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DIRICHLET PROBLEM WITH MEASURABLE DATA FOR QUASILINEAR POISSON EQUATIONS

The study of the Dirichlet problem with arbitrary measurable data for harmonic functions in the unit disk $\mathbb{D}$ goes to the famous dissertation of Luzin, see e.g. its reprint [25]. His result was formulated in terms of angular limits (along nontangent paths) that are a traditional tool for the research of the boundary behavior in the geometric function theory.

Following this way, we proved in [16] Theorem 7 on the solvability of the Dirichlet problem for the Poisson equations $\triangle U = G$ with sources in classes $G \in L^p$, $p > 1$, in Jordan domains with arbitrary boundary data that are measurable with respect to the logarithmic capacity. There we assumed that the domains satisfy the quasihyperbolic boundary condition by Gehring–Martio, generally speaking, without the known $(A)-$condition by Ladyzhenskaya–Ural'tseva and, in particular, without the outer cone condition that were standard for boundary-value problems in the PDE theory. Note that such Jordan domains cannot be even locally rectifiable.

With a view to further development of the theory of boundary value problems for semi-linear equations, the present paper is devoted to the Dirichlet problem with arbitrary measurable (over logarithmic capacity) boundary data for quasilinear Poisson equations in such Jordan domains.

For this purpose, it is first constructed completely continuous operators generating nonclassical solutions of the Dirichlet boundary-value problem with arbitrary measurable data for the Poisson equations $\triangle U = G$ with the sources $G \in L^p$, $p > 1$.

The latter makes it possible to apply the Leray-Schauder approach to the proof of theorems on the existence of regular nonclassical solutions of the measurable Dirichlet problem for quasilinear Poisson equations of the form $\triangle U(z) = H(z) \cdot Q(U(z))$ for multipliers $H \in L^p$ with $p > 1$ and continuous functions $Q : \mathbb{R} \to \mathbb{R}$ with $Q(t)/t \to 0$ as $t \to \infty$.

As consequences, we give applications to some concrete quasilinear equations of mathematical physics, arising under modelling various physical processes such as diffusion with absorption, plasma states, stationary burning etc. These results can be also applied to semi-linear equations of mathematical physics in anisotropic and inhomogeneous media.

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1. Introduction.

Recall that a path in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ terminating at $\zeta \in \partial\mathbb{D}$ is called **nontangential** if its part in a neighborhood of ζ lies inside of an angle in $\mathbb{D}$ with the vertex at ζ. Hence the limit along all nontangential paths at $\zeta \in \partial\mathbb{D}$ also named **angular** at the point. The latter is a traditional tool of the geometric function theory, see e.g. monographs [6, 9, 14, 19, 25, 29] and [30].

The research of boundary-value problems with arbitrary measurable data is due to the famous dissertation of Luzin, see its original text [24], and its reprint [25] with comments of his pupils Bari and Men'shov. Namely, he has established that, for each measurable a.e. finite $2\pi-$periodic function $\varphi(\vartheta):\mathbb{R}\to\mathbb{R}$, there is a harmonic function U in the unit disk $\mathbb{D}$ such that $U(z)\to\varphi(\vartheta)$ for a.e. ϑ as $z\to\zeta:=e^{i\vartheta}$ along all nontangential paths to $\partial\mathbb{D}$. The latter was based on his other deep result on the antiderivatives stated that, for any measurable function $\psi:[0,1]\to\mathbb{R}$, there is a continuous function $\Psi:[0,1]\to\mathbb{R}$ with $\Psi'=\psi$ a.e., see e.g. his papers [23] and [26], Theorem VII(2.3) in the Saks monograph [32].

Following this way, we proved in [16] Theorem 7 on the solvability of the Dirichlet problem for the Poisson equations $\triangle U = G$ with sources in classes $G\in L^p$, $p>1$, in Jordan domains with arbitrary boundary data that are measurable with respect to the logarithmic capacity. There we assumed that the domains satisfy the quasihyperbolic boundary condition by Gehring–Martio, generally speaking, without the known $(A)-$condition by Ladyzhenskaya–Ural'tseva and, in particular, without the outer cone condition that were standard for boundary-value problems in the PDE theory. Note that such Jordan domains cannot be even locally rectifiable.

