UDK 517.5

©2015. **V. Ryazanov**

ON THEORY OF MULTIVALENT SOLUTIONS FOR RIEMANN–HILBERT PROBLEM

It is proved the existence of multivalent solutions for the Riemann–Hilbert problem in the general settings of finitely connected domains bounded by mutually disjoint Jordan curves, measurable coefficients and measurable boundary data. The theorem is formulated in terms of harmonic measure and principal asymptotic values. It is also given the corresponding reinforced criterion for domains with rectifiable boundaries stated in terms of the natural parameter and nontangential limits. Furthemore, it is shown that the dimension of the spaces of these solutions is infinite.

Keywords: *Riemann–Hilbert problem, Jordan curves, harmonic measures, principal asymptotic values, nontangential limits.*

1. Introduction.

This note is a continuation of the paper [16] where the Riemann–Hilbert problem was resolved in these general settings for simply connected domains. At the present paper, on the basis of [16] and a theorem due to Poincare, see e.g. Section VI.1 in [6], it is given a resolution of the problem for finitely connected domains.

Recall that boundary value problems for analytic functions are due to the Riemann dissertation (1851), also to works of Hilbert (1904, 1912, 1924) and Poincaré (1910). The Riemann dissertation contained a general setting of a problem on finding analytic functions with a connection between their real and imaginary parts on the boundary.

The first concrete problem of such a type has been proposed by Hilbert (1904) and called by the Hilbert problem or the Riemann–Hilbert problem. That consists in finding an analytic function f inside of a domain bounded by a rectifiable Jordan curve C with the boundary condition

$$\lim_{z\to\zeta} \text{Re}\,\{\overline{\lambda(\zeta)}\cdot f(z)\} \;=\; \varphi(\zeta) \qquad\qquad \forall\,\zeta\in C \tag{1}$$

where it was assumed by him that the functions λ and φ are continuously differentiable with respect to the natural parameter s on C and, moreover, $|\lambda|\neq 0$ everywhere on C. Hence without loss of generality one can assume that $|\lambda|\equiv 1$ on C.

The first way for solving this problem based on the theory of singular integral equations was proposed by Hilbert (1904), see [7]. This attempt was not quite successful because of the theory of singular integral equations has been not yet enough developed at that time. However, just that way became the main approach in this research direction, see e.g. the monographs [3, 11] and [19]. In particular, the existence of solutions to this problem was in that way proved for Hölder continuous λ and φ, see e.g. [3]. But subsequent weakening conditions on λ and φ in this way led to strengthening conditions on the contour C, say to the Lyapunov curves or to the Radon condition of bounded rotation or even to smooth curves.

However, Hilbert (1905) has resolved his problem with the above settings to (1) in the second way based on the reduction it to solving the corresponding two Dirichlet problems, see e.g. [8]. It was recently shown in [16] that the latter approach makes possible to obtain perfectly general results in the problem for the arbitrary Jordan domains with coefficients λ and boundary data φ that are only measurable with respect to the harmonic measure.

The key was the following Gehring result on the Dirichlet problem for harmonic functions: if $\varphi : \mathbb{R} \to \mathbb{R}$ is 2π-periodic, measurable and finite a.e. with respect to the Lebesgue measure, then there is a harmonic function in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $u(z) \to \varphi(\vartheta)$ for a.e. ϑ as $z \to e^{i\vartheta}$ along any nontangential path, see [5], see also [17]. But the way of the reduction of the Riemann–Hilbert problem to the corresponding 2 Dirichlet problems was original in [16].

2. The case of circular domains.

Let us start from the simplest kind of multiply connected domains. Recall that a domain D in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called **circular** if its boundary consists of finite number of mutually disjoint circles and points. We call such a domain **nondegenerate** if its boundary consists only of circles.

