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ANALOGS OF FRICKE'S THEOREMS FOR ANALYTIC VECTOR-VALUED FUNCTIONS IN THE UNIT BALL HAVING BOUNDED L-INDEX IN JOINT VARIABLES

In this paper, we present necessary and sufficient conditions of boundedness of $\mathbb{L}$-index in joint variables for vector-functions analytic in the unit ball, where $\mathbf{L} = (l_1, l_2) : \mathbb{B}^2 \to \mathbb{R}^2_+$ is a positive continuous vector-function, $\mathbb{B}^2 = \{z \in \mathbb{C}^2 : |z| = \sqrt{|z_1|^2 + |z_2|^2} \le 1\}$. Particularly, we deduce analog of Fricke's theorems for this function class, give estimate of maximum modulus on the skeleton of bidisc. The first theorem concerns sufficient conditions. In this theorem we assume existence of some radii, for which the maximum of norm of vector-function on the skeleton of bidisc with larger radius does not exceed maximum of norm of vector-function on the skeleton of bidisc with lesser radius multiplied by some costant depending only on these radii. In the second theorem we show that boundedness of $\mathbf{L}$-index in joint variables implies validity of the mentioned estimate for all radii.

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1. Introduction.

This paper is addendum to [1]. There was proposed the definition of $\mathbf{L}$-index boundedness in joint variables obtained some criteria of $\mathbf{L}$-index boundedness in joint variables for vector-valued analytic functions in the unit ball. Here we pose the following goal: *to obtain and to prove criteria admitting easy application to system of partial differential equations.* Particularly, we should like to deduce analogs of Fricke's theorem and Hayman's theorem for this function class. The first theorem describes estimate of maximum modulus of entire function having bounded index. It plays important role in the proof of the second theorem. And the second theorem allows to estimate modulus of derivative of some order by moduli of derivatives of lesser order. For vector-valued function we will replace the modulus of the function by the sup-norm. Analogs of Hayman's theorem for various classes of analytic functions have many applications in analytic theory of differential equations: entire functions of bounded L-index in direction [2], entire functions of bounded $\mathbf{L}$-index in joint variables [3], analytic functions in the unit ball having bounded $\mathbf{L}$-index in joint variables [4], entire bivariate vector-valued function of bounded index [5].

2. Notations, definitions and auxiliary propositions.

Here we use notations from [1, 4, 6]. We need some standard notations. Let $\mathbb{R}_+ = [0; +\infty)$, $\mathbf{0} = (0, 0) \in \mathbb{R}^2_+$, $\mathbf{1} = (1, 1) \in \mathbb{R}^2_+$, $R = (r_1, r_2) \in \mathbb{R}^2_+$, $|(z, \omega)| = \sqrt{|z|^2 + |\omega|^2}$. For $A = (a_1, a_2) \in \mathbb{R}^2$, $B = (b_1, b_2) \in \mathbb{R}^2$, we will use formal notations without violation of the existence of these expressions: $AB = (a_1b_1, a_2b_2)$, $A/B = (a_1/b_1, a_2/b_2)$,

$A^B = (a_1^{b_1}, a_2^{b_2})$, and the notation $A < B$ means that $a_j < b_j$, $j \in \{1, 2\}$; the relation $A \le B$ is defined in the similar way. For $K = (k_1, k_2) \in \mathbb{Z}_+^2$ let us denote $K! = k_1! \cdot k_2!$. Addition, multiplication by scalar and conjugation in $\mathbb{C}^2$ is defined componentwise. For $z \in \mathbb{C}^2$, $w \in \mathbb{C}^2$ we define $\langle z, w\rangle = z_1\overline{w}_1 + z_2\overline{w}_2$, where $\overline{w}_1, \overline{w}_2$ is the complex conjugate of w_1, w_2.

The polydisc $\{(z,\omega) \in \mathbb{C}^2 : |z-z_0| < r_1, |\omega-\omega_0| < r_2\}$ is denoted by $\mathbb{D}^2((z_0,\omega_0), R)$, its skeleton $\{(z,\omega) \in \mathbb{C}^2 : |z-z_0| = r_1, |\omega-\omega_0| = r_2\}$ is denoted by $\mathbb{T}^2((z_0,\omega_0), R)$, the closed polydisc $\{(z,\omega) \in \mathbb{C}^2 : |z-z_0| \le r_1, |\omega-\omega_0| \le r_2\}$ is denoted by $\mathbb{D}^2[(z_0,\omega_0), R]$, $\mathbb{D}^2 = \mathbb{D}^2(\mathbf{0};\mathbf{1})$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The open ball $\{(z,\omega) \in \mathbb{C}^2 : \sqrt{|z-z_0|^2 + |\omega-\omega_0|^2} < r\}$ is denoted by $\mathbb{B}^2((z_0,\omega_0), r)$, the sphere $\{(z,\omega) \in \mathbb{C}^2 : \sqrt{|z-z_0|^2 + |\omega-\omega_0|^2} = r\}$ is denoted by $\mathbb{S}^2((z_0,\omega_0), r)$, and the closed ball $\{z \in \mathbb{C}^2 : \sqrt{|z-z_0|^2 + |\omega_0-\omega_0|^2} \le r\}$ is denoted by $\mathbb{B}^2[(z_0,\omega_0), r]$, $\mathbb{B}^2 = \mathbb{B}^2(\mathbf{0},\mathbf{1})$, $\mathbb{D} = \mathbb{B}^1 = \{z \in \mathbb{C} : |z| < 1\}$.

