n\in\mathbb{N}}\big(\varphi^{-n}D \cap \varphi^{-n}A\big) = \\ = \cup_{n\in\mathbb{N}}\big(D \cap \varphi^{-n}A\big) = D \cap \big(\cup_{n\in\mathbb{N}}\varphi^{-n}A\big) = D \cap X = D, \tag{4.21}$$

meaning that the mapping $\varphi : X \to X$ is ergodic too.

Similarly one also states that the converse statement is also true. Really, if the mapping $\varphi : X \to X$ is ergodic and a set $E \subset A$, $\mu(E) > 0$, is φ_A-invariant, then

$$\varphi_A^{-1}E = \sqcup_{n\in\mathbb{N}}\big(\varphi^{-n}E \cap A_n\big) = E. \tag{4.22}$$

Taking into account the invariance condition (4.22), let us construct the set $F := E \cup \sqcup_{n\in\mathbb{N}}\big(B_n \cap \cap\, \varphi^{-n}E\big)$ and calculate its φ-mapping inverse:

$$\begin{aligned} \varphi^{-1}F &= \varphi^{-1}E \cup \sqcup_{n\in\mathbb{N}}\Big(\varphi^{-1}B_n \cap \varphi^{-(n+1)}E\Big) = \\ &= \varphi^{-1}E \cup \sqcup_{n\in\mathbb{N}}\Big((B_{n+1} \sqcup A_{n+1}) \cap \varphi^{-(n+1)}E)\Big) = \\ &= \varphi^{-1}E \cup \Big(\sqcup_{n\in\mathbb{N}}\Big(B_{n+1} \cap \varphi^{-(n+1)}E\Big)\Big) \bigcup \\ &\bigcup \varphi^{-1}E \cup \Big(\sqcup_{n\in\mathbb{N}}\Big(A_{n+1} \cap \varphi^{-(n+1)}E\Big)\Big) = \\ &= \big(A_1 \cap \varphi^{-1}E \sqcup B_1 \cap \varphi^{-1}E\big) \cup \Big(\sqcup_{n\in\mathbb{N}}\Big(B_{n+1} \cap \varphi^{-(n+1)}E\Big)\Big) \bigcup \\ &\bigcup \big(A_1 \cap \varphi^{-1}E \sqcup B_1 \cap \varphi^{-1}E\big) \cup \Big(\sqcup_{n\in\mathbb{N}}\Big(A_{n+1} \cap \varphi^{-(n+1)}E\Big)\Big) = \\ &= \big(\sqcup_{n\in\mathbb{N}}\big(B_n \cap \varphi^{-n}E\big)\big) \bigcup \big(\sqcup_{n\in\mathbb{N}}\big(A_n \cap \varphi^{-n}E\big)\big) = \\ &= \big(\sqcup_{n\in\mathbb{N}}\big(B_n \cap \varphi^{-n}E\big)\big) \cup E = F, \end{aligned} \tag{4.23}$$

that is $\varphi^{-1}F = F$, meaning its invariance with respect to the mapping $\varphi : X \to X$. Then, owing to its ergodicity, one finds that $F = X$ modulo a zero subset of X. Having now taken into account that, by construction, the subset $\sqcup_{n\in\mathbb{N}}\big(B_n \cap \varphi^{-n}E\big) \subset A^c$, one ensues that the set $A \subseteq E$ modulo a zero subset of X. Insomuch as, by assumption, $E \subset A$, one derives finally that $A = E$, meaning respectively that the induced mapping $\varphi_A : A \to A$ is ergodic too.

If now one tries to apply the measure-theoretic construction devised in work [33] for proving ergodicity of the two-dimensional Boole mapping (4.2), one soon arrives at very cumbersome technical complications being so hard to get overcome. Thus, it would be reasonable to apply to this ergodicity problem the analytical approach based on Theorem 2.1, if to take into account that the two-dimensional Boole mapping (4.2) is related to the following two-dimensional transformation $T_\varphi\colon [0,1)^2 \ni (s,t) \to (\{2s\},\{2t\}) \in [0,1)^2$ on the square $Y=[0,1)^2 \subset \mathbb{R}^2$, owing to the following commutative diagram:

