UDC 517.9

QUATERNION-VALUED DYNAMIC EQUATIONS AND HENSTOCK – KURZWEIL DELTA-INTEGRALS ON TIME SCALES: A SURVEY*

КВАТЕРНІОННОЗНАЧНІ ДИНАМІЧНІ РІВНЯННЯ І ДЕЛЬТА-ІНТЕГРАЛИ ХЕНСТОКА – КУРЦВЕЙЛЯ НА ЧАСОВИХ ШКАЛАХ: ОГЛЯД

G. Qin, C. Wang

Dep. Math., Yunnan Univ.
Kunming, Yunnan 650091, China
e-mail: guangzhouqin@163.com
chaowang@ynu.edu.cn

R. P. Agarwal

Dep. Math., Texas A&M Univ.-Kingsville
TX 78363-8202, Kingsville, TX, USA,
Florida Inst. Technol.
150 W Univ. Blvd, Melbourne, FL 32901, USA
e-mail: ravi.agarwal@tamuk.edu

Quaternionic calculus is significant in applied mathematics and it closely refers to mathematical physics and engineering sciences. Our purpose of this survey paper is to present some advances in quaternion dynamic equations on time scales and Henstock – Kurzweil Δ-integral (short for HK-Δ-integral) and to introduce the corresponding quaternionic version of HK-Δ-integral (short for HK$^{\mathbb{Q}}$-Δ-integral). Some basic properties of HK$^{\mathbb{Q}}$-Δ-integral are demonstrated which will be helpful in future research related to this topic.

Кватерніонне числення дуже важливе в прикладній математиці і тісно пов’язане з математичною фізикою і технічними науками. У цьому огляді наведено деякі досягнення в дослідженні кватерніонних динамічних рівнянь на часовій шкалі та Δ-інтеграла Хенстока – Курцвейля (скорочено HK-Δ-інтеграл) і подано відповідну кватерніонну версію HK-Δ-інтеграла (скорочено HK$^{\mathbb{Q}}$-Δ-інтеграл). Сформульовано деякі базові властивості HK$^{\mathbb{Q}}$-Δ-інтеграла, які будуть корисні в подальших дослідженнях у цій галузі.

1. Introduction. Time scale theory was put forward by Hilger to unify discrete and continuous analysis in 1988 [1] and it has a wide range of applications including various types of dynamic equations and models in real-world applications [2 – 6]. Time scale is an arbitrary closed subset

* This work is supported by NSFC (No. 11961077) and CAS “Light of West China” Program of Chinese Academy of Sciences.

of the real line $\mathbb{R}$, and the calculus on a time scale includes the classical Riemann integral and the discrete sum. Moreover, it also covers different forms of calculus between the classical continuous case and the discrete one such as q-calculus and its generalizations which demonstrate enormous advantages. In recent years, there are many research works in this area. For example, the measure theory on time scales [7 – 9], the fuzzy calculus and dynamic equations on hybrid domain [10, 11], function analysis and functional dynamic equations [12 – 16]. For dynamic models, the almost periodic solutions of Lasota – Wazewska time scale model [8, 17], the neural networks model on time scales [18 – 21] and Nicholson's blowflies model [22, 23] were investigated. Moreover, by establishing the translation closeness theory of time scales [24 – 28], the pseudo almost periodic functions [29 – 32] and almost automorphic functions [33 – 36] including their generalizations [37 – 41] were investigated and applied to study dynamic equations on time scales [42 – 45]. Time scale theory have become a powerful tool in pure and applied mathematics.

On the other hand, the concept of quaternions which is a noncommutative extension of complex numbers was introduced by Irish mathematician Hamilton in 1843, since quaternion algebra does not conform to the commutative law and it has great superiority in describing rotations and complex physical motions, it has been widely applied in various fields such as robotics, multi-body system mechanics, and attitude control of artificial aircraft [46]. In 1995, Adler investigated quaternionic quantum mechanics and quantum fields [47]. In 2014, Rodman discussed the linear algebra in the framework of quaternion analysis [48]. In [49], Georgiev and Jday studied Brownian motion under quaternionic background. With deep development of applications of quaternion algebra, in 2021, Li and Wang et al. studied the Hyers – Ulam – Rassias stability of fuzzy nonlinear difference equations with impulses [50]. Meanwhile, they established the general theory of higher-order quaternion linear difference equations by using the complex adjoint matrix and the quaternion characteristic polynomial [51].

It is natural to consider quaternion functions and dynamic equations on time scales. Through combining these two powerful tools, some quaternionic problems can be considered and solved on hybrid domains. In [52, 53], the authors considered Cauchy matrix and Liouville formula of quaternion dynamic equations on time scales. In [54], the quaternion matrix dynamic equation on time scales was discussed and some real applications were demonstrated. In addition, the Hyers – Ulam – Rassias stability was extended to quaternion fuzzy nonlinear dynamic equations on time scales [55]. In 2022, Wang, Li, et al. proposed a new type of quaternionic hyper-complex space in which some basic functions and the geometric features of dynamic equations were demonstrated on time scales [56].

Henstock – Kurzweil integral which is widely applied in differential equations is a kind of generalization of Riemann integral, and in some cases it is broader than Lebesgue integral [57]. Henstock – Kurzweil points were first introduced by French mathematician Denjoy in the early twentieth century. In 1957, Czech mathematician Kurzweil gave a more elegant definition, which is similar to the definition of Riemann integral. Kurzweil called it "Gauge Integral". Henstock developed and perfected this integral theory. Based on the contributions of these two mathematicians, this kind of integral is now generally called Henstock – Kurzweil integral. In 2006, Peterson and Thompson established Henstock – Kurzweil delta and nabla integrals on time scales, and further promoted the Henstock – Kurzweil integral, making it widely used in dynamic equations. Some recent related research work can be referred to the literatures [58, 59]. Motivated by the above, it is meaningful to consider the quaternion-valued form of Henstock – Kurzweil integral on the time scales. However, some difficult problems will appear when quaternion dynamic equations on time scales is considered under Henstock – Kurzweil integral setting, the first is that there is no notion of quaternion-valued Henstock – Kurzweil integral. In this survey paper, we will introduce the definition of this kind of integral in quaternionic analysis and establish some of its properties, which may lay a foundation for solving problems of broader quaternion dynamic equations on time scales.

2. Quaternions and time scales. A time scale is a closed nonempty subset of $\mathbb{R}$ and we denote a time scale by $\mathbb{T}$.

Definition 2.1 [4]. *Let* $\mathbb{T}$ *be a time scale and* $t \in \mathbb{T}$, *then define the forward jump operator* $\sigma(t)$ *and the backward jump operator* $\rho(t)$ *at* t *by*

$$\sigma(t) := \inf\{\gamma > t : \gamma \in \mathbb{T}\}, \qquad \rho(t) := \sup\{\gamma < t : \gamma \in \mathbb{T}\},$$

where $\inf \varnothing := \sup \mathbb{T}, \ \sup \varnothing := \inf \mathbb{T}$ *and* $\varnothing$ *denotes the empty set.*

The graininess function $\mu \colon \mathbb{T} \to [0, +\infty)$ *and* $\nu \colon \mathbb{T} \to [0, \infty)$ *are defined by*

$$\mu(t) = \sigma(t) - t, \qquad \nu(t) = t - \rho(t).$$

For $t \in \mathbb{T}$, *if* $\mu(t) > 0$, *we say* t *is a right-scattered point, otherwise right-dense point, and if* $\nu(t) > 0$, *we call* t *the left-scattered point, otherwise left-dense point.*

For the convenience of discussion, the following notations for intervals on time scales will be used:

$$(a, b)_{\mathbb{T}} := \{t \in \mathbb{T} \colon a < t < b\}, \qquad (a, b]_{\mathbb{T}} := \{t \in \mathbb{T} \colon a < t \leq b\},$$

$$[a, b)_{\mathbb{T}} := \{t \in \mathbb{T} \colon a \leq t < b\}, \qquad [a, b]_{\mathbb{T}} := \{t \in \mathbb{T} \colon a \leq t \leq b\}.$$

Definition 2.2 [46]. *A quaternion algebra is defined by*

$$\mathbb{H} := \{q = q_0 + iq_1 + jq_2 + kq_3 \colon q_i \in \mathbb{R}, \ i = 0, 1, 2, 3\},$$

where $i, \ j, \ k$ *satisfy the multiplication rules:*

$$i^2 = j^2 = k^2 = -1, \quad ij = k = -ji,$$

$$jk = i = -kj, \quad ki = j = -ik.$$

The conjugation of q *over* $\mathbb{H}$ *is given by*

$$\bar{q} = q_0 - iq_1 - jq_2 - kq_3$$

and the norm is defined by $\|q\| = \sqrt{\bar{q}q} = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$.

Remark 2.1. Let m and n be two quaternions, then we have $\|m\| - \|n\| \leq \|m + n\| \leq \leq \|m\| + \|n\|$. Indeed,

$$\|m + n\|^2 = (m + n)(\bar{m} + \bar{n}) = m\bar{m} + m\bar{n} + n\bar{m} + n\bar{n} =$$

$$= \|m\|^2 + \|n\|^2 + 2(m_0n_0 + m_1n_1 + m_2n_2 + m_3n_3) \le$$

$$\le \|m\|^2 + \|n\|^2 + 2\sqrt{\left(m_0^2 + m_1^2 + m_2^2 + m_3^2\right)\left(n_0^2 + n_1^2 + n_2^2 + n_3^2\right)} =$$

$$= \|m\|^2 + \|n\|^2 + 2\|m\|\,\|n\| = (\|m\| + \|n\|)^2,$$

i.e., $\|m - n + n\| \le \|m - n\| + \|n\|$. Then the inequality is obtained.

Let $f\colon \mathbb{T} \to \mathbb{H}$ be a quaternion-valued function, then we can decompose f into

$$f(t) = f_0(t) + if_1(t) + jf_2(t) + kf_3(t), \tag{2.1}$$

with $f_i\colon \mathbb{T} \to \mathbb{R},\ i = 0, 1, 2, 3,$ is real-valued function.

Now we present the following Definition 2.3 and Corollary 2.1 which will be used in the our discussion.