Let $F(z,\omega) = (f_1(z,\omega), f_2(z,\omega))$ be an analytic vector-function in $\mathbb{B}^2$. Then at a point $(a,b) \in \mathbb{B}^2$ the function $F(z,\omega)$ has a bivariate Taylor expansion:

$$F(z,\omega) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} C_{kl}(z-a)^k(\omega-b)^m,$$

where $C_{km} = \frac{1}{k!m!}\left(\frac{\partial^{k+m} f_1(z,\omega)}{\partial z^k \partial\omega^m}, \frac{\partial^{k+m} f_2(z,\omega)}{\partial z^k \partial\omega^m}\right)\Big|_{z=a,\omega=b} = \frac{1}{k!m!}F^{(k,m)}(a,b)$.

Let $\mathbf{L}(z,\omega) = (l_1(z,\omega), l_2(z,\omega))$, where $l_j(z,\omega) : \mathbb{B}^2 \to \mathbb{R}_+^2$ is a positive continuous function such that

$$\forall (z,\omega) \in \mathbb{B}^2 : \quad l_j(z,\omega) > \frac{\beta}{1-\sqrt{|z|^2+|\omega|^2}}, \tag{1}$$

$j \in \{1, 2\}$, where $\beta > \sqrt{2}$ is a some constant.

The norm for the vector-function $F : \mathbb{B}^2 \to \mathbb{C}^2$ is defined as the sup-norm:

$$\|F(z,\omega)\| = \max_{1\le j\le 2}\{|f_j(z,\omega)|\}.$$

We write

$$F^{(i,j)}(z,\omega) = \frac{\partial^{i+j}F(z,\omega)}{\partial z^i\partial\omega^j} = \left(\frac{\partial^{i+j}f_1(z,\omega)}{\partial z^i\partial\omega^j}, \frac{\partial^{i+j}f_2(z,\omega)}{\partial z^i\partial\omega^j}\right).$$

An analytic vector-function $F : \mathbb{B}^2 \to \mathbb{C}^2$ is said to be of bounded $\mathbf{L}$-index (in joint variables), if there exists $n_0 \in \mathbb{Z}_+$ such that

$$\forall (z,\omega) \in \mathbb{B}^2 \ \ \forall (i,j) \in \mathbb{Z}_+^2 :$$

$$\frac{\|F^{(i,j)}(z,\omega)\|}{i!j!l_1^i(z,\omega)l_2^j(z,\omega)} \le \max\left\{\frac{\|F^{(k,m)}(z,\omega)\|}{k!m!l_1^k(z,\omega)l_2^m(z,\omega)} : k,m \in \mathbb{Z}_+, k+m \le n_0\right\}. \tag{2}$$

The least such integer n_0 is called the $\mathbf{L}$-index in joint variables of the vector-function F and is denoted by $N(F,\mathbf{L},\mathbb{B}^2)$. The concept of boundedness of $\mathbf{L}$-index in joint variables were considered for other classes of analytic functions. They are differed domains of analyticity: the unit ball [4, 6–8], the polydisc [9, 10], the Cartesian product of the unit disc and complex plane [11], n-dimensional complex space [12, 13]. Vector-valued functions of one and several complex variables having bounded index were considered in [14–18].

The function class $Q(\mathbb{B}^2)$ is defined as following: $\forall R\in\mathbb{R}^2_+, |R|\le\beta, j\in\{1,2\}:$

$$0<\lambda_{1,j}(R)\le\lambda_{2,j}(R)<\infty$$

where

$$\lambda_{1,j}(R)=\inf_{(z_0,\omega_0)\in\mathbb{B}^2}\inf\left\{\frac{l_j(z,\omega)}{l_j(z_0,\omega_0)}:(z,\omega)\in\mathbb{D}^2[(z_0,\omega_0),R/\mathbf{L}(z_0,\omega_0)]\right\},\qquad(3)$$

$$\lambda_{2,j}(R)=\sup_{(z_0,\omega_0)\in\mathbb{B}^2}\sup\left\{\frac{l_j(z,\omega)}{l_j(z_0,\omega_0)}:(z,\omega)\in\mathbb{D}^2[(z_0,\omega_0),R/\mathbf{L}(z_0,\omega_0)]\right\}.\qquad(4)$$

We need the two following propositions from [1].