Theorem 2.1 *Let $\mathbb{D}_*$ be a bounded nondegenerate circular domain and let $\lambda : \partial\mathbb{D}_* \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial\mathbb{D}_* \to \mathbb{R}$ be measurable functions. Then there exist multivalent analytic functions $f : \mathbb{D}_* \to \mathbb{C}$ such that*

$$\lim_{z\to\zeta} \ \mathrm{Re}\ \{\overline{\lambda(\zeta)} \cdot f(z)\} \ = \ \varphi(\zeta) \tag{2}$$

along any nontangential path to a.e. $\zeta \in \partial\mathbb{D}_*$.

Proof. Indeed, by the Poincare theorem, see e.g. Theorem VI.1 in [6], there is a locally conformal mapping g of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto $\mathbb{D}_*$. Let $h : \mathbb{D}_* \to \mathbb{D}$ be the corresponding multivalent analytic function that is inverse to g. $\mathbb{D}_*$ without a finite number of cuts is simply connected and hence h has there only single-valued branches that are extended to the boundary by the Caratheodory theorem.

By Section VI.2 in [6], $\partial\mathbb{D}$ without a countable set of its points consists of a countable collection of arcs every of which is a one-to-one image of a circle in $\partial\mathbb{D}_*$ without its one point under every extended branch of h. Note that by the reflection principle g is conformally extended into a neighborhood of every such arc and, thus, nontangential paths to its points go into nontangential paths to the corresponding points of circles in $\partial\mathbb{D}_*$ and inversely.

Setting $\Lambda = \lambda \circ g$ and $\Phi = \varphi \circ g$ with the extended g on the given arcs of $\partial\mathbb{D}$ we obtain measurable functions on $\partial\mathbb{D}$. Thus, by Theorem 2.1 in [16] there exist analytic functions $F : \mathbb{D} \to \mathbb{C}$ such that

$$\lim_{w\to\eta} \ \mathrm{Re}\ \{\overline{\Lambda(\eta)} \cdot F(w)\} \ = \ \Phi(\eta) \tag{3}$$

along any nontangential path to a.e. $\eta \in \partial\mathbb{D}$. By the above arguments, we see that $f = F \circ h$ are desired multivalent analytic solutions of (2). $\square$

In particular, choosing $\lambda \equiv 1$ in (2), we obtain the following statement.

Proposition 2.2. *Let $\mathbb{D}_*$ be a bounded nondegenerate circular domain and let $\varphi : \partial\mathbb{D}_* \to \mathbb{R}$ be a measurable function. Then there exist multivalent analytic functions $f : \mathbb{D}_* \to \mathbb{C}$ such that*

$$\lim_{z\to\zeta} \ \mathrm{Re}\ f(z) \ = \ \varphi(\zeta) \tag{4}$$

along any nontangential path to a.e. $\zeta \in \partial\mathbb{D}_*$.

3. The case of rectifiable Jordan curves.

To resolve the Riemann–Hilbert problem in the case of domains bounded by a finite number of rectifiable Jordan curves we should extend to this case the known results of Caratheodory (1912), Lindelöf (1917), F. and M. Riesz (1916) and Lavrentiev (1936) for Jordan's domains.

Lemma 3.1 *Let D be a bounded domain in $\mathbb{C}$ whose boundary components are Jordan curves, $\mathbb{D}_*$ be a bounded nondegenerate circular domain in $\mathbb{C}$ and let $\omega : D \to \mathbb{D}_*$ be a conformal mapping. Then*

(i) ω can be extended to a homeomorphism of $\overline{D}$ onto $\overline{\mathbb{D}_*}$;

(ii) $\arg\,[\omega(\zeta) - \omega(z)] - \arg\,[\zeta - z] \to$ *const as $z \to \zeta$ whenever ∂D has a tangent at $\zeta \in \partial D$;*

(iii) for rectifiable ∂D, length $\omega^{-1}(E) = 0$ *whenever $|E| = 0$, $E \subset \partial\mathbb{D}_*$;*

(iv) for rectifiable ∂D, $|\omega(\mathcal{E})| = 0$ whenever length $\mathcal{E} = 0$, $\mathcal{E} \subset \partial D$.