Definition 2.3 [58]. *We say a function $\theta = (\theta_L, \theta_R)$ is a Δ-gauge for $[a,b]_{\mathbb{T}}$ provided $\theta_L(t) > 0$ on $(a,b]_{\mathbb{T}}$, $\theta_R(t) > 0$ on $[a,b)_{\mathbb{T}}$, $\theta_L(a) \ge 0$, $\theta_R(b) \ge 0$. Moreover, $\theta_R(t) \ge \mu(t)$ for all $t \in [a,b)_{\mathbb{T}}$.*

A partition $\mathcal{P}$ of $[a,b]_{\mathbb{T}}$ is a a division of $[a,b]_{\mathbb{T}}$ denoted by

$$\mathcal{P} := \{a = t_0 \le \eta_1 \le st_1 \le t_{n-1} \le \eta_n \le t_n = b\},$$

with $t_{i-1} < t_i$ for $i = 1, 2, \ldots, n$ and $t_i, \eta_i \in \mathbb{T}$. Then we call t_i the end points and η_i the tag points.

Let θ be a Δ-gauge for $[a,b]_{\mathbb{T}}$. Then we say a partition $\mathcal{P} := \{a = t_0 \le \eta_1 \le t_1 \le \ldots \ldots \le t_{n-1} \le \eta \le t_n = b\}$ is θ-fine partition of $[a,b]_{\mathbb{T}}$ if

$$\eta_i - \theta_L(\eta_i) \le t_{i-1} < t_i \le \eta_i + \theta_R(\eta_i),$$

holds for $1 \le i \le n$.

Corollary 2.1 [58]. *Let ϑ and φ be Δ-gauge for $[a,b]_{\mathbb{T}}$ such that $0 < \vartheta_L(t) \le \varphi_L(t)$ for $t \in (a,b]_{\mathbb{T}}$ and $0 < \theta_R(t) \le \varphi_R(t)$ for $t \in [a,b)_{\mathbb{T}}$ (write $\vartheta \le \varphi$ and we say ϑ is finer than φ). If $\mathcal{P}$ is a ϑ-fine partition of $[a,b]_{\mathbb{T}}$, then $\mathcal{P}$ is a φ-fine partition of $[a,b]_{\mathbb{T}}$.*

3. Henstock – Kurzweil delta-integrals in quaternion analysis. In this section, a notion of Henstock – Kurzweil delta-integrals in quaternion analysis will be given. First, we begin with some basic concepts.

Definition 3.1. *Let $F = F^0 + iF^1 + jF^2 + kF^3$, $f = f^0 + if^1 + jf^2 + kf^3$; we say $F^\Delta = f$, if $(F^l)^\Delta = f^l$, $l \in \{0,1,2,3\}$.*

Definition 3.2. *A function $\vartheta = \left(\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3\right)$ is called Δ-gauge, if ϑ^l is Δ-gauge, $l \in \in \{0,1,2,3\}$. Let $\left[a^l, b^l\right] \subset \mathbb{T}$, $l \in \{0,1,2,3\}$; we call a partition $\mathcal{P} = \mathcal{P}^0 \times \mathcal{P}^1 \times \mathcal{P}^2 \times \mathcal{P}^3$ is ϑ-fine partition of*

$$\Pi_{\mathbb{T}} = \left[a^0, b^0\right]_{\mathbb{T}} \times \left[a^1, b^1\right]_{\mathbb{T}} \times \left[a^2, b^2\right]_{\mathbb{T}} \times \left[a^3, b^3\right]_{\mathbb{T}},$$

if $\mathcal{P}^l$ is ϑ^l-fine partition of $\left[a^l, b^l\right]_{\mathbb{T}}$, $l \in \{0,1,2,3\}$.

Remark 3.1. For $\theta = \left(\theta^0, \theta^1, \theta^2, \theta^3\right)$ and $\varphi = \left(\varphi^0, \varphi^1, \varphi^2, \varphi^3\right)$, assume that θ^l and φ^l are Δ-gauge for $\left[a^l, b^l\right]_{\mathbb{T}}$, and θ^l is finer than φ^l, from Corollary 2.1. Then $\mathcal{P} = (\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3)$ is a φ-fine partition if $\mathcal{P}$ is a θ-fine partition. Moreover, let $f\colon [a,b]_{\mathbb{T}} \to \mathbb{H}$ be a quaternionic function with the form (2.1), and let $\vartheta = \left(\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3\right)$ be Δ-gauge for $\Pi_{\mathbb{T}}$ with $\left[a^l, b^l\right]_{\mathbb{T}} = [a,b]_{\mathbb{T}}$, $\mathcal{P}^l$ be ϑ^l-fine partition; we can define $\psi = (\psi_L, \psi_R)$ with $\psi_L(t) = \min\{\vartheta_L^l(t)\}$, $\psi_R(t) = \min\{\vartheta_R^l(t)\}$, then any ψ-fine partition is ϑ^l-fine $l = 0, 1, 2, 3$.

Next, we introduce a lemma which will be useful in the process of proving following theorems and properties.

Lemma 3.1 [58]. *Let θ be a Δ-gauge for $[a,b]_{\mathbb{T}}$. Then there is a θ-fine partition $\mathcal{P}$ of $[a,b]_{\mathbb{T}}$.*

We give the concept of Henstock – Kurzweil delta-integral in the framework of quaternion analysis.

Definition 3.3. *Let $[a,b]\subset\mathbb{T}$. Then we call $f\colon [a,b]_{\mathbb{T}}\to\mathbb{H}$ is quaternion-valued Henstock – Kurzweil delta-integrable or HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$ with value $J = HK^Q\int_a^b f(t)\Delta t$ if, for any $\varepsilon>0$, there exists a Δ-gauge ϑ for $[a,b]_{\mathbb{T}}$ such that*

$$\left\| J-\sum_{i=1}^{n} f(\eta_i)\odot(t_i-t_{i-1})\right\| < \varepsilon,$$

for all ϑ-fine partition $\mathcal{P}$ of $[a,b]_{\mathbb{T}}$, where

$$f(\eta_i)\odot(t_i-t_{i-1}) = \big[f^0(\eta_i)+if^1(\eta_i)+jf^2(\eta_i)+kf^3(\eta_i)\big](t_i-t_{i-1}),\quad i=1,2,\ldots,n.$$

Lemma 3.2 [60]. *For every $\delta>0$ there exists at least one partition $\mathcal{P}=\{a=t_0<t_1<\,<t_2<\ldots<t_n=b\}$ of $[a,b)_{\mathbb{T}}$ such that for each $i\in\{1,2,\ldots,n\}$ either $t_i-t_{i-1}<\delta$ or $t_i-t_{i-1}>\delta$ and $\rho(t_i)=t_{i-1}$, where ρ denotes the backward jump operator in $\mathbb{T}$.*

We denote the set which possesses the properties of Lemma 3.2 by $\mathcal{P}_\delta$.

Definition 3.4 [60]. *Let f be a bound function on $[a,b)_{\mathbb{T}}$ and let $\mathcal{P}=\{a=t_0<t_1<t_2<\ldots\,\ldots<t_n=b\}$ be a partition of $[a,b)_{\mathbb{T}}$. In each interval $[t_{i-1},t_i)$, where $i=1,2,\ldots,n$, we choose an arbitrary point ξ_i and form the sum*

$$S=\sum_{i=1}^{n} f(\xi_i)(t_i-t_{i-1}).$$

We call S a Riemann Δ-sum of f corresponding to the partition $\mathcal{P}$.

We say that f is Riemann Δ-integrable from a to b if there exists a number I such that, for each $\varepsilon>0$, there exists $\delta>0$ such that $\|S-I\|<\varepsilon$, for every Riemann Δ-sum S of f corresponding to a partition $\mathcal{P}\in\mathcal{P}_\delta$, independent of the way in which we choose $\xi_i\in[t_{i-1},t_i)$, $i=1,2,\ldots,n$. Then we denote the Riemann Δ-integral of f by $\int_a^b f(t)\,\Delta t$.

Remark 3.2. If f^l is Riemann Δ-integrable on $[a,b]_{\mathbb{T}}$, according to the Riemann Δ-sum for the bounded function f^l on $[a,b]_{\mathbb{T}}$ in Definition 3.4, $l\in\{0,1,2,3\}$, then $f=f^0+if^1+jf^2+kf^3$ is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$ with

$$\mathrm{HK}^Q\int_a^b f(t)\Delta t=\int_a^b f^0(t)\Delta t+i\int_a^b f^1(t)\Delta t+j\int_a^b f^2(t)\Delta t+k\int_a^b f^3(t)\Delta t.$$

In fact, we just need to let $\mathcal{P}\in\mathcal{P}_\delta$ and $\vartheta_L(t)=\varepsilon>0$ a constant for points in $[a,b]_{\mathbb{T}}$, and $\vartheta_R(t)=\varepsilon>0$ for all right-dense points in $[a,b]_{\mathbb{T}}$, $\vartheta_R(t)=\mu(t)$ for the right-scattered points in $[a,b]_{\mathbb{T}}$, then ϑ is Δ-gauge for $[a,b]_{\mathbb{T}}$, according to Definition 3.3, the result follows.

Lemma 3.3 [3]. *Let* $a, b \in \mathbb{T},\ a < b,\ [a,b]_{\mathbb{T}}$ *consists of only isolated points, and if* f *is continuous at right-dense points in* $\mathbb{T}$ *and its left limits exist (and are finite) at left-dense points in* $\mathbb{T}$, *then*

$$\int\limits_a^b f(t)\Delta t = \sum_{t\in[a,b]_{\mathbb{T}}} \mu(t)f(t).$$

Example 3.1. Let $\mathbb{T} = \left\{t = \dfrac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ and $f\colon \mathbb{T}\to\mathbb{H}$ be defined by $f(t) = f^0(t) + + if^1(t) + jf^2(t) + kf^3(t)$, where

$$f^0(t) = \begin{cases} 1, & t \neq 0, \\ 0, & t = 0, \end{cases} \quad f^1(t) = \begin{cases} \dfrac{1}{\sqrt{t} + \sqrt{\sigma(t)}}, & t \neq 0, \\ 0, & t = 0, \end{cases}$$

$$f^2(t) = 0, \quad f^3(t) = 0.$$

Then we can obtain $\mathrm{HK}^Q \displaystyle\int_0^1 f(t)\Delta t = 1 + i$.