$$\begin{array}{ccccc} [0,1)^2 & \overset{\cot(\pi\circ)}{\rightarrow} & \mathbb{R}^2 & \overset{\varphi}{\rightarrow} & \mathbb{R}^2 \\ S\downarrow & & & & \downarrow \cot^{-1}\pi\,, \\ [0,1)^2 & \overset{T_\varphi(\circ)}{\rightarrow} & [0,1)^2 & \overset{\alpha^{-1}}{\rightarrow} & [0,1)^2 \end{array} \tag{4.24}$$

where we have denoted by $\alpha^{-1}\colon [0,1)^2 \to \mathbb{R}^2$ the mapping

$$\alpha^{-1}\begin{pmatrix} s\\ t\end{pmatrix} := \begin{pmatrix} \alpha_1^{-1}(s,t)\\ \alpha_2^{-1}(s,t)\end{pmatrix} = \begin{pmatrix} \pi^{-1}\cot^{-1}\left(\dfrac{2\cot\{\pi(s+t)\}}{1+\sin\{\pi(s-t)\}/\sin\{\pi(s+t)\}}\right)\\ \pi^{-1}\cot^{-1}\left(\dfrac{2\cot\{\pi(s-t)\}}{-1+\sin\{\pi(s+t)\}/\sin\{\pi(s-t)\}}\right)\end{pmatrix} \tag{4.25}$$

owing to changing the variables $x=\cot(\pi s),\ y=\cot(\pi t),\ (s,t)\in[0,1)^2,\ (x,y)\in\mathbb{R}^2$, subject to the new coordinates $(s,t)\in[0,1)^2$ and the transformation $S^{-1}\colon [0,1)^2 \ni (s,t) \to (\{s+ +t\},\{s-t\}) \in [0,1)^2$. This approach was proved to be successful and allowed to obtain a proof of the ergodicity theorem of the two-dimensional Boole transformation (4.1), announced before in [35–37].

Theorem 4.3. *The two-dimensional Boole transformation (4.1) is ergodic with respect to the invariant Lebesgue measure λ on $\mathbb{R}^2$.*

Proof. One can now construct the proper cylindrical sets $I_n := I_n(k_0,k_{1,},\ldots,k_{n-1};l_0,l_{1,},\ldots \ldots,l_{n-1}) \subset [0,1)^2,\ n\in\mathbb{Z}_+$:

$$I_n = \left\{ \overset{\rightarrow}{\prod_{j=\overline{0,n-1}}} \left(S^{-1}\circ\sigma_{k_j,l_j}\circ\alpha\right)\colon (u,v)\in[0,1)^2\right\} \tag{4.26}$$

for the diffeomorphically equivalent $\tilde\varphi$-mapping $\tilde\varphi = \left(\tilde\varphi_1,\tilde\varphi_2\right)^\intercal\colon [0,1)^2 \to [0,1)^2$, where, by definition, $T_\varphi\circ\sigma_{k_j,l_j} = Id : [0,1)^2 \to [0,1)^2,\ \sigma_{k_j,l_j} := (\sigma_{k_j},\sigma_{l_j})^{-1},\ k_j,l_j\in\{0,1\},\ j=\overline{0,n-1}$, $\sigma_0(s)=s/2$, if $s\in[0,1/2)$, $\sigma_1(s)=(1+s)/2$, if $s\in[1/2,1)$ and

$$(\tilde\varphi_1,\tilde\varphi_2)^\intercal = \cot^{-1}(\pi\circ)\alpha^{-1}\circ T_\varphi\circ S, \tag{4.27}$$

satisfying the obvious conditions

$$\tilde\varphi_1\circ\left(S^{-1}\circ\pi^{-1}\cot^{-1}\circ\pi\sigma_{k_j}\circ\alpha_1\circ\cot(\pi\circ)\right)(u,v)=u,$$

$$\tilde\varphi_2\circ\left(S^{-1}\circ\pi^{-1}\cot^{-1}\circ\pi\sigma_{k_j}\circ\alpha_1\circ\cot(\pi\circ)\right)(u,v)=v$$

for every $(u,v)\in[0,1)^2,\ k_j,l_j\in\overline{0,1},\ j\in\overline{0,n-1}$. Now the Lebesgue measure of the cylindrical interval (4.26) can be now easily estimated as follows. We have, by definition, of the Lebesgue interval measure