Proof. (i) Indeed, we are able to transform $\mathbb{D}_*$ into a simply connected domain $\mathbb{D}^*$ through a finite sequence of cuts. Thus, we come to the desired conclusion applying the Caratheodory theorems to simply connected domains $\mathbb{D}^*$ and $D^* := \omega^{-1}(\mathbb{D}^*)$, see e.g. Theorem 9.4 in [2] and Theorem II.C.1 in [9].

(ii) In the construction from the previous item, we may assume that the point ζ is not the end of the cuts in D generated by the cuts in $\mathbb{D}_*$ under the extended mapping ω^{-1}. Thus, we come to the desired conclusion twice applying the Caratheodory theorems, the reflection principle for conformal mappings and the Lindelöf theorem for the Jordan domains, see e.g. Theorem II.C.2 in [9].

Points (iii) and (iv) are proved similarly to the last item on the basis of the corresponding results of F. and M. Riesz and Lavrentiev for Jordan domains with rectifiable boundaries, see e.g. Theorem II.D.2 in [9], and [10], see also the point III.1.5 in [14]. □

Theorem 3.2. *Let D be a bounded domain in $\mathbb{C}$ whose boundary components are rectifiable Jordan curves and $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable functions with respect to the natural parameter on ∂D. Then there exist multivalent analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that along any nontangential path*

$$\lim_{z\to\zeta} \ \mathrm{Re}\ \{\overline{\lambda(\zeta)} \cdot f(z)\} \ = \ \varphi(\zeta) \qquad \textit{for a.e.} \quad \zeta \in \partial D \tag{5}$$

with respect to the natural parameters of the boundary components of D.

Proof. This case is reduced to the case of a bounded nondegenerate circular domain $\mathbb{D}_*$ in the following way. First, there is a conformal mapping ω of D onto a circular domain $\mathbb{D}_*$, see e.g. Theorem V.6.2 in [6]. Note that $\mathbb{D}_*$ is not degenerate because isolated singularities of conformal mappings are removable that is due to the well-known Weierstrass theorem, see e.g. Theorem 1.2 in [2]. Without loss of generality, we may assume that $\mathbb{D}_*$ is bounded.

By point (i) in Lemma 3.1 ω can be extended to a homeomorphisms of $\overline{D}$ onto $\overline{\mathbb{D}_*}$. If ∂D is rectifiable, then by point (iii) in Lemma 3.1 length $\omega^{-1}(E) = 0$ whenever $E \subset \partial\mathbb{D}_*$ with $|E| = 0$, and by (iv) in Lemma 3.1, conversely, $|\omega(\mathcal{E})| = 0$ whenever $\mathcal{E} \subset \partial D$ with length $\mathcal{E} = 0$.

In the last case ω and ω^{-1} transform measurable sets into measurable sets. Indeed, every measurable set is the union of a sigma-compact set and a set of measure zero, see e.g. Theorem III(6.6) in [18], and continuous mappings transform compact sets into compact sets. Thus, a function $\varphi : \partial D \to \mathbb{R}$ is measurable with respect to the natural parameter on ∂D if and only if the function $\Phi = \varphi \circ \omega^{-1} : \partial\mathbb{D}_* \to \mathbb{R}$ is measurable with respect to the natural parameter on $\partial\mathbb{D}_*$.

By point (ii) in Lemma 3.1, if ∂D has a tangent at a point $\zeta \in \partial D$, then $\arg\,[\omega(\zeta) - \omega(z)] - \arg\,[\zeta - z] \to \text{const}$ as $z \to \zeta$. In other words, the conformal images of sectors in D with a vertex at ζ is asymptotically the same as sectors in $\mathbb{D}_*$ with a vertex at $w = \omega(\zeta)$. Thus, nontangential paths in D are transformed under ω into nontangential paths in $\mathbb{D}_*$ and inversely. Finally, a rectifiable Jordan curve has a tangent a.e. with respect to the natural parameter and, thus, Theorem 3.2 follows from Theorem 2.1. □

In particular, choosing $\lambda \equiv 1$ in (5), we obtain the following statement.