Indeed, if we let $0 < \varepsilon < 1$ be given, and suppose that ϑ is a Δ-gauge for $[0,1]_{\mathbb{T}}$ satisfying $\theta_L(t) = \dfrac{1}{3}\nu(t)$ on $t \in (0,1]_{\mathbb{T}}$ and $\theta_R(t) = \mu(t)$ on $t \in (0,1)_{\mathbb{T}}$ with $\theta_R(0) = \varepsilon^2$. We obtain that the first tag point is $\eta_1 = 0$, and also for $1 \le i \le n-1$ we have $\theta_R(t_i) = \mu(t_i)$, and

$$\eta_{i+1} = t_i, \quad \eta_i = t_i, \quad t_{i+1} = \sigma(t_i), \quad 1 \le i \le n-1.$$

Then, by Lemma 3.3, we have

$$\left\| (1+i) - \sum_{i=1}^n f(\eta_i) \odot (t_i - t_{i-1}) \right\| =$$

$$= \left\| 1 + i - \sum_{i=1}^n f^0(\eta_i)(t_i - t_{i-1}) - i\sum_{i=1}^n f^1(\eta_i)(t_i - t_{i-1}) \right\| =$$

$$= \left\| 1 - \int\limits_0^1 f^0(t)\Delta t + i - i\int\limits_0^1 f^1(t)\Delta t \right\| < \varepsilon$$

since the fact that $F^0(t) = t$ is the delta antiderivative of $f^0(t)$ and $F^1(t) = \sqrt{t}$ is the delta antiderivative of $f^1(t)$ on $(0,1)_{\mathbb{T}}$, respectively.

Theorem 3.1. *Let* $f = f^0 + if^1 + jf^2 + kf^3$. *If* $f^l,\ l = 0,1,2,3,$ *is continuous on* $[a,b]_{\mathbb{T}}$, *then* f *is HK*^{Q}-Δ*-integrable on* $[a,b]_{\mathbb{T}}$.

Proof. First, let ε_n be monotone decreasing to 0 and θ^n be a Δ-gauge for $[a,b]_{\mathbb{T}}$ with $\theta_L^n(t) = \varepsilon_n$, and $\theta_R^n(t) = \varepsilon_n$ for all right-dense points of $[a,b]_{\mathbb{T}}$, $\theta_R^n(t) = \mu(t)$ for all right-scattered points of $[a,b]_{\mathbb{T}}$. Thus, it follows that every right-scattered point is a tag point. Therefore, for each $[t_{i-1}, t_i]$, either $\|t_i - t_{i-1}\| < 2\varepsilon_n$ or $\|t_i - t_{i-1}\| = \mu(t_{i-1})$ and t_{i-1} is a right-scattered point.

Hence, let E_1 be the union of all $[t_{i-1}, t_i]$ with t_{i-1} is a right-scattered point, and let $E_2 = [a,b]_{\mathbb{T}} \backslash E_1$, and A denote Riemann sum $\sum_{t_{i-1}\in E_1} f(\eta) \odot (t_i - t_{i-1})$. Then whenever $t \in [\eta - \theta^n_L(\eta), \eta + \theta^n_R(\eta)] \subset E_2$ we have

$$\left\| f^l(t) - f^l(\eta) \right\| < \varepsilon_n \quad l = 0, 1, 2, 3.$$

We may assume that $\varepsilon_{n+1} < \varepsilon_n$ for all n. Let s^l_n denote a Riemann sum over a θ^n-fine partition, where s^l_n, $n = 1, 2, \ldots$, are fixed. Take a θ^m-fine partition

$$\mathcal{P}^m = \{a = t_0 \leq \eta_1 \leq t_1 \leq \ldots \leq \eta_n \leq t_n = b\}$$

and a θ^n-fine partition

$$\mathcal{P}^n = \left\{a = t'_0 \leq \eta'_1 \leq t'_1 \leq \ldots \leq \eta'_n \leq t'_n = b\right\}.$$

If $[t_{i-1}, t_i] \cap [t'_{i-1}, t'_i]$ is nonempty and contains t, then we have

$$\left\| f(\eta) - f(\eta') \right\| \leq \| f(\eta) - f(t) \| + \left\| f(t) - f(\eta') \right\| < \varepsilon_m + \varepsilon_n.$$

It follows that

$$\left\| s^l_m - s^l_n \right\| < (\varepsilon_m + \varepsilon_n)(b - a),$$

and hence $J^l = \lim_{n\to\infty} s^l_n$ exists. Therefore, for any $\varepsilon > 0$, there is Δ-gauge θ^n with $\varepsilon < \varepsilon_n$ and $\left\| s^l_n - J^l \right\| < \varepsilon$ such that over any θ^n-fine division $\mathcal{P}^n$ we have

$$\left\| \sum_{i=1}^{n} f(\eta_i) \odot (t_i - t_{i-1}) - \left(A + J^0 + iJ^1 + jJ^2 + kJ^3\right) \right\| \leq$$

$$\leq \left\| \sum_{t\in E_2} f(\eta_i) \odot (t_i - t_{i-1}) - \left(s^0_n + is^1_n + js^2_n + ks^3_n\right) \right\| +$$

$$+ \left\| s^0_n + is^1_n + js^2_n + ks^3_n - \left(J^0 + iJ^1 + jJ^2 + kJ^3\right) \right\| <$$

$$< 2\varepsilon(b - a) + 2\varepsilon.$$

That is, f is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$.

Theorem 3.1 is proved.

4. Some properties of quaternion-valued HK^Q-Δ-integral. In this section, we shall present some main results and prove them. We first establish some basic properties and give their proofs.

Theorem 4.1. *Let $f, g\colon \mathbb{T} \to \mathbb{H}$, if f and g are HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$, then so are $f + g$ and βf, where β is a real number. Furthermore,*

$$HK^Q \int_a^b (f(t) + g(t))\Delta t = HK^Q \int_a^b f(t)\Delta t + HK^Q \int_a^b g(t)\Delta t,$$

$$HK^Q \int_a^b \beta f(t)\Delta t = \beta\left(HK^Q \int_a^b f(t)\Delta t \right). \tag{4.1}$$

Proof. Let A and B denote the HK^Q-Δ-integrals of f and g on $[a,b]_{\mathbb{T}}$, respectively. Then, for any $\varepsilon > 0$, there is a Δ-gauge φ such that, for any φ-fine partition $\mathcal{P}_1$, we have

$$\left\| A - \sum_{i=1}^{n} f(\xi_i) \odot (t_i - t_{i-1}) \right\| < \frac{\varepsilon}{2}$$

and there is a Δ-gauge γ such that, for any γ-fine partition $\mathcal{P}_2$, we get

$$\left\| B - \sum_{i=1}^{m} f(\eta_i) \odot (t_i - t_{i-1}) \right\| < \frac{\varepsilon}{2}.$$

Now, we let $\vartheta_L = \min\{\varphi_L, \gamma_L\}$, $\vartheta_R = \min\{\varphi_R, \gamma_R\}$; then let $\mathcal{P}$ be a ϑ-fine partition for $[a,b]_{\mathbb{T}}$, since ϑ is finer than φ and γ, $\mathcal{P}$ also a φ-fine and γ-fine partition for $[a,b]_{\mathbb{T}}$ by Corollary 2.1. Thus we get

$$\left\| (A+B) - \sum_{i=1}^{p} [f(\eta_i + g(\eta_i)] \odot (t_i - t_{i-1}) \right\| \le$$

$$\le \left\| A - \sum_{i=1}^{p} f(\eta_i) \odot (t_i - t_{i-1}) \right\| + \left\| B - \sum_{i=1}^{p} g(\eta_i) \odot (t_i - t_{i-1}) \right\| < \varepsilon.$$

Assume that βf has the HK^Q-Δ-integral C on $[a,b]_{\mathbb{T}}$. For any given $\varepsilon > 0$, we have

$$\left\| C - \sum_{i=1}^{n} \beta f(\eta_i) \odot (t_i - t_{i-1}) \right\| < \frac{\varepsilon}{\beta},$$

which implies that

$$\left\| \frac{C}{\beta} - \sum_{i=1}^{n} f(\eta_i) \odot (t_i - t_{i-1}) \right\| < \varepsilon.$$

Hence (4.1) holds.

Theorem 4.1 is proved.

Theorem 4.2. *Let $a < c < b$, if f is HK^Q-Δ-integrable on $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$, then so is it on $[a,b]_{\mathbb{T}}$ with*

$$HK^Q \int_a^b f(t)\Delta t = HK^Q \int_a^c f(t)\Delta t + HK^Q \int_c^b f(t)\Delta t.$$

Proof. Let

$$A := \mathrm{HK}^Q \int_a^c f(t)\Delta t, \qquad B := \mathrm{HK}^Q \int_c^b f(t)\Delta t.$$

Let $\varepsilon > 0$ be given, then there is a Δ-gauge $\gamma = (\gamma_L, \gamma_R)$ for $[a,c]_{\mathbb{T}}$ and a Δ-gauge $\vartheta = (\vartheta_L, \vartheta_R)$ for $[c,b]_{\mathbb{T}}$ such that for all γ-fine partition $\mathcal{P}$ of $[a,c]_{\mathbb{T}}$ and all ϑ-fine partition $\mathcal{P}'$ of $[c,b]_{\mathbb{T}}$, it follows that

$$\left\| A - \sum_{i=1}^{n} f(\eta_i) \odot (t_i - t_{i-1}) \right\| < \frac{\varepsilon}{2},$$

and

$$\left\| B - \sum_{i=1}^{m} f(\eta_i') \odot (t_i' - t_{i-1}') \right\| < \frac{\varepsilon}{2}.$$

Now, define $\theta_L(t) = \gamma_L(t)$ on $[a,c)_{\mathbb{T}}$, $\theta_L(t) = \min\left\{\vartheta_L(t), \dfrac{t-c}{2}\right\}$ on $(c,b]_{\mathbb{T}}$, and

$$\theta_L(c) = \begin{cases} \gamma_L(c), & \rho(c) = c, \\ \min\left\{\gamma_L, \dfrac{\nu(c)}{2}\right\}, & \rho(c) < c. \end{cases}$$

Similarly, define $\theta_R(t) = \min\left\{\gamma_R(t), \dfrac{c-t}{2}\right\}$ on $[a,c)_{\mathbb{T}}$, $\theta_R(t) = \vartheta_R(t)$ on $(c,b]_{\mathbb{T}}$ and

$$\theta_R(c) = \begin{cases} \vartheta_R(c), & \mu(c) = 0, \\ \min\{\vartheta_R(c), \mu(c)\}, & \mu(c) > 0. \end{cases}$$

Next, let $\mathcal{P}''$ be a θ-fine partition of $[a,b]_{\mathbb{T}}$. Note that c is always an end point for $\mathcal{P}''$. Thus we have

$$\left\| (A+B) - \sum_{i=1}^{p} f\big(\eta_i''\big) \odot \big(t_i'' - t_{i-1}''\big) \right\| \le$$

$$\le \left\| A - \sum_{i=1}^{k-1} f\big(\eta_i''\big) \odot \big(t_i'' - t_{i-1}''\big) - f\big(\eta_k''\big) \odot \big(c - t_{k-1}''\big) \right\| +$$

$$+ \left\| B - \sum_{i=k+1}^{p} f\big(\eta_i''\big) \odot \big(t_i'' - t_{i-1}''\big) - f\big(\eta_{k+1}''\big) \odot \big(t_{k+1}'' - c\big) \right\| < \varepsilon.$$

Since θ is finer than γ on $[a,c]_{\mathbb{T}}$, and finer than ϑ on $[c,b]_{\mathbb{T}}$, then from Corollary 2.1, $\mathcal{P}''$ is γ-fine partition for $[a,c]_{\mathbb{T}}$ and is ϑ-fine partition for $[c,b]_{\mathbb{T}}$, then we get the desired result.