$$\lambda(I_n) = \int\limits_{I_n} dudv = \int\limits_{[0,1)^2} \chi_{I_n}(u,v)dudv =$$

$$= \int\limits_{[0,1)^2} \left| J_{(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha)\circ(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha)\circ\ldots\circ(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha)}(u,v)\right| dudv =$$

$$= \int\limits_{[0,1)^2} \prod_{j=\overline{0,n-1}} |J_{S^{-1}}| J_{\sigma_{k_j,l_j}} |J_\alpha(u_j,v_j)| dudv = \frac{1}{4^n} \int\limits_{[0,1)^2} \prod_{j=\overline{0,n-1}} |J_\alpha(u_j,v_j)| dudv, \qquad (4.28)$$

where, by definition, $S^{-1}\circ\sigma_{k_j,l_j}\circ\alpha(u_j,v_j) := (u_{j+1},v_{j+1}) \in [2^{-(j+1)},2^{-j})^2,\ \tilde{\varphi}(u_{j+1},v_{j+1}) = = (u_j,v_j),\ j=\overline{0,n-1},\ (u_0,v_0) := (u,v)\in[0,1)^2$. Taking into account that

$$\begin{gathered}\big(S^{-1}\circ\sigma_{k_{j-1},l_{j-1}}\circ\alpha\big)\circ\big(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha\big)\circ\ldots \\ \ldots\circ\big(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha\big)\big([0,1)^2\big)\subset[1/2^{j+1},1/2^j)\end{gathered} \qquad (4.29)$$

and $2t\le\sin\pi t\le\pi t,\ 2s\le\sin\pi s\le\pi s$ for all $s,t\in[0,1/2]$, the subintegral Jacobian product of (4.28) can be represented as

$$\prod_{j=\overline{0,n-1}} J_\alpha(u_j,v_j) =$$

$$= \prod_{j=\overline{0,n-1}} \frac{\left[\cos^2\pi(u_j+v_j)+\sin^2\pi u_j\cos^2\pi v_j\right]\left[\cos^2\pi(u_j-v_j)+\sin^2\pi v_j\cos^2\pi u_j\right]}{(1-\sin^2\pi u_j-\ \sin^2\pi v_j+2\sin^2\pi u_j\sin^2\pi v_j)} =$$

$$= \prod_{j=\overline{0,n-1}} \frac{\left[1-\sin^2\pi v_j+\sin^2\pi u_j\sin^2\pi v_j-1/2\sin^2 2\pi u_j\sin 2\pi v_j\right]}{\big(1-\sin^2\pi u_j-\sin^2\pi v_j+2\sin^2\pi u_j\sin^2\pi v_j\big)}\times$$

$$\times\prod_{j=\overline{0,n-1}}\left[1-\sin^2\pi u_j+\sin^2\pi u_j\sin^2\pi v_j+1/2\sin^2 2\pi u_j\sin 2\pi v_j\right], \qquad (4.30)$$

one easily obtains its estimation as

$$\left(\frac{3\pi^2}{4}+\frac{1}{4^2}-1\right)\left(\frac{\pi^2}{4}-\frac{3}{2}+\frac{1}{4^2}\right)\exp\left[\sum_{j\in\mathbb{Z}_+}\left(\frac{1-\pi^2}{4^j}+\frac{2(1-\pi^4)}{16^j}\right)\right]\le$$

$$\le\prod_{j=\overline{0,n-1}} J_\alpha(u_j,v_j)\le\left[1-\frac{\pi^2}{2}+\left(\frac{\pi}{2}\right)^4\right]^{-1}\exp\left[\sum_{j\in\mathbb{Z}_+}\left(\frac{2\pi^2+6}{4^{j+1}}+\frac{1}{16^{j+1}}\right)\right]. \qquad (4.31)$$