With a view to further development of the theory of boundary value problems for semi-linear equations, the present paper is devoted to the Dirichlet problem with arbitrary measurable (over logarithmic capacity) boundary data for quasilinear Poisson equations in such Jordan domains.

For this purpose, it is first constructed completely continuous operators generating nonclassical solutions of the Dirichlet boundary-value problem with arbitrary measurable data for the Poisson equations $\triangle U = G$ with the sources $G\in L^p$, $p>1$.

The latter makes it possible to apply the Leray-Schauder approach to the proof of theorems on the existence of regular nonclassical solutions of the measurable Dirichlet problem for quasilinear Poisson equations of the form $\triangle U(z) = H(z)\cdot Q(U(z))$ for multipliers $H\in L^p$ with $p>1$ and continuous functions $Q:\mathbb{R}\to\mathbb{R}$ with $Q(t)/t\to 0$ as $t\to\infty$.

As consequences, we give applications to some concrete quasilinear equations of mathematical physics, arising under modelling various physical processes such as diffusion with absorption, plasma states, stationary burning etc. These results can be also applied to semi-linear equations of mathematical physics in anisotropic and inhomogeneous media.

2. Definitions and preliminary remarks.

First of all, recall that a **completely continuous** mapping from a metric space M_1 into a metric space M_2 is defined as a continuous mapping on M_1 which takes bounded subsets of M_1 into relatively compact ones of M_2, i.e. with compact closures in M_2. When a continuous mapping takes M_1 into a relatively compact subset of M_1, it is nowadays said to be **compact** on M_1.

Leray and Schauder extend as follows the Brouwer degree to compact perturbations of the identity I in a Banach space B, i.e. a complete normed linear space. Namely, given an open bounded set $\Omega\subset B$, a compact mapping $F:B\to B$ and $z\notin\Phi(\partial\Omega)$, $\Phi:=I-F$, the **(Leray–Schauder) topological degree** $\deg[\Phi,\Omega,z]$ of Φ in Ω over z is constructed from the Brouwer degree by approximating the mapping F over Ω

by mappings F_ε with range in a finite-dimensional subspace B_ε (containing z) of B. It is showing that the Brouwer degrees $\deg[\Phi_\varepsilon, \Omega_\varepsilon, z]$ of $\Phi_\varepsilon := I_\varepsilon - F_\varepsilon$, $I_\varepsilon := I|_{B_\varepsilon}$, in $\Omega_\varepsilon := \Omega \cap B_\varepsilon$ over z stabilize for sufficiently small positive ε to a common value defining $\deg[\Phi, \Omega, z]$ of Φ in Ω over z.

This topological degree "algebraically counts" the number of fixed points of $F(\cdot) - z$ in Ω and conserves the basic properties of the Brouwer degree as additivity and homotopy invariance. Now, let a be an isolated fixed point of F. Then the **local (Leray–Schauder) index** of a is defined by $\operatorname{ind}[\Phi, a] := \deg[\Phi, B(a, r), 0]$ for small enough $r > 0$. $\operatorname{ind}[\Phi, 0]$ is called by **index** of F. In particular, if $F \equiv 0$, correspondingly, $\Phi \equiv I$, then the index of F is equal to 1.

The fundamental Theorem 1 in [22] can be formulated in the following way:

Proposition 1. *Let B be a Banach space, and let $F(\cdot, \tau) : B \to B$ be a family of operators with $\tau \in [0, 1]$. Suppose that the following hypotheses hold:*

(H1) *$F(\cdot, \tau)$ is completely continuous on B for each $\tau \in [0, 1]$ and uniformly continuous with respect to the parameter $\tau \in [0, 1]$ on each bounded set in B;*

(H2) *the operator $F := F(\cdot, 0)$ has finite collection of fixed points whose total index is not equal to zero;*

(H3) *the collection of all fixed points of the operators $F(\cdot, \tau)$, $\tau \in [0, 1]$, is bounded in B.*

Then the collection of all fixed points of the family of operators $F(\cdot, \tau)$ contains a continuum along which τ takes all values in $[0, 1]$.

Let us also recall with the following analog of the Luzin theorem on the antiderivatives in terms of logarithmic capacity, see Theorem 3.1 in [10].