Theorem 4.2 is proved.

Lemma 4.1. *A quaternionic function $f : \mathbb{T} \to \mathbb{H}$ is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$ if and only if for every $\varepsilon > 0$ there is a Δ-gauge ϑ for $[a,b]_{\mathbb{T}}$ such that for any ϑ-fine partitions $\mathcal{P}$ and $\mathcal{P}'$ we have*

$$\left\| \sum_{i=1}^{n} F^*(t_{i-1}, t_i) - \sum_{i=1}^{n} F^*(t_{i-1}', t_i') \right\| < \varepsilon,$$

where $\sum_{i=1}^{n} F^*(t_{i-1}, t_i)$ and $\sum_{i=1}^{m} F^*(t_{i-1}', t_i')$ denote the sums

$$\sum_{i=1}^{n} f(\eta_i) \odot (t_i - t_{i-1}), \qquad \sum_{i=1}^{m} f(\eta_i') \odot (t_i' - t_{i-1}'),$$

respectively.

Proof. If f is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$, for given $\varepsilon > 0$, there is a number J and Δ-gauge θ such that

$$\left\| J - \sum_{i=1}^{n} f(\eta_i) \odot (t_i - t_{i-1}) \right\| < \frac{\varepsilon}{2}$$

for any θ-fine partition $\mathcal{P}$. Hence we have

$$\left\| \sum_{i=1}^{m} F^*(t_{i-1}, t_i) - \sum_{i=1}^{n} F^*(t'_{i-1}, t'_i) \right\| \le$$

$$\le \left\| \sum_{i=1}^{m} F^*(t_{i-1}, t_i) - J \right\| + \left\| J - \sum_{i=1}^{n} F^*\big(t'_{i-1}, t'_i\big) \right\| < \varepsilon.$$

For any θ-fine partitions $\mathcal{P}$ and $\mathcal{P}'$, if

$$\left\| \sum_{i=1}^{m} F^*(t_{i-1}, t_i) - \sum_{i=1}^{n} F^*(t'_{i-1}, t'_i) \right\| < \varepsilon$$

holds, let ε_n be monotone decreasing to 0, and assume that $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$, J_n denotes $\sum_{i=1}^{m_n} F_n^*(t_{i-1}, t_i)$. For sufficiently large n, take a θ-fine partition $\mathcal{P}_k$ and a θ-fine partition $\mathcal{P}_s$, $s > k > n$. Then

$$\|J_s - J_k\| < \varepsilon_n,$$

it implies that $J = \lim_{n\to\infty} J_n$ exists. Therefore, given $\varepsilon > 0$, there is a Δ-gauge ϑ with $\varepsilon_n < \varepsilon$ and $\|J_n - J\| < \varepsilon$ such that for any ϑ-fine partition we have

$$\left\| J - \sum_{i=1}^{n} F^*(t_{i-1}, t_i) \right\| \le \left\| \sum_{i=1}^{n} F^*(t_{i-1}, t_i) - J_k \right\| + \|J_k - J\| < \varepsilon_n + \varepsilon.$$

That is, f is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$.

Lemma 4.1 is proved.

Theorem 4.3. *If f is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$, then so is it on a subset $[c,d]_{\mathbb{T}} \subset [a,b]_{\mathbb{T}}$.*

Proof. If f is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$, assume that θ is a Δ-gauge for $[a,b]_{\mathbb{T}}$, and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two θ-fine partitions for $[c,d]_{\mathbb{T}}$, then we may let s_1 and s_2 be the HK^Q-Δ-integrals of f. Similarly, let $\mathcal{P}_3$ be a θ-fine partition on $[a,c]_{\mathbb{T}} \cup [d,b]_{\mathbb{T}}$ and denote by s_3 the corresponding HK^Q-Δ-integral. Then the union $\mathcal{P}_1 \cup \mathcal{P}_3$ forms a θ-fine partition for $[a,b]_{\mathbb{T}}$. And it follows that the HK^Q-Δ-integral on $[a,b]_{\mathbb{T}}$ forms $s_1 + s_3$. Also, for $\mathcal{P}_2 \cup \mathcal{P}_3$, it becomes $s_2 + s_3$, then from Lemma 4.1 we obtain

$$\|s_1 - s_2\| \le \|(s_1 + s_3) - (s_2 + s_3)\| < \varepsilon.$$

Lemma 4.1 is proved.

Theorem 4.4. *Let $[a,b] \subset \mathbb{T}$, assume $f\colon [a,b]_{\mathbb{T}} \to \mathbb{H}$. If f is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$, then the value of $\sum_{i=1}^{n} f(\xi_i) \odot (t_i - t_{i-1})$ does not depend on $f(b)$. Let $c \in [a,b]_{\mathbb{T}}$ be a right-scattered point, then the value of $\sum_{i=1}^{n} f(\eta_i) \odot (t_i - t_{i-1})$ does depend on $f(c)\mu(c)$.*

Proof. Assume that $f = f^0 + if^1 + jf^2 + kf^3$ is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$. Then we consider the two cases: $\rho(b) < b$ and $\rho(b) = b$. If $\rho(b) < b$, we let $\theta_L(b) < \nu(b)$, then $b \notin \{\xi_i\}_1^n$ for any θ-fine partition, thus $\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})$ does not depend on the value $f(b)$. If $\rho(b) = b$, for any given $\varepsilon > 0$ we let

$$\theta_L(b) < p = \min\left\{\frac{\varepsilon}{\|f^0(b)\|+1}, \frac{\varepsilon}{\|f^1(b)\|+1}, \frac{\varepsilon}{\|f^2(b)\|+1}, \frac{\varepsilon}{\|f^3(b)\|+1}\right\},$$

then if $b = \xi_n$, we have

$$\|f(\xi_n)(t_n - t_{n-1})\| = \|f(b) \odot (b - t_{n-1})\| \leq \|f(b)\theta_L(b)\| < \|f(b)p\| < \varepsilon.$$

Now assume that $c \in [a,b)_{\mathbb{T}}$ and c is right-scattered, then form Theorem 4.2 and Remark 3.2, we obtain

$$\mathrm{HK}^Q\int_a^b f(t)\Delta t = \mathrm{HK}^Q\int_a^c f(t)\Delta t + \mathrm{HK}^Q\int_c^{\sigma(c)} f(t)\Delta t + \mathrm{HK}^Q\int_{\sigma(c)}^b f(t)\Delta t =$$

$$= \mathrm{HK}^Q\int_a^c f(t)\Delta t + \mathrm{HK}^Q\int_{\sigma(c)}^b f(t)\Delta t +$$

$$+ f^0(c)\mu(c) + if^1(c)\mu(c) + jf^2(c)\mu(c) + kf^k(c)\mu(c).$$

Lemma 4.1 is proved.

Remark 4.1. According to the proof of Theorem 4.4, without loss of generality, we can assume that $\xi_n \neq b$ in the definition of HK^Q-Δ-integral.

Theorem 4.5. *Let $F\colon [a,b]_{\mathbb{T}} \to \mathbb{H}$ be continuous, $f\colon [a,b]_{\mathbb{T}} \to \mathbb{H}$, and there is a set D with $M_\mu \subset D \subset [a,b]_{\mathbb{T}}^{\kappa}$ such that $F^{\Delta}(t) = f(t)$ for $t \in D$ and $[a,b]_{\mathbb{T}} \backslash D$ is countable, $l \in \{0,1,2,3\}$. Then $f = f^0 + if^1 + jf^2 + kf^3$ is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$ with*

$$HK^Q\int_a^b f(t)\,\Delta t = J,$$

where

$$J = F^0(b) - F^0(a) + i\big(F^1(b) - F^1(a)\big) + j\big(F^2(b) - F^2(a)\big) + k\big(F^3(b) - F^3(a)\big),$$

and M_μ denotes all right-scattered points of $[a,b]_{\mathbb{T}}$, i.e.,

$$M_\mu = \{z_j \in [a,b)_{\mathbb{T}} \colon \mu(z_j) > 0\}.$$

According to the hypothesis, let D be a set with $M_\mu \subset D \subset [a,b]_{\mathbb{T}}^{\kappa}$ such that $(F^l)^{\Delta}(t) = f^l(t)$ for $t \in D$ and $[a,b]_{\mathbb{T}} \backslash D$ is countable. Then we let

$$P := [a,b]_{\mathbb{T}} \backslash D = \{p_1, p_2, \ldots\},$$

and let $\varepsilon > 0$ be given, then we define a Δ-gauge ϑ for $[a,b]_{\mathbb{T}}$.

First, let $t \in M_\mu$, thus, we define $\theta_R(t) = \mu(t)$, since F^l is delta differentiable at $t \in D$, it follows that there is a $\gamma_L(t) > 0$ such that

$$\left\| F^l(\sigma(t_j)) - F^l(s) - (F^l)^\Delta(t_j)(\sigma(t_j) - s) \right\| \leq \frac{\varepsilon}{4(b-a)} \|\sigma(t_j) - s\| \tag{4.2}$$

for all $s \in [t_j - \gamma_L(t_j), t_j]_\mathbb{T}$. Meanwhile, since F^l is continuous at $t \in D$, it implies that there is a $\gamma'_L(t_j) > 0$ such that

$$\left\| F^l(t_j) - F^l(s) - (F^l)^\Delta(t_j)(t_j - s) \right\| \leq \frac{\varepsilon}{2^{j+2}} \tag{4.3}$$

for all $s \in [t_j - \gamma'_L(t_j), t_j]_\mathbb{T}$. Thus, we define $\theta_L(t_j) = \min\{\gamma_L(t_j), \gamma'_L(t_j)\}$ such that (4.2) and (4.3) both hold for $s \in [t_j - \theta_L(t_j), t_j]_\mathbb{T}$.