Thus, based on the estimations (4.30), one ensues the following inequalities for the measure (4.28):

$$\frac{C_1}{4^n}\le\lambda(I_n)\le\frac{C_2}{4^n} \qquad (4.32)$$

for any $n\in\mathbb{Z}_+$, where the bounded constants

$$\begin{gathered} C_1 := \left(\frac{3\pi^2}{4}+\frac{1}{4^2}-1\right)\left(\frac{\pi^2}{4}-\frac{3}{2}+\frac{1}{4^2}\right)\exp\left[\sum_{j\in\mathbb{Z}_+}\left(\frac{1-\pi^2}{4^j}+\frac{2(1-\pi^4)}{16^j}\right)\right], \\ C_2 := \left[1-\frac{\pi^2}{2}+\left(\frac{\pi}{2}\right)^4\right]^{-1}\exp\left[\sum_{j\in\mathbb{Z}_+}\left(\frac{2\pi^2+6}{4^{j+1}}+\frac{1}{16^{j+1}}\right)\right]. \end{gathered} \qquad (4.33)$$

The estimation (4.32) means that we can apply Lemmas 2.1 and 2.2. So, let a measurable set $B \subset [0,1)^2$ is invariant: $B = \varphi^{-1}B = \varphi^{-n}B,\ n \in \mathbb{N}$, and calculate the following Lebesgue measure:

$$\lambda(B \cap I_n) = \int\limits_{[0,1)^2} \chi_{B\cap I_n}(u,v)dudv =$$

$$= \int\limits_{[0,1)^2} \chi_B(u,v)\chi_{I_n}(u,v)dudv = \int\limits_{I_n} \chi_B(u,v)dudv =$$

$$= \int\limits_{[0,1)^2} \chi_B\Big(\big(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha\big)\circ\big(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha\big)\circ\ldots$$

$$\ldots\circ\big(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha\big)(t)\Big)d\lambda\Big(\big(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha\big)\circ$$

$$\circ\big(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha\big)\circ\ldots\circ\big(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha\big)(u,v)\Big) =$$

$$= \int\limits_{[0,1)} \chi_{\varphi^{-n}B}\Big(\big(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha\big)\circ\big(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha\big)\circ\ldots$$

$$\ldots\circ\big(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha\big)\Big)(u,v)d\lambda\Big(\big(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha\big)\circ$$

$$\circ\big(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha\big)\circ\ldots\circ\big(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha\big)\big)(u,v)) =$$

$$= \int\limits_{[0,1)} \chi_B\Big(\tilde{\varphi}^n\circ\big(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha\big)\circ\big(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha\big)\circ\ldots$$

$$\ldots\circ\big(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha\big)(u,v)\Big)d\lambda\big(\big(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha\big)\circ$$

$$\circ\big(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha\big)\circ\ldots\circ\big(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha\big)\Big)(u,v)) =$$

$$= \int\limits_{[0,1)^2} \chi_B(u,v)\Big|J_{(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha)\circ(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha)\circ\ldots\circ(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha)}(u,v)\Big|dudv =$$

$$= \int\limits_{[0,1)^2} \chi_B(u,v)\prod_{j=\overline{0,n-1}}|J_{S^{-1}}|J_{\sigma_{k_j,l_j}}|J_\alpha(u_j,v_j)|dudv = \frac{1}{4^n}\int\limits_B \prod_{j=\overline{0,n-1}} J_\alpha(u_j,v_j)dudv, \tag{4.34}$$

where we made use of the property that the composition $\varphi\circ\big(S^{-1}\circ\sigma_{k_j,l_j}\circ\alpha\big) = Id$ for any $j = \overline{0,n-1}$. Now from (4.32) and one ensues that

$$\lambda(B\cap I_n) = \frac{1}{4^n}\int\limits_{[0,1)^2}\chi_B(u,v)\prod_{j=\overline{0,n-1}} J_\alpha(u_j,v_j) \ \geq \frac{C_1}{4^n\lambda(I_n)}\lambda(I_n)\lambda(B) \geq C_1C_2^{-1}\lambda(I_n)\lambda(B)$$

that is the Lebesgue measure $\lambda(I_n)\lambda(B) \leq C\lambda(B\cap I_n)$ for all $n\in\mathbb{Z}_+$, where the constant $C := C_2C_1^{-1}$. Thus, owing to Lemma 2.2, either the Lebesgue measure $\lambda(B) = 1$ or $\lambda(B) = 0$, meaning simultaneously the ergodicity of the two-dimensional Boole mapping (4.1) with respect to the same invariant Lebesgue measure λ on $\mathbb{R}^2$, finishing the proof.