Lemma 1. *Let $\varphi : [a, b] \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity. Then there is a continuous function $\Phi : [a, b] \to \mathbb{R}$ with $\Phi'(x) = \varphi(x)$ q.e. on (a, b). Furthermore, Φ can be chosen with $\Phi(a) = \Phi(b) = 0$ and $|\Phi(x)| \le \varepsilon$ for all $x \in [a, b]$ under arbitrary prescribed $\varepsilon > 0$.*

Remark 1. In view of arbitrariness of $\varepsilon > 0$ in Lemma 1, for each φ, there is the infinite collection of such Φ. Furthermore, it is easy to see by Lemma 3.1 in [10] that the space of such functions Φ has the infinite dimension.

Corollary 1. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity. Then the space of continuous functions $\Phi : \partial\mathbb{D} \to [-1, 1]$ with $\Phi(1) = 0$, $|\Phi(\zeta)| \le \varepsilon$ for all $\zeta \in \partial\mathbb{D}$ under arbitrary prescribed $\varepsilon > 0$, and $\Phi'(e^{it}) = \varphi(e^{it})$ q.e. on $\mathbb{R}$ has the infinite dimension.*

On this basis, we obtain the following result, see e.g. Theorem 4.1 in [10].

Proposition 2. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity. Then there is a space of harmonic functions U in the unit disk $\mathbb{D}$ of the infinite dimension with the angular limits*

$$\lim_{z \to \zeta} u(z) \;=\; \varphi(\zeta) \qquad q.e. \text{ on } \partial\mathbb{D} \; . \tag{1}$$

Remark 2. By the proof of Theorem 4.1 in [10], $u(z) = \frac{\partial}{\partial\vartheta}\, U(z)$, where

$$U(re^{i\vartheta}) \;=\; \frac{1}{2\pi}\int\limits_{0}^{2\pi}\frac{1-r^2}{1-2r\cos(\vartheta-t)+r^2}\;\Phi(e^{it})\;dt\;, \tag{2}$$

i.e., for any function Φ from Corollary 1, u can be calculated in the explicit form

$$u(re^{i\vartheta}) \;=\; -\,\frac{r}{\pi}\int\limits_{0}^{2\pi}\frac{(1-r^2)\sin(\vartheta-t)}{(1-2r\cos(\vartheta-t)+r^2)^2}\;\Phi(e^{it})\;dt\;. \tag{3}$$

Later on, it was shown by Theorems 1 and 3 in [31] that the functions $u(z)$ can be represented as the **Poisson–Stieltjes integrals**

$$\mathbb{U}_\Phi(z) \;=\; \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}P_r(\vartheta-t)\;d\,\Phi(e^{it}) \qquad \forall\; z=re^{i\vartheta},\; r\in(0,1)\;,\; \vartheta\in[-\pi,\pi]\;, \tag{4}$$

where $P_r(\Theta)=(1-r^2)/(1-2r\cos\Theta+r^2), r<1, \Theta\in\mathbb{R}$, is the **Poisson kernel**.

The corresponding analytic functions $\mathcal{A}(z)$ in $\mathbb{D}$ with the real parts $u(z)$ can be represented as the **Schwartz–Stieltjes integrals**

$$\mathbb{S}_\Phi(z) \;=\; \frac{1}{2\pi}\int\limits_{\partial\mathbb{D}}\frac{\zeta+z}{\zeta-z}\;d\,\Phi(\zeta)\;, \qquad z\in\mathbb{D}\;, \tag{5}$$

because of the Poisson kernel is the real part of the (analytic in the variable z) **Schwartz kernel** $(\zeta+z)/(\zeta-z)$. Integrating (5) by parts, see Lemma 1 and Remark 1 in [31], we obtain also the more convenient form of the representation

$$\mathbb{S}_\Phi(z) \;=\; \frac{z}{\pi}\int\limits_{\partial\mathbb{D}}\frac{\Phi(\zeta)}{(\zeta-z)^2}\;d\,\zeta\;, \qquad z\in\mathbb{D}\;. \tag{6}$$

Note that by Corollary 1 the spaces of solutions of the Dirichlet problem in the classes of harmonic and analytic functions generating by integral operators $\mathbb{U}_\Phi$ and $\mathbb{S}_\Phi$, correspondingly, under each fixed boundary date φ that is measurable with respect to logarithmic capacity have the infinite dimension.

