УДК 539.3

GREEN'S FUNCTION FOR AN ELASTIC LAYER WITH TEMPERATURE DEPENDENT PROPERTIES

S. J. MATYSIAK [1], *D. M. PERKOWSKI* [2]

[1] *University of Warsaw, Poland;*

[2] *Białystok University of Technology, Poland*

The distributions of stresses and displacements in a thermoelastic layer with temperature dependent properties are investigated. The problem is considered for the case of antiplane state of strains. The boundary planes are assumed to be kept at constant temperatures. The upper boundary plane is free of loading, and the lower plane is loaded by a concentrated force. The solution is found in the form of integrals and the singularities of stresses are determined.

Keywords: *temperature, displacements, stresses, elasticity, temperature dependent properties, concentrated load.*

The study of the behavior of stresses in elastic materials with temperature dependent properties is of importance in many engineering applications. Some elastic materials change their mechanical modulus under temperature effect. In these cases the application of Hookean strain-stress relations is not appropriate to describe stress distributions. The theory of termoelasticity of materials with temperature-dependent properties seems to be the most adjusted for modelling of the interaction between mechanical and thermal fields. One of the first investigators, who considerably developed the theoretical basis of elastic bodies with temperature dependent modulus was J. L. Nowiński [1–3], and monograph [4]. Many experimental results for determination of mechanical properties of solids as functions of temperature are presented in monograph [5] (mainly for metals), as well as in papers [6–11]. Some theoretical investigations of solid mechanics with temperature-dependent properties are given in [12–15].

In this paper the antiplane state of strain of an elastic layer with temperature dependent properties is considered. The boundary planes are assumed to be kept at given constant temperatures, what leads to the linear temperature distribution in the considered layer. The lower boundary plane is loaded by a concentrated force; the upper boundary plane is free of loading. The shear modulus μ as a function of temperature θ is taken into account in the form of linear function. The assumption connected with the temperature dependence of shear modulus leads to the problem of FGM layer with material properties continuously dependent on space variables. It can be observed that in the case of classical thermoelasticity for homogeneous, isotropic bodies in the antiplane state of strain the distributions of stresses are independent of temperature, what is different from the considered problem.

Formulation and solution of the problem. Consider an isotropic elastic layer with thickness h. Let (x_1, x_2, x_3) be the Cartesian coordinate system such that the planes $x_2 = 0$ and $x_2 = h$ are boundaries of the body, and the axis $0x_3$ is perpendicular to the boundaries. Let the lower and upper boundary planes are kept at given constant temperatures θ_0 and θ_1, respectively.

Corresponding author: S. J. MATYSIAK, e-mail*:* s.j.matysiak@uw.edu.pl

Moreover, the considered layer is loaded by a linearly distributed along the axis $0x_3$ and concentrated force with intensity P acting in $0x_3$ direction. The shear modulus μ is assumed to be function of temperature θ as follows

$$\mu(\theta) = \mu_0(1 - A\theta) , \tag{1}$$

where μ_0, A are constant. The form of shear modulus dependence (1) agrees with the experimental results presented in [5].

The assumptions taken into account lead to the antiplane state of strain described by the displacement vector $\mathbf{u}(x_1, x_2) = (0, 0, u_3(x_1, x_2))$ and the considered problem is stationary and independent of x_3. The temperature $\theta = \theta(x_1, x_2)$ satisfies the following equation

$$\frac{\partial^2\theta}{\partial x_1^2} + \frac{\partial^2\theta}{\partial x_2^2} = 0, \qquad x_1 \in R, \qquad x_2 \in (0,\ h) ,$$

and boundary conditions

$$\theta(x_1, 0) = \theta_0 , \quad \theta(x_1, h) = \theta_1 , \quad x_1 \in R ,$$

causing the distribution of temperature

$$\theta(x_1, x_2) = (\theta_1 - \theta_0)x_2 / h + \theta_0 , \quad x_1 \in R, \ \ x_2 \in <0, h> . \tag{2}$$

From equations (1) and (2) it follows that

$$\mu(x_1, x_2) = \mu_0(\alpha_0 + \alpha_1 x_2), \quad \alpha_0 = 1 - A\theta_0 \ , \quad \alpha_1 = -A(\theta_1 - \theta_0)/h . \tag{3}$$

The state of stresses is described by nonzero components σ_{13} and σ_{23} in the form

$$\sigma_{13}(x_1, x_2) = \mu_0(\alpha_0 + \alpha_1 x_2)\partial u_3 / \partial x_1, \qquad \sigma_{23}(x_1, x_2) = \mu_0(\alpha_0 + \alpha_1 x_2)\partial u_3 / \partial x_2 . \tag{4}$$