Secondly, we consider the case $t \in D \backslash M_\mu$. Since F^l is delta differentiable at t, we obtain that there is a $\beta_1(t) > 0$ such that

$$\left\| F^l(t) - F(s) - (F^l)^\Delta(t)(t - s) \right\| \leq \frac{\varepsilon}{4(b-a)} \|t - s\| \tag{4.4}$$

for $s \in [t - \beta_1(t), t + \beta_1(t)]_\mathbb{T}$, then we define $\theta_L(t) = \theta_R(t) = \beta_1(t)$.

Next, suppose that $t \in P$, then $t = p_j$ for some j. In this situation, since F^l is continuous at p_j, there is a $\lambda(p_j) > 0$ such that

$$\left\| F^l(r) - F^l(s) - f^l(p_j)(r - s) \right\| \leq \frac{\varepsilon}{2^{j+2}} \tag{4.5}$$

for all $r, s \in [p_j - \lambda(p_j), p_j + \lambda(p_j)]_\mathbb{T}$. Hence, we define $\theta_R(p_j) = \theta_L(p_j) = \lambda(p_j)$, then we get $\theta_L(t)$ and $\theta_R(t)$ for $t \in [a, b]_\mathbb{T}$. Therefore, we get a Δ-gauge $\theta = (\theta_L, \theta_R)$ for $[a, b]_\mathbb{T}$.

Now suppose that $\mathcal{P}$ is a θ-fine partition of $[a, b]_\mathbb{T}$, and now consider $\eta_i \in M_\mu$, $\eta_i \in D \backslash M_\mu$, $\eta_i \in P$, combining (4.2) to (4.5), we obtain

$$\begin{aligned} &\left\| F^l(b) - F^l(a) - \sum_{i=1}^n f^l(\eta_i)(t_i - t_{i-1}) \right\| = \\ &= \left\| \sum_{i=1}^n \Big(F^l(t_i) - F^l(t_{i-1}) + f^l(\eta_i)(t_i - t_{i-1}) \Big) \right\| \leq \\ &\leq \sum_{i=1}^n \left\| F^l(t_i) - F^l(t_{i-1}) + f^l(\eta_i)(t_i - t_{i-1}) \right\| < \varepsilon. \end{aligned} \tag{4.6}$$

Then we have

$$\left\| J - \sum_{i=1}^n f(\eta_i)(t_i - t_{i-1}) \right\| = \left\| J - \sum_{i=1}^n F^\Delta(\eta_i)(t_i - t_{i-1}) \right\| < \varepsilon.$$

Lemma 4.1 is proved.

Definition 4.1 [58]. *If $F \colon [a, b]_\mathbb{T} \to \mathbb{R}$ is a delta antiderivative of f on $[a, b]_\mathbb{T}$, then we say f is CN-delta integrable on $[a, b]_\mathbb{T}$ and we define*

$$CN \int_a^b f(t)\Delta t := F(b) - F(a).$$

Remark 4.2. It follows from Theorem 4.5 that if $f^l,\ l = 0, 1, 2, 3,$ is CN-delta integrable function f on $[a, b]_{\mathbb{T}}$, then $f = f^0 + if^1 + if^2 + kf^3$ is HK^Q-Δ-integrable on $[a, b]_{\mathbb{T}}$ and

$$\mathrm{HK}^Q \int_a^b f(t)\Delta t = CN \int_a^b f^0(t)\Delta t + i\left(CN \int_a^b f^1(t)\Delta t\right) + \\ + j\left(CN \int_a^b f^2(t)\Delta t\right) + k\left(CN \int_a^b f^3(t)\Delta t\right).$$

Hence, the class of HK^Q-Δ-integrable functions on $[a, b]_{\mathbb{T}}$ contains the class of Riemann delta integrable functions on $[a, b]_{\mathbb{T}}$.

Definition 4.2 [3]. *A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with region D if:*

(1) $D \subset \mathbb{T}^\kappa$;

(2) $\mathbb{T}^\kappa \backslash D$ *is countable and contain no right-scattered elements of* $\mathbb{T}$;

(3) f *is delta differentiable at each* $t \in D$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated if its right-side limits exist (and are finite) at all right-dense points in $\mathbb{T}$ and its left-side limits exist (and are finite) at all left-dense points in $\mathbb{T}$. F is called Δ pre-antiderivative of f, provided it satisfies above properties.

Lemma 4.2 [3]. *Let $f : \mathbb{T} \to \mathbb{R}$ be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^\Delta(t) = f(t) \quad \textit{holds for all} \quad t \in D.$$

Lemma 4.3 [4]. *Let f be a Δ-integrable function on $[a, b]_{\mathbb{T}}$. If f has a Δ pre-antiderivative $F : [a, b]_{\mathbb{T}} \to \mathbb{T}$ with region of differentiation D, then*

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Corollary 4.1. *If $f^l : \mathbb{T} \to \mathbb{R},\ l = 0, 1, 2, 3,$ is regulated and $a, b \in \mathbb{T}$, then $f = f^0 + if^1 + jf^2 + kf^3$ is HK^Q-Δ-integrable on $[a, b]_{\mathbb{T}}$ and*

$$HK^Q \int_a^b f(t)\Delta t = \int_a^b f^0(t)\Delta t + i\left(\int_a^b f^1(t)\Delta t\right) + j\left(\int_a^b f^2(t)\Delta t\right) + k\left(\int_a^b f^3(t)\Delta t\right).$$

Proof. By Lemma 4.2, since f^l is regulated, then there is a function $F^l : [a, b]_{\mathbb{T}} \to \mathbb{R}$ which is continuous on $[a, b]_{\mathbb{T}},\ l \in \{0, 1, 2, 3\}$, and there is a set D with $M_\mu \subset D \subset [a, b]^\kappa_{\mathbb{T}}$ such that $(F^l)^\Delta(t) = f^l(t)$ for $t \in D$ and $[a, b]_{\mathbb{T}} \backslash D$ is countable.

According to Theorem 4.5, it follows that f is HK^Q-Δ-delta integrable on $[a, b]_{\mathbb{T}}$ with

$$\mathrm{HK}^Q \int_a^b f(t)\Delta t = \big(F^0(b) - F^0(a)\big) + i\big(F^1(b) - F^1(a)\big) + \\ + j\big(F^2(b) - F^2(a)\big) + k\big(F^3(b) - F^3(a)\big). \tag{4.7}$$

Since f^l is regulated, by Lemma 4.2 and Lemma 4.3 we obtain

$$\int\limits_a^b f^l(t)\Delta t = F^l(b) - F^l(a). \tag{4.8}$$

Thus, from (4.7) and (4.8) we get the desired result.

Lemma 4.1 is proved.

Definition 4.3 [58]. *Let S be a subset of a time scale $\mathbb{T}$, then we say it has Δ-measure zero provided S contains no right-scattered points and S has Lebesgue measure zero. We say a property A holds Δ-almost everywhere (Δ-a.e.) on $\mathbb{T}$ provided there is a subset S of $\mathbb{T}$ such that the property A holds for all $t \notin S$ and S has Δ-measure zero.*

Theorem 4.6. *If $f(t) = 0$, Δ-almost everywhere in $[a, b]_{\mathbb{T}}$, i.e., for every $t \in [a, b]_{\mathbb{T}}$ except a set D of Δ-measure zero, then $f = f^0 + if^1 + jf^2 + kf^3$ is HK^Q-Δ-integrable to 0 on $[a, b]_{\mathbb{T}}$.*

Proof. In fact, D is the union of D_i, $i = 1, 2, \ldots,$ where D_i is a subset of D with $i - 1 < \|f(d)\| \le i$, for $d \in D_i$. Then, we have that each D_i is also of Δ-measure zero. Hence, given $\varepsilon > 0$, for each i there is a G_i which is the union of a countable number of open intervals with the total length less than $\varepsilon 2^{-i} i^{-1}$ and such that $D_i \subset G_i$. Then define $\theta = (\theta_L, \theta_R)$ such that

$$(\eta - \theta_L(\eta), \eta + \theta_R(\eta)) \subset G_i,$$

for $\eta \in D_i$, $i = 1, 2, \ldots$. Hence for any θ-fine partition $\mathcal{P}$ we have

$$\left\| \sum_{i=1}^{n} f(\eta_i) \odot (t_i - t_{i-1}) \right\| < \varepsilon.$$

Lemma 4.1 is proved.

Theorem 4.7. *If $f, g \colon \mathbb{T} \to \mathbb{H}$ are HK^Q-Δ-integrable on $[a, b]_{\mathbb{T}}$ with $\|f^l(t)\| \le \|g^l(t)\|$, $l = 0, 1, 2, 3$, Δ-a.e. on $[a, b]_{\mathbb{T}}$, then*

$$\left\| HK^Q \int\limits_a^b f(t)\Delta t \right\| \le \left\| HK^Q \int\limits_a^b g(t)\Delta t \right\|,$$

where $f(t) = f^0(t) + if^1(t) + jf^2(t) + kf^3(t)$, $g(t) = g^0(t) + ig^1(t) + jg^2(t) + kg^3(t)$.