Theorem 1. ([1]) *Let $\mathbf{L}\in Q(\mathbb{B}^2)$. An analytic vector-function $F:\mathbb{B}^2\to\mathbb{C}^2$ has bounded $\mathbf{L}$-index in joint variables if and only if for every $R\in\mathbb{R}^2$, $|R|\le\beta$ there exist $n_0\in\mathbb{Z}_+$, $p>0$ such that for all $(z_0,\omega_0)\in\mathbb{B}^2$ there exists 2-tuple $(k_0,m_0)\in\mathbb{Z}^2_+$, $k_0+m_0\le n_0$, satisfying inequality*

$$\max\left\{\frac{\|F^{(k,m)}(z,\omega)\|}{k!m!l_1^k(z,\omega)l_2^m(z,\omega)}:k+m\le n_0,(z,\omega)\in\mathbb{D}^2[(z_0,\omega_0),R/\mathbf{L}(z_0,\omega_0)]\right\}\le$$

$$\le p_0\frac{\|F^{(k_0,m_0)}(z_0,\omega_0)\|}{k_0!m_0!l_1^{k^0}(z_0,\omega_0)l_2^{m^0}(z_0,\omega_0)}.\qquad(5)$$

Let $\mathbf{L}_1$, $\mathbf{L}_2\in Q(\mathbb{B}^2)$. We say that $\mathbf{L}_1\asymp\mathbf{L}_2$ if there exist two constants $0<C_1<C_2<+\infty$ such that

$$C_1\mathbf{L}_1(z,w)\le\mathbf{L}_2(z,w)\le C_2\mathbf{L}_1(z,w)$$

for every point $(z,w)\in\mathbb{B}^2$.

Lemma 2. ([1]) *Let $\mathbf{L}\in Q(\mathbb{B}^2)$, $\mathbf{L}\asymp\widetilde{\mathbf{L}}$, $\beta>1$. An analytic in $\mathbb{B}^2$ vector-function $F:\mathbb{B}^2\to\mathbb{C}^2$ has bounded $\widetilde{\mathbf{L}}$-index in joint variables if and only if, she has bounded $\mathbf{L}$-index in joint variables.*

3. Estimate of maximum modulus on the skeleton of analytic vector-function in ball.

For an analytic vector-function $F:\mathbb{B}^2\to\mathbb{C}^2$ we put

$$M(R,(z_0,\omega_0),F)=\max\left\{\|F(z,\omega)\|:(z,\omega)\in\mathbb{T}^2((z_0,\omega_0),R)\right\},$$

where $(z_0, \omega_0) \in \mathbb{B}^2$, $R \in \mathbb{R}^2_+$. Then

$$M(R, (z_0, \omega_0), F) = \max \left\{ \|F(z, \omega)\| : (z, \omega) \in \mathbb{D}^2((z_0, \omega_0), R) \right\},$$

because the maximum modulus of the analytic vector-function in a closed bidisc is attained on its skeleton.

In theory of functions having bounded index a very important role has Hayman's theorem. It was obtained by W. Hayman [19] for entire functions of one variable having bounded index. Later it was generalized for various classes of analytic functions [6,9,11,20]. This theorem is applicable in analytic theory of partial differential equations [3, 4, 21]. It allows to deduce conditions by the coefficients of equation providing index boundedness of each analytic solution. Moreover, the idea of proof was used to obtain growth estimate for functions from this class. To prove an analogue of Hayman's theorem, we need the following theorem. The theorem gives sufficient conditions by the estimate of maximum modulus on the skeleton of bidisc.

Theorem 3. *Let $\mathbf{L} \in Q(\mathbb{B}^2)$, $F : \mathbb{B}^2 \to \mathbb{C}^2$ be an analytic vector-function. If there exist $R', R'' \in \mathbb{R}^2_+$, $R' < R''$,$|R''| < \beta$ and $p_1 = p_1(R', R'') \geq 1$ such that for each $(z_0, \omega_0) \in \mathbb{B}^2$*

$$M\left(\frac{R''}{\mathbf{L}(z_0, \omega_0)}, (z_0, \omega_0), F\right) \leq p_1 M\left(\frac{R'}{\mathbf{L}(z_0, \omega_0)}, (z_0, \omega_0), F\right) \tag{6}$$

then F has bounded $\mathbf{L}$-index in joint variables.

Proof. The current proof is similar to proof for functions analytic in the unit ball in [6]. Suppose that $\mathbf{0} < R' < \mathbf{1} < R''$.