Proposition 3.3 *Let D be a bounded domain in $\mathbb{C}$ whose boundary components are rectifiable Jordan curves and let $\varphi : \partial D \to \mathbb{R}$ be measurable. Then there exist multivalent analytic functions $f : D \to \mathbb{C}$ such that*

$$\lim_{z\to\zeta} \operatorname{Re} f(z) \;=\; \varphi(\zeta) \qquad \textit{for a.e.} \quad \zeta \in \partial D \tag{6}$$

along any nontangential path with respect to the natural parameters of the boundary components of ∂D.

4. The case of arbitrary Jordan curves.

The conceptions of a harmonic measure introduced by R. Nevanlinna in [12] and a principal asymptotic value based on one nice result of F. Bagemihl [1] make possible with a great simplicity and generality to formulate the existence theorems for the Dirichlet and Riemann–Hilbert problems.

First of all, given a measurable set $E \subseteq \partial\mathbb{D}$ and a point $z \in \mathbb{D}$, a *harmonic measure* of E at z relative to $\mathbb{D}$ is the value at z of the bounded harmonic function u in $\mathbb{D}$ with the boundary values 1 a.e. on E and 0 a.e on $\partial\mathbb{D} \setminus E$. In particular, by the mean value theorem for harmonic functions, the harmonic measure of E at 0 relative to $\mathbb{D}$ is equal to $|E|/2\pi$. In general, the geometric sense of the harmonic measure of E at z_0 relative to $\mathbb{D}$ is the angular measure of view of E from the point z_0 in radians divided by 2π. Hence the harmonic measure on $\partial\mathbb{D}$ has also the corresponding probabilistic interpretation. The harmonic measure in domains D bounded by finite collections of Jordan curves is defined in a similar way.

Next, a Jordan curve generally speaking has no tangents. Hence we need a replacement for the notion of a nontangential limit. In this connection, recall Theorem 2 in [1], see also Theorem III.1.8 in [13], stating that, for any function $\Omega : \mathbb{D} \to \overline{\mathbb{C}}$, for all pairs of arcs γ_1 and γ_2 in $\mathbb{D}$ terminating at $\zeta \in \partial\mathbb{D}$, except a countable set of $\zeta \in \partial\mathbb{D}$,

$$C(\Omega, \gamma_1) \ \cap \ C(\Omega, \gamma_2) \ \neq \ \varnothing \tag{7}$$

where $C(\Omega, \gamma)$ denotes the *cluster set of* Ω *at* ζ *along* γ, i.e.,

$$C(\Omega, \gamma) \ = \ \{w \in \overline{\mathbb{C}} \ : \ \Omega(z_n) \to w, \ z_n \to \zeta, \ z_n \in \gamma\} \ .$$

Applying the Poincare mapping, branches of its inverse mapping and their boundary behavior, see e.g. Theorem VI.1 and Section VI.2 in [6], we extend this result to arbitrary domains D bounded by a finite number of Jordan curves, cf. the proof of Theorem 2.1.

Now, given a function $\Omega : D \to \overline{\mathbb{C}}$ and $\zeta \in \partial D$, denote by $P(\Omega, \zeta)$ the intersection of all cluster sets $C(\Omega, \gamma)$ for arcs γ in D terminating at ζ. Later on, we call the points of the set $P(\Omega, \zeta)$ *principal asymptotic values* of Ω at ζ. Note that, if Ω has a limit along at least one arc in D terminating at a point $\zeta \in \partial D$ with the property (7), then the principal asymptotic value is unique.