As it was mentioned above, the Lebesgue measure on $\mathbb{R}^3$ is also invariant with respect to the three-dimensional Boole-type transformations (4.3), which are plausibly ergodic too, yet proofs of this statement are still under search.

5. Conclusion. We demonstrated that the Schweiger's smooth fibered approach based on the Bernoulli type shift transformations technique is an effective tool for proving the ergodicity of discrete measure invariant dynamical systems. In particular, we proved that one- and two-dimensional Boole type transformations are ergodic due to the infinite Lebesgue measures.

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References

1. J. Aaronson, *An introduction to infinite ergodic theory*, American Mathematical Society, Providence, RI (1997).
2. R. Bayless, J. Hawkings, *A special class of infinite measure-preserving quadratic rational maps*, Dyn. Syst., **34**, № 2, 218 – 233 (2019).
3. N. N. Bogoljubov, J. A. Mitropolskii, A. M. Samoilenko, *Method of accelerated convergence in nonlinear mechanics*, Hindustan Publ. Corp., Delhi; Springer-Verlag, Berlin, New York (1976).
4. G. Boole, *On the comparison of transcendents with certain applications to the theory of definite integrals*, Philos. Trans. Roy. Soc. London Ser. B, **147**, 745 – 803 (1857).
5. W. Bosma, K. Dajani, C. Kraaikamp, *Entropy quotient and correct digits in number-theoretical expansions*, Dynamics and Stochastics, Inst. Math. Statist., Beachwood, OH, 176 – 188 (2006).
6. R. Bowen, *Invariant measures for Markov maps of the interval*, Comm. Math. Phys., **69**, 1 – 17 (1979).
7. B. Leal, G. Mata, D. Ramirez, *Traslaciones de transformaciones tipo boole robustamente transitivas*, Revista Digital Novasinergia, **1**, № 1, 6 – 13 (2018).
8. T. Meyerovitch, *Ergodicity of Poisson products and applications*, Ann. Probab., **41**, № 5, 3181 – 3200 (2013).
9. Y. A. Prykarpatsky, D. Blackmore, Y. Golenia, A. K. Prykarpatski, *Invariant measures for discrete dynamical systems and ergodic properties of generalized Boole type transformations*, Ukr. Math. Zh., **65**, № 1, 44 – 57 (2013); ***English translation:*** Ukr. Math. J., **65**, № 1, 47 – 63 (2013).
10. A. M. Samoilenko et al., *A geometrical approach to quantum holonomic computing algorithms*, Math. Comput. Simulation, **66**, 1 – 20 (2004).
11. T. Schindler, *A central limit theorem for the Birkhoff-sum of the Riemann zeta-function over a Boolean type transformation*, arXiv:2003.02118v1 [math.NT] 4 Mar (2020).
12. F. Schweiger, *Ergodic theory of fibred systems and metric number theory*, The Clarendon Press, Oxford Univ. Press, New York (1995).
13. T. O. Banakh, A. K. Prykarpatsky, *Ergodic deformations of nonlinear Hamilton systems and local homeomorphism of metric spaces*, J. Math. Sci. (N.Y.), **241**, № 1, 27 – 35 (2019).
14. P. Billingsley, *Ergodic theory and information*, John Wiley & Sons, Inc., New York, London, Sydney (1965).
15. P. R. Halmosh, *Lectures on ergodic theory*, Chelsea Publ. Co., New York (1960).
16. J. Lebowitz, O. Penrose, *Modern ergodic theory*, Phys. Today, № 2, 23 – 29 (1973).
17. P. J. Mitkowski, W. Mitkowski, *Ergodic theory approach to chaos: Remarks and computational aspects*, Int. J. Appl. Math. Comput. Sci., **22**, № 2, 259 – 267 (2012); DOI: 10.2478/v10006-012-0019-4.
18. K. Okubo, K. Umeno, *Infinite ergodicity that preserves the Lebesgue measure*, Chaos, **31**, Paper No. 033135 (2021); Doi: 10.1063/5.0029751.
19. P. Oprocha, *Distributional chaos revisited*, Trans. Amer. Math. Soc., **361**, № 9, 4901 – 4925 (2009).