The equilibrium equation in the case of stresses given by (4) can be written as

$$\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\alpha_1}{\alpha_0 + \alpha_1 x_2}\frac{\partial u_3}{\partial x_2} = 0, \quad x_1 \in R, \quad x_2 \in (0, h) . \tag{5}$$

The boundary conditions

$$\sigma_{23}(x_1, 0) = P\delta(x_1), \qquad \sigma_{23}(x_1, h) = 0, \qquad x_1 \in R , \tag{6}$$

where $\delta(\cdot)$ is the Dirac delta function. By using integral Fourier transform [16] with respect to variable x_1 and denoting by

$$\tilde{u}_3(s, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_3(x_1, x_2)\, e^{-isx_1} dx_1 \ ,$$

from equation (5) it follows that

$$\frac{d^2\tilde{u}_3(s, x_2)}{dx_2^2} + \frac{\alpha_1}{\alpha_0 + \alpha_1 x_2}\frac{d\,\tilde{u}_3(s, x_2)}{dx_2} - s^2\tilde{u}_3(s, x_2) = 0 . \tag{7}$$

The linear, ordinary differential second-order equation (7) is reduced to the form

$$\frac{d^2\tilde{u}_3}{d\omega^2} + \frac{1}{\omega}\frac{d\,\tilde{u}_3}{d\omega} - \frac{s^2}{\alpha_1^2}\tilde{u}_3 = 0 \ , \tag{8}$$

where

$$\omega = \alpha_0 + \alpha_1 x_2 . \tag{9}$$

The general solution of equation (8) can be written as [17]:

$$\tilde{u}_3(s,\omega) = C_1 J_0(i\omega s/\alpha_1) + C_2 Y_0(i\omega s/\alpha_1), \tag{10}$$

where C_1, C_2 are constants; $J_0(x)$, $Y_0(x)$ are Bessel functions of the first and the second kind. From relations [8]:

$$J_0(i\,x) = I_0(x), \quad Y_0(i\,x) = K_0(x), \quad x \in R,$$

where $I_0(\cdot)$, $K_0(\cdot)$ are modified Bessel functions, and using (9), (10), the general solution of equation (7) can be written as follows

$$\tilde{u}_3(s,x_2) = C_1 I_0\left[\frac{s\omega}{\alpha_1}\right] + C_2 K_0\left[\frac{s\omega}{\alpha_1}\right]. \tag{11}$$

The constants C_1, C_2 will be determined from boundary conditions (6). From equations $(4)_1$, (6), (11) and using relations [19]:

$$\frac{d}{dz} I_0(z) = I_1(z), \quad \frac{d}{dz} K_0(z) = -K_1(z), \tag{12}$$

where $I_1(x)$, $K_1(x)$ are modified Bessel functions, it follows that a_1, a_2 should satisfied the following system of algebraic equations

$$a_1 I_1\left[\frac{s\omega^*}{\alpha_1}\right] - a_2 K_1\left[\frac{s\omega^*}{\alpha_1}\right] = 0, a_1 I_1\left(\frac{s\,\alpha_0}{\alpha_1}\right) - a_2 K_1\left(\frac{s\,\alpha_0}{\alpha_1}\right) = \frac{P}{\sqrt{2\pi}\,s\mu_0\alpha_0}, \tag{13}$$

where $\omega^* = \alpha_0 + \alpha_1 h$.

From equations (13) and (11), and applying inverse Fourier transform relation [16], the displacement u_3 can be written in the form

$$u_3(x_1, x_2) = -\frac{P}{\pi\alpha_0\,\mu_0}\int_0^\infty \frac{1}{sW}\left\{K_1\left[\frac{s\omega^*}{\alpha_1}\right] I_0\left[\frac{s\omega}{\alpha_1}\right] + I_1\left[\frac{s\omega^*}{\alpha_1}\right] K_0\left[\frac{s\omega}{\alpha_1}\right]\right\}\cos(s\,x_1)\,ds, \tag{14}$$

where

$$W = I_1\left[\frac{s\omega^*}{\alpha_1}\right] K_1\left(\frac{s}{\alpha_1}\alpha_0\right) - K_1\left[\frac{s\omega^*}{\alpha_1}\right] I_1\left(\frac{s\,\alpha_0}{\alpha_1}\right).$$

The stress components σ_{13}, σ_{23} will be calculated from equations (4) and (14).