Let $(z_0, \omega_0) \in \mathbb{B}^2$ be an arbitrary point. We expand the vector-function F in power series

$$F(z, \omega) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} B_{k,m}(z - z_0)^k (\omega - \omega_0)^m, \tag{7}$$

where $B_{k,m} = (b_{1,k,m}, b_{2,k,m}) = (\frac{f_1^{(k,m)}(z_0,\omega_0)}{k!m!}, \frac{f_2^{(k,m)}(z_0,\omega_0)}{k!m!})$.

Let $\mu(R, (z_0, \omega_0), F) = \max\{\|B_{k,m}\| r_1^k r_2^m : k + m \geq 0\}$ be the maximal term of series (7) and $\nu(R) = \nu(R, (z_0, \omega_0), F) = (\nu_1(R), \nu_2(R))$ be a set of indices such that $\mu(R, (z_0, \omega_0), F) = \|B_{\nu(R)}\| r_1^{\nu_1(R)} r_2^{\nu_2(R)}$,

$$\nu_1(R) + \nu_2(R) = \max\{k + m : k + m \geq 0, \|B_{k,m}\| r_1^k r_2^m = \mu(R, (z_0, \omega_0), F)\}.$$

Then in view of inequality (7) we obtain that for every $|R| < 1 - \sqrt{|z_0|^2 + |\omega_0|^2}$:

$$\mu(R, (z_0, \omega_0), F) \leq M(R, (z_0, \omega_0), F).$$

Then for the given R' and R'' with $0 < |R'| < 1 < |R''| < \beta$ we deduce

$$M(R'R,(z_0,\omega_0),F) \le \sum_{k\ge 0}\sum_{m\ge 0}\|B_{k,m}\|(r_1'r_1)^k(r_2'r_2)^m \le$$
$$\le \sum_{k\ge 0}\sum_{m\ge 0}\mu(R,(z_0,\omega_0),F)(r_1')^k(r_2')^m =$$
$$= \mu(R,(z_0,\omega_0),F)\sum_{k\ge 0}\sum_{m\ge 0}(r_1')^k(r_2')^m = \prod_{j=1}^{2}\frac{1}{1-r_j'}\mu(R,(z_0,\omega_0),F).$$

And

$$\ln\mu(R,(z_0,\omega_0),F) = \ln\left\{\|B_{\nu(R)}\|r_1^{\nu_1(R)}r_2^{\nu_2(R)}\right\} =$$
$$= \ln\left\{\|B_{\nu(R)}\|(r_1r_1')^{\nu_1(R)}(r_2r_2')^{\nu_2(R)}\frac{1}{(r_1'')^{\nu_1(R)}(r_2'')^{\nu_2(R)}}\right\} =$$
$$= \ln\left\{\|B_{\nu(R)}\|(r_1r_1')^{\nu_1(R)}(r_2r_2')^{\nu_2(R)}\right\} + \ln\left\{\frac{1}{(r_1'')^{\nu_1(R)}(r_2'')^{\nu_2(R)}}\right\} \le$$
$$\le \ln\mu(R''R,(z_0,\omega_0),F) - (\nu_1(R)+\nu_2(R))\ln\min_{1\le j\le 2} r_j''.$$

Hence, we have

$$\nu_1(R)+\nu_2(R) \le \frac{1}{\ln\min_{1\le j\le 2} r_j''}(\ln\mu(R''R,(z_0,\omega_0),F) - \ln\mu(R,(z_0,\omega_0),F)) \le$$
$$\le \frac{(\ln M(R''R,(z_0,\omega_0),F) - \ln(\prod_{j=1}^{2}(1-r_j)\ln M(R'R,(z_0,\omega_0),F))}{\ln\min_{1\le j\le 2} r_j''} \le$$
$$\le \frac{1}{\ln\min_{1\le j\le 2} r_j''}(\ln M(R''R,(z_0,\omega_0),F) - \ln M(R'R,(z_0,\omega_0),F)) -$$
$$-\frac{\sum_{j=1}^{2}\ln(1-r_j)}{\min_{1\le j\le 2} r_j''} = \frac{1}{\min_{1\le j\le 2} r_j''}\ln\frac{M(R''R,(z_0,\omega_0),F)}{M(R'R,(z_0,\omega_0),F)} - \frac{\sum_{j=1}^{2}\ln(1-r_j)}{\min_{1\le j\le 2} r_j''}. \quad (8)$$

Put $R = \frac{1}{\mathbf{L}(z_0,\omega_0)}$. Let $N(F,(z_0,\omega_0),\mathbf{L})$ be the $\mathbf{L}$-index of vector-function F in joint variables at the point (z_0,ω_0).