Next, given a bounded Borel set E in the plane $\mathbb{C}$, a **mass distribution** on E is a nonnegative completely additive function ν of a set defined on its Borel subsets with $\nu(E)=1$. The function

$$U^\nu(z) := \int\limits_{E}\log\left|\frac{1}{z-\zeta}\right|\,d\nu(\zeta) \tag{7}$$

is called a **logarithmic potential** of the mass distribution ν at a point $z \in \mathbb{C}$. A **logarithmic capacity** $C(E)$ of the bounded Borel set E is the quantity

$$C(E) = e^{-V} \ , \qquad V \ = \ \inf_{\nu} \ V_\nu(E) \ , \qquad V_\nu(E) \ = \ \sup_{z} \ U^\nu(z) \ . \tag{8}$$

It is also well-known the following geometric characterization of the logarithmic capacity, see e.g. the point 110 in [27]:

$$C(E) \ = \ \tau(E) \ := \ \lim_{n\to\infty} \ V_n^{\frac{2}{n(n-1)}} \tag{9}$$

where V_n denotes the supremum of the product

$$V(z_1, \ldots, z_n) \ = \ \prod_{k<l}^{l=1,\ldots,n} \ |z_k - z_l| \tag{10}$$

taken over all collections of points $z_1, \ldots, z_n$ in the set E. Following Fékete, see [11], the quantity $\tau(E)$ is called the **transfinite diameter** of the set E.

Remark 3. Thus, we see that if $C(E) = 0$, then $C(f(E)) = 0$ for an arbitrary mapping f that is Hölder continuous.

In order to introduce sets that are measurable with respect to logarithmic capacity, we define, following [5], **inner C_* and outer C^* capacities:**

$$C_*(E) \ : \ = \ \sup_{F\subseteq E} \ C(E), \qquad C^*(E) \ : \ = \ \inf_{E\subseteq O} \ C(O) \tag{11}$$

where supremum is taken over all compact sets $F \subset \mathbb{C}$ and infimum is taken over all open sets $O \subset \mathbb{C}$. A set $E \subset \mathbb{C}$ is called **measurable with respect to the logarithmic capacity** if $C^*(E) = C_*(E)$, and the common value of $C_*(E)$ and $C^*(E)$ is still denoted by $C(E)$.

A function $\varphi : E \to \mathbb{C}$ defined on a bounded set $E \subset \mathbb{C}$ is called **measurable with respect to logarithmic capacity** if, for all open sets $O \subseteq \mathbb{C}$, the sets

$$\Omega = \{z \in E : \varphi(z) \in O\} \tag{12}$$

are measurable with respect to logarithmic capacity. It is clear from the definition that the set E is itself measurable with respect to logarithmic capacity.

Note also that sets of logarithmic capacity zero coincide with sets of the so-called **absolute harmonic measure** zero introduced by Nevanlinna, see Chapter V in [27]. Hence a set E is of (Hausdorff) length zero if $C(E) = 0$, see Theorem V.6.2 in [27]. However, there exist sets of length zero having a positive logarithmic capacity, see e.g. Theorem IV.5 in [5].

Remark 4. It is known that Borel sets and, in particular, compact and open sets are measurable with respect to logarithmic capacity, see e.g. Lemma I.1 and Theorem III.7

in [5]. Moreover, as it follows from the definition, any set $E \subset \mathbb{C}$ of finite logarithmic capacity can be represented as a union of a sigma-compactum (union of countable collection of compact sets) and a set of logarithmic capacity zero. Thus, by Remark 3 functions that are measurable with respect to the logarithmic capacity are invariant under Hölder continuous mappings.

It is also known that the Borel sets and, in particular, compact sets are measurable with respect to all Hausdorff's measures and, in particular, with respect to measure of length, see e.g. Theorem II(7.4) in [32]. Consequently, any set $E \subset \mathbb{C}$ of finite logarithmic capacity is measurable with respect to measure of length. Thus, on such a set any function $\varphi : E \to \mathbb{C}$ being measurable with respect to logarithmic capacity is also measurable with respect to measure of length on E. However, there exist functions that are measurable with respect to measure of length but not measurable with respect to logarithmic capacity, see e.g. Theorem IV.5 in [5].