Substituting (14) into (4) it follows that

$$\sigma_{13}(x_1, x_2) = \frac{P\omega}{\pi\alpha_0}\int_0^\infty \frac{1}{W}\left\{K_1\left[\frac{s\omega^*}{\alpha_1}\right] I_0\left[\frac{s\omega}{\alpha_1}\right] + I_1\left[\frac{s\omega^*}{\alpha_1}\right] K_0\left[\frac{s\omega}{\alpha_1}\right]\right\}\sin(s\,x_1)\,ds, \tag{15}$$

$$\sigma_{23}(x_1, x_2) = \frac{P\omega}{\pi\alpha_0}\int_0^\infty \frac{1}{W}\left\{-K_1\left[\frac{s\omega^*}{\alpha_1}\right] I_1\left[\frac{s\omega}{\alpha_1}\right] + I_1\left[\frac{s\omega^*}{\alpha_1}\right] K_1\left[\frac{s\omega}{\alpha_1}\right]\right\}\cos(s\,x_1)\,ds. \tag{16}$$

Equations (14)–(16) are the fundamental solution (Green's function) to the considered problem in the integral form.

From the view point of mechanics, the singularities of stresses at the point of concentrated force acting should be investigated. For this purpose an asymptotic behavior of integrand functions in (15) and (16) will be analyzed.

Stress singularities. By using relations [19]

$$I_\nu(x) \underset{x\to\infty}{\approx} \frac{e^x}{\sqrt{2\pi x}}, \quad K_\nu(x) \underset{x\to\infty}{\approx} \sqrt{\frac{\pi}{2x}}e^{-x}, \quad I_0(x) \underset{x\to 0}{\approx} 1,$$

$$I_1(x) \underset{x\to 0}{\approx} \frac{1}{2}x, \qquad K_1(x) \underset{x\to 0}{\approx} \frac{1}{x}, \qquad K_0(x) \underset{x\to 0}{\approx} \ln\frac{2}{x}, \tag{17}$$

from equation (14) it follows that

$$W \underset{s\to 0}{\approx} \frac{\alpha_1 h(2\alpha_0 + \alpha_1 h)}{2\alpha_0(\alpha_0 + \alpha_1 h)} = \text{const}, \quad W \underset{s\to\infty}{\approx} \frac{\alpha_1}{\sqrt{\alpha_0(\alpha_0 + \alpha_1 h)}} \frac{\sinh(sh)}{s}, \tag{18}$$

Denoting by

$$L_0 \equiv K_1\left[\frac{s\omega^*}{\alpha_1}\right] I_0\left[\frac{s\omega}{\alpha_1}\right] + I_1\left[\frac{s\omega^*}{\alpha_1}\right] K_0\left[\frac{s\omega}{\alpha_1}\right],$$

$$L_1 \equiv -K_1\left[\frac{s\omega^*}{\alpha_1}\right] I_1\left[\frac{s\omega}{\alpha_1}\right] + I_1\left[\frac{s\omega^*}{\alpha_1}\right] K_1\left[\frac{s\omega}{\alpha_1}\right], \tag{19}$$

and using (17) and (18) we obtain

$$\frac{L_0}{W} \underset{s\to\infty}{\approx} \sqrt{\frac{\alpha_0}{\omega}}\, e^{-sx_2}, \qquad \frac{L_1}{W} \underset{s\to\infty}{\approx} \sqrt{\frac{\alpha_0}{\omega}}\, e^{-sx_2}. \tag{20}$$

From equations (15), (16), (19) and (20) the singularities of stress components can be written in the form

$$\sigma_{13}(x_1, x_2) = \frac{P}{\pi}\sqrt{\frac{\omega}{\alpha_0}} \frac{x_1}{x_1^2 + x_2^2} + 0(1), \quad \sigma_{23}(x_1, x_2) = \frac{P}{\pi}\sqrt{\frac{\omega}{\alpha_0}} \frac{x_2}{x_1^2 + x_2^2} + 0(1)\ . \tag{21}$$

From equation (21) it follows that the order of singularities of stress components σ_{13}, σ_{23} is the same as in the elastic homogeneous and isotropic layer, however the difference is observed in the coefficients of singularities.