Theorem 4.1. *Let D be a bounded domain in $\mathbb{C}$ whose boundary components are Jordan curves and let $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable functions with respect to harmonic measures in D. Then there exist multivalent analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that*

$$\lim_{z \to \zeta} \ \mathrm{Re}\ \{\overline{\lambda(\zeta)} \cdot f(z)\} \ = \ \varphi(\zeta) \qquad \textit{for a.e.} \quad \zeta \in \partial D \tag{8}$$

with respect to harmonic measures in D in the sense of the unique principal asymptotic value.

Proof. By the reasons of the first item in the proof of Theorem 3.2, there is a conformal mapping ω of D onto a bounded nondegenerate circular domain $\mathbb{D}_*$ in $\mathbb{C}$. Set $\Lambda = \lambda\circ\Omega$ and $\Phi = \varphi\circ\Omega$ where $\Omega := \omega^{-1}$ extended to $\partial\mathbb{D}_*$ by point (i) in Lemma 3.1.

Note that harmonic measure zero is invariant under conformal mappings. Thus, arguing as in the third item of the proof to Theorem 3.2, we conclude that the functions Λ and Φ are measurable with respect to harmonic measures in $\mathbb{D}_*$.

By Theorem 2.1 there exist multivalent analytic functions $F : \mathbb{D}_* \to \mathbb{C}$ such that

$$\lim_{w \to \eta} \ \mathrm{Re}\ \{\overline{\Lambda(\eta)} \cdot F(w)\} \ = \ \Phi(\eta)$$

along any nontangential path to a.e. $\eta \in \partial\mathbb{D}_*$.

By the construction the functions $f := F \circ \omega$ are desired multivalent analytic solutions of (8) in view of the Bagemihl result. □

In particular, choosing $\lambda \equiv 1$ in (8), we obtain the following consequence.

Proposition 4.2. *Let D be a bounded domain in $\mathbb{C}$ whose boundary components are Jordan curves and let $\varphi : \partial D \to \mathbb{R}$ be a measurable function with respect to harmonic measures in D. Then there exist multivalent analytic functions $f : D \to \mathbb{C}$ such that*

$$\lim_{z \to \zeta} \mathrm{Re}\, f(z) \;=\; \varphi(\zeta) \qquad \textit{for a.e.} \quad \zeta \in \partial D \tag{9}$$

with respect to harmonic measures in D in the sense of the unique principal asymptotic value.

5. On dimension of spaces of solutions.

By the Lindelöf maximum principle, see e.g. Lemma 1.1 in [4], it follows the uniqueness theorem for the Dirichlet problem in the class of bounded harmonic functions on the unit disk. Our multivalent analytic solutions are generally speaking not bounded and we have the new phenomena.

Theorem 5.1 *The spaces of solutions of the Riemann–Hilbert problem in Theorems 2.1, 3.2 and 4.1 and in Propositions 2.2, 3.3 and 4.2 have the infinite dimension.*

Proof. By Theorem 5.1 in [16] the space of solutions of the problem (3) has the infinite dimension. Thus, the conclusion follows by the construction of these solutions in the given theorems through the reduction to (3). □