Proof. From the proof of Theorem 4.6, we may suppose that $\|f^l(t)\| \le \|g^l(t)\|$ for all $t \in [a, b]_{\mathbb{T}}$. For given $\varepsilon > 0$, there is a Δ-gauge, $\gamma = (\gamma_L, \gamma_R)$, such that for any γ-fine partition $\mathcal{P}_1$, we have

$$\left\| A - \sum_{i=1}^{n} f(\eta_i) \odot (t_i - t_{i-1}) \right\| < \frac{\varepsilon}{2},$$

and there is a Δ-gauge, $\beta = (\beta_L, \beta_R)$, such that for any β-fine partition $\mathcal{P}_2$, we have

$$\left\| B - \sum_{i=1}^{m} g(\eta_i) \odot (t_i - t_{i-1}) \right\| < \frac{\varepsilon}{2}.$$

Now, we let $\theta_L = \min\{\beta_L, \gamma_L\}$, $\theta_R = \min\{\beta_R, \gamma_R\}$, then let $\mathcal{P}$ be a θ-fine partition of $[a,b]_{\mathbb{T}}$, since θ is finer than β and γ, then $\mathcal{P}$ also a β-fine and γ-fine partition of $[a,b]_{\mathbb{T}}$ by Corollary 2.1, thus we get

$$\left\|A - \sum_{i=1}^{p} f(\eta_i) \odot (t_i - t_{i-1})\right\| < \frac{\varepsilon}{2}, \qquad \left\|B - \sum_{i=1}^{p} g(\eta_i) \odot (t_i - t_{i-1})\right\| < \frac{\varepsilon}{2}$$

holds for all θ-fine partitions. Then

$$\left\|\mathrm{HK}^Q \int_a^b f(t)\Delta t\right\| = \left\|\sum_{i=1}^{p} \left[f^0(\eta_i) + i f^1(\eta_i) + j f^2(\eta_i) + k f^3(\eta_i)\right](t_i - t_{i-1})\right\| + \frac{\varepsilon}{2} \leq$$

$$\leq \left\|\sum_{i=1}^{p} \left[g^0(\eta_i) + i g^1(\eta_i) + j g^2(\eta_i) + k g^3(\eta_i)\right](t_i - t_{i-1})\right\| + \frac{\varepsilon}{2} =$$

$$= \left\|\mathrm{HK}^Q \int_a^b g(t)\Delta t\right\| + \varepsilon.$$

According to arbitrariness of ε, the desired result follows.

Lemma 4.1 is proved.

Theorem 4.8. *Assume* $f\colon \mathbb{T} \to \mathbb{H}$ *is* HK^Q-Δ*-integrable on* $[a,b]_{\mathbb{T}}$*. Then given any* $\varepsilon > 0$ *there is a* Δ*-gauge* ϑ *for* $[a,b]_{\mathbb{T}}$ *such that*

$$\sum_{i=1}^{n} \left\| HK^Q \int_{t_{i-1}}^{t_i} f(t)\Delta t - f(\eta_i) \odot (t_i - t_{i-1})\right\| < \varepsilon$$

for all ϑ*-fine partitions* $\mathcal{P}$ *of* $[a,b]_{\mathbb{T}}$.

Proof. Assume that f is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$. Let θ be a Δ-gauge for $[a,b]_{\mathbb{T}}$ and $\mathcal{P} = \{a = t_0 \leq \eta_1 \leq t_1 \leq \ldots \leq t_n \leq b\}$ be a θ-fine partition of $[a,b]_{\mathbb{T}}$, $E_1 = [t_0, t_1]$ and $\mathrm{HK}^Q \int_{E_1} f(t)\Delta t$ is the HK^Q-Δ-integrable of f on E_1. Assume that E_2 is the closure of $[a,b]_{\mathbb{T}} \backslash E_1$. Then, by Theorem 4.3, f is HK^Q-Δ-integrable on E_1 and E_2. Given $\varepsilon > 0$, there is a Δ-gauge θ^1 such that for any θ^1-fine partition $\mathcal{P}^1$ such that

$$\left\|\mathrm{HK}^Q \int_{E_1} f(t)\Delta t - \sum_{i=1}^{p} f(\eta_i) \odot (t_i - t_{i-1})\right\| < \frac{\varepsilon}{2}.$$

Also, there is a Δ-gauge θ^2 such that for any θ^2-fine partition $\mathcal{P}^2$ such that

$$\left\|\mathrm{HK}^Q \int_{E_2} f(t)\Delta t - \sum_{i=1}^{m} f(\eta_i) \odot (t_i - t_{i-1})\right\| < \frac{\varepsilon}{2},$$

where $\mathrm{HK}^Q \int_{E_2} f(t)\Delta t$ is HK^Q-Δ-integral of f on E_2. Let $\theta_L(t) = \theta_L^1(t),\ \theta_R(t) = \theta_R^1(t),$ $t \in E_1$ and $\theta_L(t) = \theta_L^2(t),\ \theta_R(t) = \theta_R^2(t),\ t \in E_2$. Then we have

$$\mathrm{HK}^Q \int_a^b f(t)\Delta t = \mathrm{HK}^Q \int_{E_1} f(t)\Delta t + \mathrm{HK}^Q \int_{E_2} f(t)\Delta t.$$

According to Theorem 4.2 we have

$$\left\| \mathrm{HK}^Q \int_a^b f(t)\Delta t - \sum_{i=1}^{n} f(\eta_i) \odot (t_i - t_{i-1}) \right\| \le$$

$$\le \left\| \mathrm{HK}^Q \int_{E_1} f(t)\Delta t - f(\eta_i) \odot (t_1 - t_0) \right\| +$$

$$+ \left\| \mathrm{HK}^Q \int_{E_2} f(t)\Delta t - \sum_{i=1}^{m} f(\eta_i) \odot (t_i - t_{i-1}) \right\| < \varepsilon.$$

Lemma 4.1 is proved.

Theorem 4.9. *Let* $f_k, f : [a,b]_{\mathbb{T}} \to \mathbb{H}$ *and assume that*

(i) f_k *is* HK^Q-Δ*-integrable on* $[a,b]_{\mathbb{T}},\ k \in \mathbb{N}$;

(ii) $\|f_k - f\| \to 0,\ \Delta$*-a.e. on* $[a,b]_{\mathbb{T}}$;

(iii) $\left\|f_k^l\right\| \le \left\|f_{n+1}^l\right\|,\ l = 0,1,2,3,\ \Delta$*-a.e. on* $[a,b]_{\mathbb{T}},\ k \in \mathbb{N}$;

(iv) $\lim_{k\to\infty} HK^Q \int_a^b f_k(t)\Delta t = J.$

Then f *is* HK^Q-Δ*-integrable on* $[a,b]_{\mathbb{T}}$ *and*

$$J = HK^Q \int_a^b f(t)\Delta t.$$

Proof. Without loss of generality, we can assume that $f_k^l(t) \ge 0,\ \Delta$-a.e. on $[a,b]_{\mathbb{T}}$, then according to Theorem 4.6, we will replace the condition (ii) by

$$\|f_k - f\| \to 0 \quad \text{for each} \quad t \in [a,b)_{\mathbb{T}}, \tag{4.9}$$

and replace (iii) by $\|f_k^l(t)\| \le \|f_{k+1}^l(t)\|,\ l = 0,1,2,3,\ t \in [a,b)_{\mathbb{T}}$. Let $\varepsilon > 0$ be given, we suppose that there is a positive integer s_0 such that

$$\left\| J - \mathrm{HK}^Q \int_a^b f_k(t)\Delta t \right\| < \frac{\varepsilon}{3}$$

for all $k \ge s_0$. From (4.9), there is a positive integer $m(\varepsilon,t) \ge s_0$ for each $t \in [a,b]_{\mathbb{T}}$ such that

$$\left\| f_{m(\varepsilon,t)}^l(t) - f^l(t) \right\| < \frac{\varepsilon}{3(b-a)}, \quad l = 0,1,2,3. \tag{4.10}$$

Since each f_k is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$, from Theorem 4.8, there is a Δ-gauge, θ_k, for $[a,b]_{\mathbb{T}}$ such that

$$\sum_{i=1}^{n}\left\|\mathrm{HK}^Q\int\limits_{t_{i-1}}^{t_i} f_k(t)\Delta t - f_k(\eta_i)\odot(t_i-t_{i-1})\right\| < \frac{\varepsilon}{3\cdot 2^k} \tag{4.11}$$

holds for each θ_k-fine partition for $[a,b]_{\mathbb{T}}$. Thus, we define a Δ-gauge, θ, on $[a,b]_{\mathbb{T}}$ by

$$\theta(t) := \theta_{m(\varepsilon,t)}(t).$$

Then let $\mathcal{P}$ be a θ-fine partition, by using (4.10) and (4.11), we obtain

$$\left\|\sum_{i=1}^{n} f(\eta_i)\odot(t_i-t_{i-1}) - J\right\| \le \sum_{i=1}^{n}\left\|f(\eta_i)-f_{m(\varepsilon,\eta_i)}(\eta_i)\right\|\odot(t_i-t_{i-1}) +$$

$$+\sum_{i=1}^{n}\left\|f_{m(\varepsilon,\eta_i)}(\eta_i)\odot(t_i-t_{i-1}) - \mathrm{HK}^Q\int\limits_{t_{i-1}}^{t_i} f_{m(\varepsilon,\eta_i)}(t)\Delta t\right\| +$$

$$+\left\|\sum_{i=1}^{n}\mathrm{HK}^Q\int\limits_{t_{i-1}}^{t_i} f_{m(\varepsilon,\eta_i)}(t)\Delta t - J\right\| <$$

$$< \frac{\varepsilon}{3(b-a)}(b-a) + \frac{\varepsilon}{3}\sum_{i=1}^{\infty}\frac{1}{2^i} +$$

$$< \left\|\sum_{i=1}^{n}\mathrm{HK}^Q\int\limits_{t_{i-1}}^{t_i} f_{m(\varepsilon,\eta_i)}(t)\Delta t - J\right\| =$$

$$= \frac{2\varepsilon}{3} + \left\|\sum_{i=1}^{n}\mathrm{HK}^Q\int\limits_{t_{i-1}}^{t_i} f_{m(\varepsilon,\eta_i)}(t)\Delta t - J\right\|.$$

To get our desired result, we need to show that the last term above is less than $\varepsilon/3$. In fact, from Theorem 4.2, we obtain

$$\mathrm{HK}^Q\int\limits_{a}^{b} f_{m(\varepsilon,\eta_i)}(t)\Delta t = \sum_{i=1}^{n}\mathrm{HK}^Q\int\limits_{t_{i-1}}^{t_i} f_{m(\varepsilon,\eta_i)}(t)\Delta t,$$

and since f_k is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$, therefore

$$\left\|\mathrm{HK}^Q\int\limits_{a}^{b} f_{m(\varepsilon,\eta_i)}\Delta t - J\right\| < \frac{\varepsilon}{3},$$

then the result follows.

Theorem 4.9 is proved.