The integrals represented the stresses σ_{13} and σ_{23} given by equations (15) and (16) and will be calculated numerically. For this purpose the following dimensionless variables will be used

$$\breve{x}_1 = x_1/h, \quad \breve{x}_2 = x_2/h, \quad \breve{s} = sh,$$

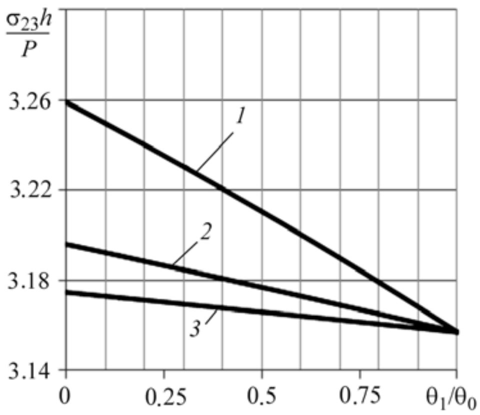

Fig. 1. The dimensionless stresses $\sigma_{23}(\breve{x}_1, \breve{x}_2)h/P$ as a function of parameter θ_1/θ_0 for $\theta_0 = 819$ K, $\breve{x}_1 = 0.0$, $\breve{x}_2 = 0.1$: *1* – A = 0.00051 K^{-1}; *2* – 0.00025 K^{-1}; *3* – 0.000125 K^{-1}.

The physical data taken into account are the same as in [20], where copper material has been considered. Fig. 1 allows to observe the influence of parameter A and differences between the boundary temperatures σ_0 and θ_1 on the stresses σ_{23}. The dimensionless stresses σ_{23} at point $\breve{x}_1 = 0.0$, $\breve{x}_2 = 0.1$ as a function of ratio θ_1/θ_0 is presented in Fig. 1 for three cases of parameter A. It can be observed that the component of stresses is dependent linearly on the ratio θ_1/θ_0 and for θ_1/θ_0 the solutions are reduced to the case of a homogeneous body with constant material properties.

In Fig. 2 the cases A = 0 are adequate for the homogeneous elastic body. Small changes of the stresses σ_{23} with respect to the boundary temperatures near the boundary plane loaded by a concentrated force can be observed for thee cases of A: A = = 0.00051 K^{-1}; A = 0.00025 K^{-1}; A = 0.

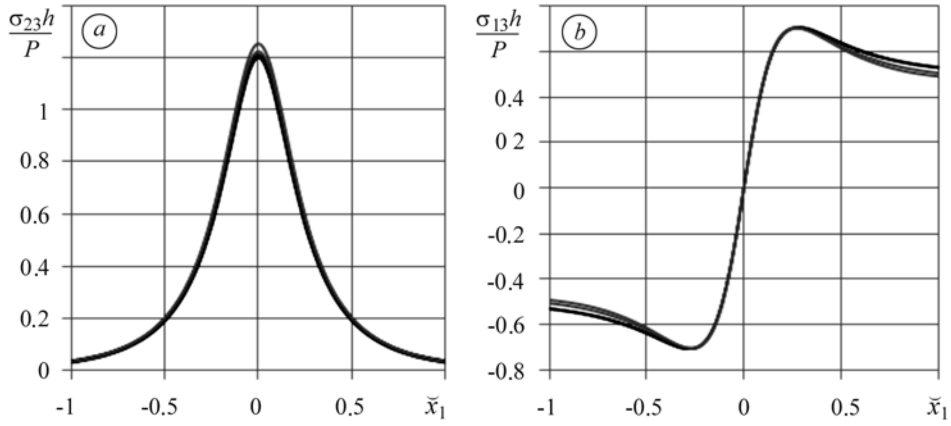

Fig. 2. The dimensionless stresses as a function of parameter A:
a – $\sigma_{23}h/P$; b – $\sigma_{13}h/P$: θ_0 = 819 K, $\theta_1 = 0.5\ \theta_0$, $\breve{x}_2 = 0.25$.

The stresses σ_{13} change the sign at $\breve{x}_1 = 0$ (the curve represented σ_{13} is antisymmetric, the curve represented σ_{23} is symmetric). The maximal values of σ_{23} are achieved at a point of the concentrated force action.

CONCLUSIONS

The problem of stress distribution in the thermoelastic layer with temperature dependent properties loaded by a concentrated force on the boundary plane is solved under condition of antiplane state of strain. It was assumed that the shear modulus is dependent linearly on temperature. The obtained results for stresses at a point of concentrated force action are characterized by the singularity of order, which is the same as for the case of an isotropic homogeneous body with constant material properties. The difference between the singularities for two mentioned materials is the singularity coefficients. Moreover, it can be underlined that for the case of the classical elasticity (when the shear modulus is constant), the boundary temperatures do effect the stresses σ_{13} and σ_{23}. In the considered problem of the layer with temperature dependent properties, the temperature is coupled with the displacement u_3.