1. *Bagemihl F.* Curvilinear cluster sets of arbitrary functions // Proc. Nat. Acad. Sci. U.S.A. – 2015. – V. 41. – P. 379-382.
2. *Collingwood E.F., Lohwator A.J.* The theory of cluster sets // Cambridge Tracts in Math. and Math. Physics, No. 56, Cambridge Univ. Press, Cambridge, 1966.
3. *Gakhov F.D.* Boundary value problems. – Dover Publications Inc., New York, 1990.
4. *Garnett J.B., Marshall D.E.* Harmonic Measure. – Cambridge Univ. Press, Cambridge, 2005.
5. *Gehring F.W.* On the Dirichlet problem // Michigan Math. J. – 1955—1956. – V. 3. – P. 201.
6. *Goluzin G.M.* Geometric theory of functions of a complex variable. – Transl. of Math. Monographs, Vol. 26, American Mathematical Society, Providence, R.I. 1969.
7. *Hilbert D.* Über eine Anwendung der Integralgleichungen auf eine Problem der Funktionentheorie. – Verhandl. des III Int. Math. Kongr., Heidelberg, 1904.
8. *Hilbert D.* Grundzüge einer algemeinen Theorie der Integralgleichungen. – Leipzig, Berlin, 1912.
9. *Koosis P.* Introduction to H_p spaces. – 2nd ed., Cambridge Tracts in Mathematics, 115, Cambridge Univ. Press, Cambridge, 1998.
10. *Lavrentiev M.* On some boundary problems in the theory of univalent functions // Mat. Sbornik N.S. – 1936. – V. 1, No.43. – P. 815–846 [in Russian].
11. *Muskhelishvili N.I.* Singular integral equations. Boundary problems of function theory and their application to mathematical physics. – Dover Publications. Inc., New York, 1992.
12. *Nevanlinna R.* Eindeutige analytische Funktionen. – Ann Arbor, Michigan, 1944.
13. *Noshiro K.* Cluster sets. – Springer-Verlag, Berlin etc., 1960.
14. *Priwalov I.I.* Randeigenschaften analytischer Funktionen. – Hochschulbücher für Mathematik, Bd. 25, Deutscher Verlag der Wissenschaften, Berlin, 1956.
15. *Riesz M.* Sur les functions conjuguees // Math. Z. – 1927. – Vol. 27, No. 2. – P. 218-244.

16. *Ryazanov V.* On the Riemann-Hilbert Problem without Index // Ann. Univ. Bucharest, Ser. Math. – 2014. – Vol. 5 (LXIII), No. 1. – P. 169-178.
17. *Ryazanov V.* Infinite dimension of solutions of the Dirichlet problem // Open Math. (the former *Central European J. Math.*). – 2015. – V. 13, no. 1. – P. 348-350.
18. *Saks S.* Theory of the integral. – Warsaw, 1937; Dover Publications Inc., New York, 1964.
19. *Vekua I. N.* Generalized analytic functions. – Pergamon Press, London etc., 1962.

В. И. Рязанов

К теории многозначных решений задачи Римана–Гильберта.

Доказано существование многозначных решений задачи Римана–Гильберта при общих предположениях конечносвязных областей, ограниченных взаимно непересекающимися жордановыми кривыми, измеримых коэффициентах и измеримых граничных данных. Теорема сформулирована в терминах гармонической меры и главных асимптотических значений. Также приведен соответствующий усиленный критерий для областей со спрямляемыми границами, сформулированный в терминах натурального параметра длины и некасательных пределов. Кроме того, показано, что размерность пространства найденных решений бесконечна.

Ключевые слова: *задача Римана–Гильберта, жордановы кривые, гармоническая мера, главные асимптотические значения, некасательные пределы.*

В. І. Рязанов

До теорії многозначних рішень задачі Рімана–Гільберта.

Доведено існування багатозначних рішень задачі Рімана–Гільберта при загальних припущеннях кінцевозв'язних областей, обмежених взаємно неперетинаючими жордановими кривими, вимірних коефіцієнтах і вимірних граничних даних. Теорема сформульована в термінах гармонійної міри і головних асимптотичних значень. Також наведено відповідний посилений критерій для областей зі спрямлюваними межами, сформульований в термінах натурального параметра довжини і недотичних границь. Крім того, показано, що розмірність простору знайдених рішень нескінченна.

Ключові слова: *задача Рімана–Гільберта, жорданові криві, гармонійна міра, головні асимптотичні значення, недотичні границі.*

Ин-т прикл. математики и механики НАН Украины, Славянск
`vl.ryazanov1@gmail.com`

Received 11.10.15