Therefore,

$$N(F,(z_0,\omega_0),\mathbf{L}) \le \nu\left(\frac{1}{\mathbf{L}(z_0,\omega_0)},(z_0,\omega_0),F\right) = \nu(R,(z_0,\omega_0),F). \quad (9)$$

But

$$M(R''/\mathbf{L}(z_0,\omega_0),(z_0,\omega_0),F) \le p_1(R',R'')M(R'/\mathbf{L}(z_0,\omega_0),(z_0,\omega_0),F). \quad (10)$$

Thus, in view of (8), (9) and (10) we obtain $\forall(z_0,\omega_0)\in\mathbb{B}^2$

$$N(F,(z_0,\omega_0),\mathbf{L}) \le \frac{-\sum_{j=1}^{2}\ln(1-r_j')}{\ln\min\{r_1',r_2''\}} + \frac{\ln p_1(R',R'')}{\ln\min\{r_1',r_2''\}}.$$

Hence, we conclude that F has bounded $\mathbf{L}$-index in joint variables with $\mathbf{0} < R' < \mathbf{1} < R''$, $|R''| < \beta$.

Let us prove the theorem for all $\mathbf{0} < R' < R''$, $|R''| < \beta$. From (6) one has

$$\max\left\{\|F(z,\omega)\| : (z,\omega) \in \mathbb{T}^2\left((z_0,\omega_0), \frac{2R''}{R'+R''}\frac{R'+R''}{2\mathbf{L}(z_0,\omega_0)}\right)\right\} \leq$$
$$\leq p_1 \max\left\{\|F(z,\omega)\| : (z,\omega) \in \mathbb{T}^2\left((z_0,\omega_0), \frac{2R''}{R'+R''}\frac{R'+R''}{2\mathbf{L}(z_0,\omega_0)}\right)\right\}.$$

Put $\widetilde{\mathbf{L}}(z,\omega) = \frac{2\mathbf{L}(z,\omega)}{R'+R''}$. We obtain

$$\max\left\{\|F(z,\omega)\| : (z,\omega) \in \mathbb{T}^2\left((z_0,\omega_0), \frac{2R''}{(R'+R'')\widetilde{\mathbf{L}}(z_0,\omega_0)}\right)\right\} \leq$$
$$\leq p_1 \max\left\{\|F(z,\omega)\| : (z,\omega) \in \mathbb{T}^2\left((z_0,\omega_0), \frac{2R''}{(R'+R'')\widetilde{\mathbf{L}}(z_0,\omega_0)}\right)\right\},$$

where $\mathbf{0} < \frac{2R'}{R'+R''} < \mathbf{1} < \frac{2R''}{R'+R''}$.

Using the first part of the proof, we get that the vector-function F has bounded $\widetilde{\mathbf{L}}$-index in joint variables. Then by Lemma 2 the vector-function F is a function of bounded $\mathbf{L}$-index in joint variables. □

Using the arguments of the proof of Theorem 3, one can prove the following theorem

Theorem 4. *Let $\mathbf{L} \in Q(\mathbb{B}^2)$. If analytic vector-function $F : \mathbb{B}^2 \to \mathbb{C}^2$ has bounded $\mathbf{L}$-index in joint variables then for all $R', R'' \in \mathbb{R}^2_+$, $R' < R''$, $|R''| \leq \beta$ there exists $p_1 = p_1(R', R'') \geq 1$ such that for every $(z_0,\omega_0) \in \mathbb{B}^2$ inequality* (6) *holds.*

Proof. Let $N(F,\mathbf{L}) = N < +\infty$. Suppose that inequality (6) does not hold i.e. there exist $R' < R''$, $0 < |R'| < |R''| < \beta$, such that for each $p_* \geq 1$ and for some $z_0 = z_0(p_\star)$, $\omega_0 = \omega_0(p_*)$

$$M\left(\frac{R''}{\mathbf{L}(z_0,\omega_0)}, (z_0,\omega_0), F\right) \geq p_* M\left(\frac{R'}{\mathbf{L}(z_0,\omega_0)}, (z_0,\omega_0), F\right). \tag{11}$$

By Theorem 1, there exists $p_0 = p_0(R'') \geq 1$ such that for every $(z_0,\omega_0) \in \mathbb{B}^2$ and some $(k_0,m_0) \in \mathbb{Z}^2_+$, $k_0 + m_0 \leq N$ (that is $n_0 = N$, see proof of Theorem 1) one has :

$$M\left(\frac{R''}{\mathbf{L}(z_0,\omega_0)(z_0,\omega_0), F^{(k_0,m_0)}}\right) \leq p_0\|F^{(k_0,m_0)}(z_0,\omega_0)\|. \tag{12}$$

We put

$$b_1 = p_0\lambda_{2,2}^N(R'')N!\left(\sum_{j=1}^{2}\frac{(N-j)!}{(r_1'')^j}\right)\left(\frac{r_1''r_2''}{r_1'r_2'}\right)^N,$$
$$b_2 = p_0\left(\sum_{j=1}^{2}\frac{(N-j)!}{(r_2'')^j}\right)\left(\frac{r_2''}{r_2'}\right)^N \max\left\{1, \frac{1}{(r_1')^N}\right\},$$

and

$$p_* = (N!)^2 p_0 \left(\frac{r_1'' r''2}{r_1' r_2'}\right)^N + \sum_{k=1}^{2} b_k + 1.$$