РЕЗЮМЕ: Проаналізовано розподіли переміщень і напружень в плоскопаралельному шарі за умов антиплоскої деформації за лінійної залежності модуля зсуву від температури. На поверхнях шару підтримуються сталі температури, верхня поверхня вільна, а до нижньої прикладено зосереджену силу. Розв'язок задачі отримано в квадратурах. Досліджено особливість напружень в місці дії зосередженої сили.

РЕЗЮМЕ: Проанализированы распределения перемещений и напряжений в плоскопаралельном слое в условиях антиплоской деформации при линейной зависимости модуля сдвига от температуры. На поверхностях слоя поддерживаются постоянные температуры, верхняя поверхность свободна, а к нижней приложена сосредоточенная сила. Решение задачи получено в квадратурах. Исследовано особенность напряжений в месте действия сосредоточенной силы.

1. *Nowiński J. L.* Thermoelastic problem for an isotropic sphere with temperature dependent properties // Z. Angew. Math. Phys. – 1959. – **10**. – P. 565–575.
2. *Nowiński J. L.* A Betti-Rayleigh theorem for elastic bodies exhibiting temperature dependent properties // Appl. Sci. Res. – 1960. – **A9**. – P. 429–436.

3. *Nowiński J. L.* Transient thermoelastic problems for an infinite medium with a spherical cavity exhibiting temperature dependent properties // Trans ASME. J. Appl. Mech. – 1962. – **29**. – P. 399–407.
4. *Nowiński J. L.* Theory of thermoelasticity with applications // Alphen aan den Rijn: Sijthoff and Noordhoff. – 1978. – 836 p.
5. *Schreiber E., Anderson O. L., and Soga N.* Elastic constants and their measurement. – Mc Graw-Hill. – 1973. – 194 p.
6. *Lee H. J. and Saravanos D. A.* The effect of temperature dependent material properties on the response of piezoelectric composite materials // J. Intell. Mater. Syst. and Struct. – 1998. – **9**. – P. 503–508.
7. *Tao L. N.* The heat conduction problem with temperature-dependent material properties // Int. J. Heat and Mass Transfer. – 1989. – **32**. – P. 487–491.
8. *Temperature* dependence of the mechanical properties of tetrahedrally coordinated amorphous carbon thin films / D. A. Czaplewski, J. P. Sullivan, T. A. Friedmann, and J. R. Wendt // Appl. Phys. Lett. – 2005. – **87**. – P. 161915–161918.
9. *Speriatu L.* Temperature dependent mechanical properties of composite materials and uncertainties in experimental measurements. – University of Florida, 2005. – 159 p.
10. *Emery A. F. and Fadale T. D.* Handling temperature dependent properties and boundary conditions in stochastic finite element analysis // Numerical Heat Transfer. – 1997. – **A31**, № 1. – P. 37–51.
11. *Identification* of temperature-dependent thermal properties of solid materials / A. R. Tillmann, V. L. Borges, G. Guimaraes, de Lima e Silva A. L. F. // J. Brazil Soc. of Mech. Sci. and Engng. – 2008. – **4**. – P. 269–278.
12. *Matysiak S. J.* Wave fronts in elastic media with temperature dependent properties // Appl. Sci. Res. – 1988. – **45**. – P. 97–106.
13. *Ezzat M., Zakaria M., and Abdel-Bary A.* Generalized thermoelasticity with temperature dependent modulus of elasticity under three theories // J. Appl. Math. and Comput. – 1986. – **14**, № 1–2. – P. 193–212.
14. *Hata T.* Thermoelastic problem for a Griffith crack in a plate with temperature-dependent properties under a linear temperature distribution // J. Therm. Stresses. – 1979. – **2**, № 3–4. – P. 353–366.
15. *Hata T.* Thermoelastic problem for a Griffith crack in a plate whose shear modulus is an exponential function of the temperature // Z. Angew. Math. Mech. – 1981. – **61**, № 2. – P. 81–87.
16. *Sneddon I. N.* Fourier transforms. – Mc Graw-Hill, 1951. – 542 p.
17. *Камке Э.* Справочник по обыкновенным дифференциальным уравнениям. – М.: Наука, 1971. – 576 с.
18. *Lebiediev N. N.* Funkcje specjalne i ich zastosowania. – Warszawa: PWN, 1957. – 303 s.
19. *Whittaker E. T. and Watson G. N.* A course of modern analysis. – Cambridge University Press., 1976. – 616 p.
20. *Mukhopadhyay S. and Kumar R.* Solution of a problem of generalized thermoelasticity of an annular cylinder with variable material properties by finite difference method // Computation. Methods in Sci. and Techn. – 2009. – **15**, № 2. – P. 169–176.

Received 17.09.2012