Lemma 4.4. *Let* $f_1 = f_1^0 + if_1^1 + jf_1^2 + kf_1^3$ *and* $f_2 = f_2^0 + if_2^1 + jf_2^2 + kf_2^3$ *be HK*^{Q}*-*Δ*-integrable on* $[a,b]_{\mathbb{T}}$, *and if* $p^l(t) \le f_1^l(t) \le h^l(t)$, $p^l(t) \le f_2^l(t) \le h^l(t)$, $l = 0,1,2,3$, Δ*-almost everywhere, where* p *and* h *are also HK*^{Q}*-*Δ*-integrable on* $[a,b]_{\mathbb{T}}$, *then* $\max\{f_1,f_2\}$ *and* $\min\{f_1,f_2\}$ *are both HK*^{Q}*-*Δ*-integrable on* $[a,b]_{\mathbb{T}}$, *where*

$$\max\{f_1,f_2\} = \max\{f_1^0,f_2^0\} + i\max\{f_1^1,f_2^1\} + j\max\{f_1^2,f_2^2\} + k\max\{f_1^3,f_2^3\},$$

$$\min\{f_1,f_2\} = \min\{f_1^0,f_2^0\} + i\min\{f_1^1,f_2^1\} + j\min\{f_1^2,f_2^2\} + k\min\{f_1^3,f_2^3\}.$$

Proof. Without loss of generality, suppose $g(t) = 0$ for $t \in [a,b]_{\mathbb{T}}$. Let $F_i(u,v)$ be the HK^Q-Δ-integral of f_i, $i = 1,2$, on $[u,v]_{\mathbb{T}} \subset [a,b]_{\mathbb{T}}$, and let

$$F^*(u,v) = \max\{F_1(u,v), F_2(u,v)\}.$$

Note that F^* is not additive, i.e., if $x < y < z$ we have

$$F^*(x,z) \le F^*(x,y) + F^*(y,z).$$

Choose any partition $\mathcal{P}_1 = \{a = t_0 < t_1 < \ldots < t_n = b\}$ of $[a,b]_{\mathbb{T}}$ and we have

$$\left\|\sum_{k=1}^{n} F^*(t_{k-1},t_k)\right\| \le \left\|\mathrm{HK}^Q\int_a^b h(t)\Delta t\right\|.$$

Let $A = \sum_{k=1}^{n} F^*(t_{k-1},t_k)$ be the largest integral over such all $\sum_{k=1}^{n} F^*(t_{k-1},t_k)$. We shall show that A is the HK^Q-Δ-integral of $\max\{f_1,f_2\}$ on $[a,b]_{\mathbb{T}}$.

According to Theorem 4.8, for given $\varepsilon > 0$, there is a Δ-gauge θ for $[a,b]_{\mathbb{T}}$ such that for any θ-fine partition $\mathcal{P}$ of $[a,b]_{\mathbb{T}}$, we have

$$\sum_{k=1}^{n}\|f_i(\eta_k)\odot(t_k - t_{k-1}) - F_i(t_{k-1},t_k)\| < \varepsilon, \quad i = 1,2.$$

Now let

$$x_i(a,b) = \sup\sum_{k=1}^{n}\big\|f_i(\eta_k)\odot(t_k - t_{k-1}) - F_i(t_{k-1},t_k)\big\|, \quad i = 1,2,$$

where the supremum is over all θ-fine partitions $\mathcal{P}$ of $[a,b]_{\mathbb{T}}$. Note that

$$x_i(a,c) + x_i(c,b) \le x_i(a,b) \quad \text{for} \quad a < c < b \quad \text{and} \quad x_i(a,b) \le \varepsilon,$$

for any θ-fine partition $\mathcal{P}$ of $[a,b]_{\mathbb{T}}$ we have

$$\big\|f_i(\eta_k)\odot(t_k - t_{k-1})\big\| \le \big\|F^*(t_{k-1},t_k)\big\| + x_1(t_{k-1},t_k) + x_2(t_{k-1},t_k), \quad i = 1,2.$$

Thus, writing $f = \max\{f_1,f_2\}$ we have

$$\big\|f_i(\eta_k)\odot(t_k - t_{k-1})\big\| \le \big\|F^*(t_{k-1},t_k)\big\| + x_1(t_{k-1},t_k) + x_2(t_{k-1},t_k). \tag{4.12}$$

Similarly, we also have

$$\big\|F^*(t_{k-1},t_k)\big\| - x_1(t_{k-1},t_k) - x_2(t_{k-1},t_k) \le \big\|f_i(\eta_k)\odot(t_k - t_{k-1})\big\|. \tag{4.13}$$

Combining (4.12) and (4.13) we obtain

$$\left\|\sum_{k=1}^{n}(f(\eta_k)\odot(t_k-t_{k-1})-F^*(t_{k-1},t_k))\right\|<2\varepsilon.$$

Finally, fix a partition $\mathcal{P}'$ such that its corresponding sum

$$\left\|\sum_{k=1}^{n}F^*\left(t'_{i-1},t'_i\right)-A\right\|<\varepsilon.$$

Adjust θ in such a way that if $\mathcal{P}$ is θ-fine then it is finer than $\mathcal{P}'$, i.e., any subinterval of $\mathcal{P}$ is included in some subinterval of $\mathcal{P}'$. For any adjusted θ-fine partition $\mathcal{P}$ we have

$$\left\|A-\sum_{k=1}^{n}F^*(t_{k-1},t_k)\right\|\le\left\|A-\sum_{k=1}^{n}F^*\left(t'_{k-1},t'_k\right)\right\|<\varepsilon.$$

Applying the above inequalities we obtain

$$\left\|\sum_{k=1}^{n}f(\eta_k)\odot(t_k-t_{k-1})-A\right\|<3\varepsilon.$$

Hence we have proved the first part for f_1 being a zero function. For $\min\{f_1,f_2\}$, we can use the fact that $\min\{f_1,f_2\}=-\max\{-f_1,-f_2\}$. Then, the result follows from Theorem 4.1 directly.

Lemma 4.4 is proved.

Theorem 4.10. *Let* $f:[a,b]_{\mathbb{T}}\to\mathbb{H}$, *assume that:*

(i) $\|f_n-f\|\to0$ Δ*-a.e. on* $[a,b]_{\mathbb{T}}$;

(ii) $\left\|g^l\right\|\le\left\|f_n^l\right\|\le\left\|h^l\right\|$, $l=0,1,2,3,$ Δ*-a.e. on* $[a,b]_{\mathbb{T}}$;

(iii) f_n, g, h *are HK*^{Q}*-*Δ*-integrable on* $[a,b]_{\mathbb{T}}$.

Then f *is HK*^{Q}*-*Δ*-integrable on* $[a,b]_{\mathbb{T}}$ *and*

$$\lim_{n\to\infty}\mathit{HK}^Q\int_a^b f_n(t)\Delta t=\mathit{HK}^Q\int_a^b f(t)\Delta t.$$

Proof. By Lemma 4.4, the function $\min\{f_n:i\le n\le j\}$ is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$; denote it by f_j^* for $j=i,i+1,i+2,\ldots$. Then the sequence $-\|f_i^*\|$, $-\left\|f_{i+1}^*\right\|,\ldots$ are monotone increasing and they are bounded above. By Theorem 4.9, the limit function $\inf\{f_n:n\ge i\}$ is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$.

Similarly, we can show that $\sup\{f_n:n\ge i\}$ is also HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$. Then we have

$$\left\|\mathrm{HK}^Q\int_a^b\left(\inf_{n\ge i}f_n(t)\right)\Delta t\right\|\le\left\|\inf_{n\ge i}\mathrm{HK}^Q\int_a^b f_n(t)\Delta t\right\|\le$$

$$\le\left\|\sup_{n\ge i}\mathrm{HK}^Q\int_a^b f_n(t)\Delta t\right\|\le\left\|\mathrm{HK}^Q\int_a^b\left(\sup_{n\ge i}f_n(t)\right)\Delta t\right\|.$$

It is well-known that $f_n(x) \to f(x)$ as $n \to \infty$ if and only if

$$\lim_{i\to\infty}\left(\inf_{n\geq i} f_n(t)\right) = f(t) = \lim_{i\to\infty}\left(\sup_{n\geq i} f_n(t)\right).$$

Apply Theorem 4.9 again to sequence $\inf\{f : n > i\}$ for $i = 1, 2, \ldots$ and we obtain that f is HK^Q-Δ-integrable on $[a,b]_{\mathbb{T}}$. Consequently,

$$\left\|\mathrm{HK}^Q\int_a^b f(t)\Delta t\right\| \leq \lim_{i\to\infty}\left\|\inf_{n\geq i}\mathrm{HK}^Q\int_a^b f_n(t)\Delta t\right\| \leq$$

$$\leq \lim_{i\to\infty}\left\|\sup_{n\geq i}\mathrm{HK}^Q\int_a^b f_n(t)\Delta t\right\| \leq \left\|\mathrm{HK}^Q\int_a^b f(t)\Delta t\right\|,$$

and the result follows.

Lemma 4.1 is proved.

References

1. S. Hilger, *Analysis on measure chains — a unified approach to continuous and discrete calculus*, Results Math., **18**, 18 – 56 (1990).
2. R. P. Agarwal, M. Bohner, *Basic calculus on time scales and some of its applications*, Results Math., **35**, 3 – 22 (1999).
3. M. Bohner, A. Peterson, *Dynamic equations on time scales: an introduction with application*, Birkhäuser, Boston (2001).
4. M. Bohner, A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser, Boston (2004).
5. L. Erbe, A. Peterson, *Boundedness and oscillation for nonlinear dynamic equations on a time scale*, Proc. Amer. Math. Soc., **132**, 735 – 744 (2004).
6. G. Sh. Guseinov, B. Kaymakçalan, *Basics of Riemann delta and nabla integration on time scales*, J. Difference Equ. Appl., **8**, 1001 – 1017 (2002).
7. G. Qin, C. Wang, *Lebesgue – Stieltjes combined $\diamond_\alpha$-measure and integral on time scales*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115**, No. 2, Paper No. 50, 24 pp. (2021).
8. C. Wang, R. P. Agarwal, D. O'Regan, *Matrix measure on time scales and almost periodic analysis of the impulsive Lasota – Wazewska model with patch structure and forced perturbations*, Math. Methods Appl. Sci., **39**, 5651 – 5669 (2016).
9. C. Wang, G. Qin, R. P. Agarwal, D. O'Regan, *$\diamond_\alpha$-Measurability and combined measure theory on time scales*, Appl. Anal., **101**, № 8, 2755–2796 (2022); https://doi.org/10.1080/00036811.2020.1820997.
10. C. Wang, R. P. Agarwal, D. O'Regan, *Calculus of fuzzy vector-valued functions and almost periodic fuzzy vector-valued functions on time scales*, Fuzzy Sets and Systems, **375**, 1 – 52 (2019).
11. C. Wang, R. P. Agarwal, D. O'Regan, *Almost periodic fuzzy multidimensional dynamic systems and applications on time scales*, Chaos Solitons Fractals, **156**, 111 – 781 (2022).
12. Z. Li, C. Wang, R. P. Agarwal, *The non-eigenvalue form of Liouville's formula and α-matrix exponential solutions for combined matrix dynamic equations on time scales*, Mathematics, **7**(10), 962 (2019); doi.org/10.3390/math7100962.
13. C. Wang, R. P. Agarwal, *A survey of function analysis and applied dynamic equations on hybrid time scales*, Entropy, **23**, 450; https://doi.org/10.3390/e23040450 (2021).
14. C. Wang, R. P. Agarwal, D. O'Regan, *Compactness criteria and new impulsive functional dynamic equations on time scales*, Adv. Difference Equ., **197**, 1 – 41 (2016).