Let $z_0 = z_0(p_*)$, $\omega_0 = \omega_0(p_*)$, (z_0, ω_0) be a point for which inequality (11) holds and (k_0, m_0) be such that (12) holds and

$$M\left(\frac{R'}{\mathbf{L}(z_0,\omega_0)}, (z_0,\omega_0), F\right) = \|F(z^*,\omega^*)\|,$$
$$M\left(\frac{R''}{\mathbf{L}(z_0,\omega_0)}, (z_0,\omega_0), F^{(i,j)}\right) = \|F^{(i,j)}(z^*_{i,j},\omega^*_{i,j})\|$$

for every $(i,j) \in \mathbb{Z}^2_+$, $i+j \le N$. We apply Cauchy's inequality

$$\|F^{(i,j)}(z_0,\omega_0)\| \le i!j! \left(\frac{l_1(z_0,\omega_0)}{r_1'}\right)^i \left(\frac{l_2(z_0,\omega_0)}{r_2'}\right)^j \|F(z^*,\omega^*)\| \tag{13}$$

for estimate the difference

$$\|F^{(i,j)}(z^*_{i,j},\omega^*_{i,j}) - F^{(i,j)}(z_1^0,\omega^*_{i,j})\| = \|\int_{z_1^0}^{z^*_{i,j}} \frac{\partial^{i+j+1}F}{\partial z^{i+1}\partial\omega^j}(\xi,\omega^*_{i,j})d\xi\| \le$$
$$\le \left\|\frac{\partial^{i+j+1}F}{\partial z^{i+1}\partial\omega^j}(z^*_{i+1,j},\omega^*_{i+1,j})\right\| \frac{r_1''}{l_1(z_0,\omega_0)}. \tag{14}$$

Since $(z_1^0,\omega^*_{i,j}) \in \mathbb{D}^2\left[(z_0,\omega_0), \frac{R''}{\mathbf{L}(z_0,\omega_0)}\right]$, and $|z^*_{i,j} - z_1^0| = \frac{r_1''}{l_1(z_0,\omega_0)}$, $l_1(z_1^0, z^*_{i,2}) \le \lambda_{2,1}(R'')l_1(z_0,\omega_0)$, $|\omega^*_{i,j} - \omega_2^0| = \frac{r_2''}{l_2(z_0,\omega_0)}$, $l_2(\omega_2^0,\omega^*_{j,2}) \le \lambda_{2,2}(R'')l_2(z_0,\omega_0)$, and inequality (13) holds with $i = k_0$, $j = m_0$ by Theorem 1 we have

$$\|F^{(i,j)}(z_1^0,\omega^*_{i,j})\| \le \frac{i!j!l_1^i(z_1^0,\omega^*_{j,2})l_2^j(z_1^0,\omega^*_{j,2})}{k_0!m_0!l_1^{k_0}(z_0,\omega_0)l_2^{m_0}(z_0,\omega_0)} p_0 \|F^{(k_0,m_0)}(z_0,\omega_0)\| \le$$
$$\le \frac{i!j!l_1^i(z_0,\omega_0)l_2^j(z_0,\omega_0)\lambda^i_{2,1}(R'')\lambda^j_{2,2}(R'')}{k_0!m_0!l_1^{k_0}(z_0,\omega_0)l_2^{m_0}(z_0,\omega_0)} p_0 k_0!m_0! \frac{l_1^{k_0}(z_0,\omega_0)l_2^{m_0}(z_0,\omega_0)}{(r_1')^{k_0}(r_2')^{m_0}} \|F(z^*,\omega^*)\| =$$
$$= \frac{i!j!l_1^i(z_0,\omega_0)l_2^j(z_0,\omega_0)\lambda^i_{2,1}(R'')\lambda^j_{2,2}(R'')}{(r_1')^{k_0}(r_2')^{m_0}} \|F(z^*,\omega^*)\|. \tag{15}$$

From inequalities (14) and (15) it follows that

$$\left\|\frac{\partial^{i+j+1}F}{\partial z_{i,j}^{i+1}\partial\omega_{i,j}^j}(z^*_{i+1,j},\omega^*_{i+1,j})\right\| \ge \frac{l_1(z_0,\omega_0)}{r_1''}\left\{\|F^{i,j}(z^*_{i,j},\omega^*_{i,j})\| - \|F^{i,j}(z_1^0,\omega^*_{i,j})\|\right\} \ge$$
$$\ge \frac{l_1(z_1^0,\omega^*_{i,j})}{r_1''}\|F^{i,j}(z_1^0,\omega^*_{i,j})\| - \frac{p_0 i!j!l_1^{i+1}(z_0,\omega_0)l_2^j(z_0,\omega_0)\lambda^{i,j}_{2,2}(R'')}{r_1''(r_1')^{k_0}(r_2')^{m_0}}\|F(z^*,\omega^*)\|.$$