20. J. A. Roberts, D. T. Tran, *Algebraic entropy of integrable lattice equations and their reductions*, Nonlinearity, **32**, 622 – 653 (2019).

21. D. J. Rudolph, *Fundamentals of measurable dynamics: ergodic theory on Lebesgue spaces*, The Clarendon Press, Oxford Univ. Press, New York (1990).

22. A. M. Samoilenko, *Elements of the mathematical theory of multi-frequency oscillations*, Kluwer Acad. Publ. Group, Dordrecht (1991).

23. F. Schweiger, *Number theoretical endomorphisms with σ-finite invariant measure*, Israel J. Math., **21**, № 4, 308 – 318 (1975).

24. B. Weiss, *The isomorphism problem in ergodic theory*, Bull. Amer. Math. Soc., **78**, № 5, 668 – 684 (1972).

25. A. Skorokhod, *Homogeneous Markov chains in compact spaces*, Theory Stoch. Process, **13(29)**, № 3, 80 – 95 (2007).

26. S. Ito, M. Yuri, *Number theoretical transformations with finite range structure and their ergodic properties*, Tokyo J. Math., **10**, № 1, 1 – 32 (1987).

27. M. Yuri, *On a Bernoulli property for multi-dimensional mappings with finite range structure*, Tokyo J. Math., **9**, № 2, 459 – 485 (1986).

28. M. Pollycott, M. Yuri, *Dynamical systems and ergodic theory*, Cambridge Univ. Press, Cambridge (1998).

29. M. Einsiedler, T. Ward, *Ergodic theory with a view towards number theory*, Springer-Verlag London, Ltd., London (2011).

30. R. L. Wheedon, A. Zygmund, *Measure and integral. An introduction to real analysis*, Marcel Dekker, Inc., New York, Basel (1977).

31. A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar., **8**, 477 – 4937 (1957).

32. A. Katok, B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge Univ. Press (1999).

33. R. Adler, B. Weiss, *The ergodic, infinite measure preserving transformation of Boole*, Israel J. Math., **16**, 263 – 278 (1973).

34. F. Schweiger, *Some remarks on ergodicity and invariant measures*, Michigan Math. J., **22**, 308 – 318 (1975).

35. A. K. Prykarpatsky, *On invariant measure structure of a class of ergodic discrete dynamical systems*, Nonlinear Oscil., **3**, № 1, 78 – 83 (2000).

36. A. K. Prykarpatski, *Ergodic theory, Boole type transformations, dynamical systems theory*, Current trends in analysis and its applications, Birkhäuser/Springer, Cham, 325 – 333 (2015).

37. A. K. Prykarpatski, *On discretizations of the generalized Boole type transformations and their ergodicity*, J. Phys. Math., **7**, № 3, 1 – 3 (2016); DOI: 10.4172/2090-0902.1000199.

38. A. K. Prykarpatsky, J. Feldman, *On the ergodic and spectral properties of generalized Boole transformations. I*, Miskolc Math. Notes, **7**, № 1, 91 – 99 (2006).

39. J. Aaronson, *Ergodic theory for inner functions of the upper half plane*, Ann. Inst. H. Poincaré Sect. B (N.S.), **14**, № 3, 233 – 253 (1978).

40. J. Aaronson, *A remark on this existence of inner functions*, J. London Math. Soc. (2), **23**, № 3, 469 – 474 (1981).

41. J. Aaronson, *The eigenvalues of nonsingular transformations*, Israel J. Math., **45**, 297 – 312 (1983).

42. F. Schweiger, *Invariant measures and ergodic properties of number theoretical endomorphisms*, Dynamical systems and ergodic theory (Warsaw, 1986), PWN, Warszawa (1989).

43. F. Schweiger, *Invariant measures for maps of continued fraction type*, J. Number Theory, **39**, 162 – 174 (1991).

44. S. Kakutani, *Induced measure preserving transformations*, Proc. Imp. Acad. Jap., **12**, 82 – 84 (1936).

45. V. A. Rokhlin, *Exact endomorphism of a Lebesgue spaces*, Amer. Math. Soc. Transl. Ser. 2, **39**, № 2, 1 – 36 (1964).

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