15. C. Wang, R. P. Agarwal, D. O'Regan, *Local-periodic solutions for functional dynamic equations with infinite delay on changing-periodic time scales*, Math. Slovaca, **68**, 1397 – 1420 (2018).

16. C. Wang, R. P. Agarwal, *A classification of time scales and analysis of the general delays on time scales with applications*, Math. Methods Appl. Sci., **39**, 1568 – 1590 (2016).

17. C. Wang, R. P. Agarwal, *Almost periodic solution for a new type of neutral impulsive stochastic Lasota – Wazewska time scale model*, Appl. Math. Lett., **70**, 58 – 65 (2017).

18. C. Wang, *Almost periodic solutions of impulsive BAM neural networks with variable delays on time scales*, Commun. Nonlinear Sci. Numer. Simul., **19**, 2828 – 2842 (2014).

19. C. Wang, R. P. Agarwal, *Almost periodic dynamics for impulsive delay neural networks of a general type on almost periodic time scales*, Commun. Nonlinear Sci. Numer. Simul., **36**, 238 – 251 (2016).

20. C. Wang, *Piecewise pseudo almost periodic solution for impulsive non-autonomous highorder Hopfield neural networks with variable delays*, Neurocomputing, **171**, 1291 – 1301 (2016).

21. C. Wang, R. Sakthivel, *Double almost periodicity for high-order Hopfield neural networks with slight vibration in time variables*, Neurocomputing, **282**, 1 – 15 (2018).

22. C. Wang, *Existence and exponential stability of piecewise mean-square almost periodic solutions for impulsive stochastic Nicholson's blowflies model on time scales*, Appl. Math. Comput., **248**, 101 – 112 (2014).

23. C. Wang, R. P. Agarwal, R. Sakthivel, *Almost periodic oscillations for delay impulsive stochastic Nicholson's blowflies time scale model*, Comput. Appl. Math., **37**, 3005 – 3026 (2018).

24. C. Wang, R. P. Agarwal, D. O'Regan, R. Sakthivel, *Theory of translation closeness for time scales. With applications in translation functions and dynamic equations*, Springer, Cham (2020).

25. C. Wang, R. P. Agarwal, D. O'Regan, *A matched space for time scales and applications to the study on functions*, Adv. Difference Equ., **2017**, № 305, 28 pp. (2017).

26. C. Wang, R. P. Agarwal, D. O'Regan, G. M. N'Guérékata, *Complete-closed time scales under shifts and related functions*, Adv. Difference Equ., **429**, 19 pp. (2018).

27. C. Wang, R. P. Agarwal, D. O' Regan, *Periodicity, almost periodicity for time scales and related functions*, Nonauton. Dyn. Syst., **3**, 24 – 41 (2016).

28. C. Wang, R. P. Agarwal, D. O'Regan, R. Sakthivel, *A computation method of Hausdorff distance for translation time scales*, Appl. Anal., **99**, № 7, 1218 – 1247 (2020).

29. C. Wang, R. P. Agarwal, D. O'Regan, *Weighted piecewise pseudo double-almost periodic solution for impulsive evolution equations*, J. Nonlinear Sci. Appl., **10**, 3863 – 3886 (2017).

30. C. Wang, J. Wang, R. P. Agarwal, Z. Li, *Almost anti-periodic discrete oscillation of general* n*-dimensional mechanical system and underactuated Euler – Lagrange system*, Appl. Sci., **12**, № 4 (2022); https://doi.org/10.3390/app12041991.

31. C. Wang, R. P. Agarwal, D. O'Regan, n_0*-order* Δ*-almost periodic functions and dynamic equations*, Appl. Anal., **97**, 2626 – 2654 (2018).

32. C. Wang, R. P. Agarwal, D. O'Regan, δ*-almost periodic functions and applications to dynamic equations*, Mathematics, **7**, № 6, 525 (2019); https://doi.org/10.3390/math7060525.

33. C. Wang, R. P. Agarwal, *Almost automorphic functions on semigroups induced by complete-closed time scales and application to dynamic equations*, Discrete Contin. Dyn. Syst. Ser. B, **25**, 781 – 798 (2020).

34. C. Wang, R. P. Agarwal, *Weighted piecewise pseudo almost automorphic functions with applications to abstract impulsive* ∇*-dynamic equations on time scales*, Adv. Difference Equ., **153**, 1 – 29 (2014).

35. C. Wang, R. P. Agarwal, D. O'Regan, R. Sakthivel, *Local pseudo almost automorphic functions with applications to semilinear dynamic equations on changing-periodic time scales*, Bound. Value Probl., **133**, doi:10.1186/s13661-019-1247-4 (2019).

36. C. Wang, R. P. Agarwal, *Changing-periodic time scales and decomposition theorems of time scales with applications to functions with local almost periodicity and automorphy*, Adv. Difference Equ., **296**, 1 – 21 (2015).

37. C. Wang, R. P. Agarwal, D. O'Regan, Π*-semigroup for invariant under translations time scales and abstract weighted pseudo almost periodic functions with applications*, Dynam. Systems Appl., **25**, 1 – 28 (2016).

38. C. Wang, R. P. Agarwal, *Relatively dense sets, corrected uniformly almost periodic functions on time scales, and generalizations*, Adv. Difference Equ., **312**, 1 – 9 (2015).

39. C. Wang, G. N'Guérékata, *C_0-semigroup and Stepanov-like almost automorphic functions in matched spaces of time scales*, Studies in Evolution Equations and Related Topics, Springer, Cham (2021).
40. C. Wang, R. P. Agarwal, D. O'Regan, *Weighted pseudo d-almost automorphic functions and abstract dynamic equations*, Georgian Math. J., **28**, № 2, 313 – 330 (2021); https://doi.org/10.1515/gmj-2019-2066 (2021).
41. C. Wang, R. P. Agarwal, D. O'Regan, G. M. N'Guérékata, *n_0-order weighted pseudo Δ-almost automorphic functions and abstract dynamic equations*, Mathematics, **7**, № 9, 775 (2019); doi.org/10.3390/math7090775.
42. C. Wang, R. P. Agarwal, *A further study of almost periodic time scales with some notes and applications*, Abstr. Appl. Anal., **2014**, Article ID 267384, 1 – 11 (2014).
43. C. Wang, R. P. Agarwal, D. O'Regan, R. Sakthivel, *Discontinuous generalized doublealmost-periodic functions on almost-complete-closed time scales*, Bound Value Probl., **165**, Paper No. 165, 22 pp. (2019); doi.org/10.1186/s13661-019-1283-0.
44. C. Wang, R. P. Agarwal, *Uniformly rd-piecewise almost periodic functions with applications to the analysis of impulsive Δ-dynamic system on time scales*, Appl. Math. Comput., **259**, 271 – 292 (2015).
45. C. Wang, R. P. Agarwal, *Exponential dichotomies of impulsive dynamic systems with applications on time scales*, Math. Methods Appl. Sci., **38**, 3879 – 3900 (2015).
46. S. W. R. Hamilton, *Lectures on quaternions, Royal Irish academy*, Hodges and Smith, Dublin (1853).
47. S. L. Adler, *Quaternionic quantum mechanics and quantum fields*, The Clarendon Press, Oxford Univ. Press, New York (1995).
48. L. Rodman, *Topic in quaternion linear algebra*, Princeton Univ. Press, Princeton, NJ (2014).
49. S. G. Georgiev, R. Jday, *Brownian motion in the framework of quaternion analysis*, Stoch. Anal. Appl., **38**, 601 – 622 (2020).
50. Z. Li, C. Wang, R. P. Agarwal, R. Sakthivel, *Hyers – Ulam – Rassias stability of quaternion multidimensional fuzzy nonlinear difference equations with impulses*, Iran. J. Fuzzy Syst., **18**, 143 – 160 (2021).
51. C. Wang, D. Chen, Z. Li, *General theory of the higher-order quaternion linear difference equations via the complex adjoint matrix and the quaternion characteristic polynomial*, J. Difference Equ. Appl., **27**, 787 – 857 (2021).
52. Z. Li, C. Wang, *Cauchy matrix and Liouville formula of quaternion impulsive dynamic equations on time scales*, Open Math., **18**, 353 – 377 (2020).
53. C. Wang, Z. Li, R. P. Agarwal, *Fundamental solution matrix and Cauchy properties of quaternion combined impulsive matrix dynamic equation on time scales*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **29**, № 2, 107–130 (2021).
54. Z. Li, C. Wang, R. P. Agarwal, D. O'Regan, *Commutativity of quaternion-matrix-valued functions and quaternion matrix dynamic equations on time scales*, Stud. Appl. Math., **146**, 139 – 210 (2021).
55. C. Wang, Z. Li, R. P. Agarwal, *Hyers – Ulam – Rassias stability of high-dimensional quaternion impulsive fuzzy dynamic equations on time scales*, Discrete Contin. Dyn. Syst. Ser. S, **15**, 359 – 386 (2022).
56. C. Wang, Z. Li, R. P. Agarwal, *A new quaternion hyper-complex space with hyper argument and basic functions via quaternion dynamic equations*, J. Geom. Anal., **32**, № 2, Paper № 67, 83 pp. (2022).
57. L. Peng-Yee, *Lanzhou lectures on Henstock integration, Series in Real Analysis, 2*, World Sci. Publ. Co., Inc., Teaneck, NJ, **2** (1989).
58. A. Peterson, B. Thompson, *Henstock – Kurzweil delta and nabla integrals*, J. Math. Anal. Appl., **323**, № 1, 162 – 178 (2006).
59. G. J. Ye, *On Henstock – Kurzweil and McShane integrals of Banach space-valued functions*, J. Math. Anal. Appl., **330**, № 2, 753 – 765 (2007).
60. W. Kelley, A. Peterson, *Difference equations: an introduction with applications, second ed.*, Harcourt/Acad. Press, San Diego, CA (2001).

Received 29.11.22