Then

$$\|F^{(k_0,m_0)}(z^*_{k_0},\omega^*_{m_0})\| \geq \frac{l_1(z_0,\omega_0)}{r''_1}\left\|\frac{\partial^{(k_0+m_0)-1}f}{\partial z^{k_0-1}\partial\omega^{m_0}}(z^*_{k_0-1,m_0},\omega^*_{k_0,m_0})\right\| -$$

$$-\frac{p_0(k_0-1)!m_0!l_1^{k_0}(z_0,\omega_0)l_2^{m_0}(z_0,\omega_0)\lambda_{2,1}^{k_0}(R'')\lambda_{2,2}^{m_0}(R'')}{r''_1(r'_1)^{k_0}(r'_2)^{m_0}}\|F(z^*,\omega^*)\| \geq$$

$$\geq \frac{l_1^2(z_0,\omega_0)}{(r''_1)^2}\left\|\frac{\partial^{(k_0+m_0)-2}f}{\partial z^{k_0-2}\partial\omega^{m_0}}(z^*_{k_0-2,m_0},\omega^*_{k_0,m_0})\right\| -$$

$$-\frac{p_0(k_0-2)!m_0!l_1^{k_0}(z_0,\omega_0)l_2^{m_0}(z_0,\omega_0)\lambda_{2,1}^{k_0}(R'')\lambda_{2,2}^{m_0}(R'')}{(r''_1)^2(r'_1)^{k_0}(r'_2)^{m_0}}\|F(z^*,\omega^*)\| -$$

$$-\frac{p_0(k_0-1)!m_0!l_1^{k_0}(z_0,\omega_0)l_2^{m_0}(z_0,\omega_0)\lambda_{2,1}^{k_0}(R'')\lambda_{2,2}^{m_0}(R'')}{r''_1(r'_1)^{k_0}(r'_2)^{m_0}}\|F(z^*,\omega^*)\| \geq$$

$$\geq \frac{l_1^{k_0}(z_0,\omega_0)}{(r'')^{k_0}}\left\|\frac{\partial^{m_0}f}{\partial\omega^{m_0}}(z^*_0,\omega^*_{m_0})\right\| - \frac{p_0}{(R')^{k_0,m_0}}l_1^{k_0}(z_0,\omega_0)l_2^{m_0}(z_0,\omega_0)\times$$

$$\times\lambda_{2,2}^{m_0}(R'')m_0!\frac{(k_0-i)!}{(r''_1)^i}\|F(z^*,\omega^*)\| \geq$$

$$\geq \frac{l_1^{k_0}(z_0,\omega_0)l_2^{m_0}(z_0,\omega_0)}{(r''_1)^{k_0}(r''_2)^{m_0}}\|F(z^*_0,\omega^*_0) - F(z^*,\omega^*)\|(\widetilde{b}_1+\widetilde{b}_2), \tag{16}$$

where in view of the inequalities $\lambda_{2,1}(R'') \geq 1$, $\lambda_{2,2}(R'') \geq 1$ and $R'' \geq R'$ we have

$$\widetilde{b}_1 = \frac{p_0}{(r'_1)^{k_0}(r'_2)^{m_0}}l_1^{k_0}(z_0,\omega_0)l_2^{m_0}(z_0,\omega_0)\lambda_{2,2}^{m_0}(R'')m_0!\frac{(k_0-1)!}{r''_1} =$$

$$= \frac{l_1^{k_0}(z_0,\omega_0)l_2^{m_0}(z_0,\omega_0)}{(r''_1)^{k_0}(r''_2)^{m_0}}\left(\frac{R''}{R'}\right)^{k_0,m_0}p_0\lambda_{2,2}^{m_0}(R'')m_0!\frac{(k_0-1)!}{r''_1} \leq \frac{l_1^{k_0}(z_0,\omega_0)l_2^{m_0}(z_0,\omega_0)}{(r''_1)^{k_0}(r''_2)^{m_0}}b_1,$$

$$\widetilde{b}_2 = \frac{p_0}{(R')^{k_0,m_0}}l_1^{k_0}(z_0,\omega_0)l_2^{m_0}(z_0,\omega_0)\frac{1}{(r''_1)^{k_0}}\frac{(m_0-1)!}{r''_2} \leq \frac{l_1^{k_0}(z_0,\omega_0)l_2^{m_0}(z_0,\omega_0)}{(r''_1)^{k_0}(r''_2)^{m_0}}b_2.$$

Thus, (16) implies that:

$$\|F^{(k_0,m_0)}(z^*_{k_0,m_0},\omega^*_{k_0,m_0})\| \geq \frac{l_1^{k_0}(z_0,\omega_0)l_2^{m_0}(z_0,\omega_0)}{(r''_1)^{k_0}(r''_2)^{m_0}}\|F(z^*,\omega^*)\|\times$$

$$\times\left\{\frac{\|F(z^*_0,\omega^*_0)\|}{\|F(z^*,\omega^*)\|} - (b_1+b_2)\right\}.$$

But in view of (11) and a choice of p_* , we have $\frac{\|F(z_0^*,\omega_0^*)\|}{\|F(z^*,\omega^*)\|} \geq p_* > b_1 + b_2$. Thus, in view of (12) and (13) we obtain

$$\|F^{(k_0,m_0)}(z^*_{k_0,m_0}, \omega^*_{k_0,m_0})\| \geq \frac{l_1^{k_0}(z_0,\omega_0) l_2^{m_0}(z_0,\omega_0)}{(r_1'')^{k_0}(r_2'')^{m_0}} \|F(z^*,\omega^*)\| \{p_* - (b_1+b_2)\} \geq$$
$$\geq \frac{l_1^{k_0}(z_0,\omega_0) l_2^{m_0}(z_0,\omega_0)}{(r_1'')^{k_0}(r_2'')^{m_0}} \{p_* - (b_1+b_2)\} \frac{\|F^{(k_0,m_0)}(z_0,\omega_0)\| (R')^{k_0,m_0}}{k_0! m_0! l_1^{k_0}(z_0,\omega_0) l_2^{m_0}(z_0,\omega_0)} \geq$$
$$\geq \left(\frac{r_1' r_2}{r_1'' r_2''}\right)^N \{p_* - (b_1+b_2)\} \frac{\|F^{(k_0,m_0)}(z^*_{k_0,m_0}, \omega^*_{k_0,m_0})\|}{p_0 (N!)^2}.$$

Hence, we have $p_* \leq p_0 \left(\frac{r_1' r_2'}{r_1'' r_2''}\right)^N (N!)^2 + \sum_{j=1}^2 b_j$, but this contradicts the choice of p_*. □

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Аналоги теорем Фріке для аналітичних у кулі векторнозначних функцій обмеженого L-індексу за сукупністю змінних.

У цій статті нами отримано необхідні та достатні умови обмеженості **L**-індексу за сукупністю змінних для векторнозначних функцій, аналітичних в одиничній кулі, де $\mathbf{L}=(l_1,l_2)$ — додатна неперервна векторнозначна функція, що визначена у внутрішності одиничній кулі з двовимірного комплексного простору і кожна компонента задовольняє деяку умову в цій кулі. Точніше, при підході до межі одиничної кулі кожна компонента зростає швидше, ніж $1/(1-|z|)$, де $|z|$ — евклідова норма у двовимірному комплексному просторі. Зокрема, нами доведено аналоги теорем Фріке для цього класу функцій, які дають оцінку максимуму норми на кістяку бікруга. Перша теорема стосується достатніх умов. Згідно з цими умовами для обмеженості **L**-індексу за сукупністю змінних досить вимагати існування деяких радіусів, для яких максимум норми аналітичної векторнозначної функції на кістяку бікруга з більшим радіусом не перевищує максимуму норми векторнозначної функції на кістяку бікруга з меншим радіусом помноженого на деяку сталу, залежну лише від радіусів. Доведення першої теореми подібне до доведення відповідного твердження для аналітичних в одиничній кулі функцій та використовує властивості максимального члена, центрального індексу та коефіцієнтів степеневого розвинення в околі довільної точки з внутрішності двовимірної одиничної кулі. У другій теоремі стверджується, що з обмеженості **L**-індексу за сукупністю змінних для векторнозначної аналітичної в двовимірній одиничній кулі функції випливає справедливість згаданої оцінки для всіх радіусів. Доведення другої теореми базується на інтегральній формулі Коші та пов'язаній з нею нерівності Коші. Основою для такого доведення служить критерій обмеженості **L**-індексу за сукупністю змінних для векторнозначних функцій, аналітичних в одиничній кулі, який раніше був отриманий одним зі співавторів. Цей критерій описує локальне поводження максимумів норм частинних похідних на кістяках бікруга. З одержаних нарізно достатніх умов та необхідних умов обмеженості **L**-індексу за сукупністю змінних для векторнозначних аналітичних в одиничній кулі функцій легко одержується критерій обмеженості **L**-індексу за сукупністю змінних, який полягає у можливості оцінки максимуму норми векторнозначної функції на кістяку бікруга з більшим радіусом через максимум норми векторнозначної функції на кістяку бікруга з меншим радіусом, помножений на деяку сталу, залежну лише від радіусів і незалежну від центрів бікруга.

Ключові слова: *обмежений індекс, обмежений* **L**-*індекс за сукупністю змінних, аналітична функція, одинична куля, локальне поводження, максимум модуля.*

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