Later on, we use the **abbreviation q.e.** (**quasi-everywhere**) on a set $E \subset \mathbb{C}$ if the corresponding property holds only for all $\zeta \in E$ except its subset of zero logarithmic capacity, see e.g. [21] for this term.

3. On completely continuous Dirichlet operators.

Here we apply the designation of the **logarithmic (Newtonian) potential** $\mathcal{N}_G$ **of sources** $G \in L^p(\mathbb{C})$, $p > 1$, with compact supports given by the formula:

$$\mathcal{N}_G(z) \ := \ \frac{1}{2\pi} \int\limits_{\mathbb{C}} \ln|z-w|\, G(w)\ dm(w)\ , \tag{13}$$

where $dm(w)$ corresponds to the Lebesgue measure in the plane.

As known, N_G with G supported in $\mathbb{D}$ is continuous in $\mathbb{C}$, belongs to the class $W^{2,p}(\mathbb{D})$ and $\triangle N_G = G$ a.e. Moreover, $N_G \in W^{1,q}_{\rm loc}(\mathbb{C})$ for $q > 2$, consequently, N_G is locally Hölder continuous. If $G \in L^p(\mathbb{C})$, $p > 2$, then $N_G \in C^{1,\alpha}_{\rm loc}(\mathbb{C})$ for $\alpha := (p-2)/p$, and for all $\alpha \in (0,1)$ under $p = \infty$, see e.g. Lemma 3 in [16] or Theorem 2 in [17].

Furthermore, the collection $\{N_G\}$ is equicontinuous if the collection $\{G\}$ is bounded by the norm in $L^p(\mathbb{C})$. More precisely, on each compact set S in $\mathbb{C}$

$$\|N_G\|_C \ \leq \ M \cdot \|G\|_p\ , \tag{14}$$

where M is a constant depending only on S and, in particular, the restriction of N_G to $\overline{\mathbb{D}}$ is a completely continuous bounded linear operator, see e.g. Lemma 2 in [16] or Theorem 1 in [17].

By Proposition 2 there is a space of harmonic functions u in the unit disk $\mathbb{D}$ of the infinite dimension with the angular limits q.e. on $\partial\mathbb{D}$

$$\lim_{z\to\zeta}\ u(z) \ = \ \psi_G(\zeta) \ := \ \varphi(\zeta) \ - \ \varphi_G(\zeta)\ , \qquad \varphi_G(\zeta) := N_G(\zeta)\ . \tag{15}$$

Note that $U := u + N_G|_{\mathbb{D}}$ with such u are continuous solutions of the Poisson equation $\triangle U = G$ a.e. in the class $W^{2,p}_{\rm loc}(\mathbb{D})$ with the angular limits

$$\lim_{z\to\zeta}\ U(z) \ = \ \varphi(\zeta) \quad \text{ q.e. on } \partial\mathbb{D}\ . \tag{16}$$

By Remark 2 such a harmonic function $u : \mathbb{D} \to \mathbb{R}$ can be obtained in the form of the real part of the analytic function

$$\mathbb{S}_{\Psi}(z) \ := \ \frac{z}{\pi} \int\limits_{\partial\mathbb{D}} \frac{\Psi(\zeta)}{(\zeta - z)^2} \, d\zeta \ , \qquad z \in \mathbb{D} \ , \tag{17}$$

where Ψ is an antiderivative of the function ψ_G from Corollary 1.

Consequently, such a harmonic function u can be represented in the form

$$u(z) \ = \ u_0(z) \ - \ u_G(z) \ , \quad u_0(z) \ := \ \mathrm{Re}\, \mathbb{S}_{\Phi}(z) \ , \quad u_G(z) \ := \ \mathrm{Re}\, \mathbb{S}_{\Phi_G}(z) \ , \tag{18}$$

where Φ and Φ_G are antiderivatives of φ and φ_G in Corollary 1, correspondingly. Note that the harmonic function u_0 does not depend on the sources G at all.

Let us choose the function Φ_G in a suitable way to guarantee that the correspondence $G \mapsto u + N_G|_{\mathbb{D}}$ is a Dirichlet operator $\mathcal{D}_G$ that is completely continuous on compact sets in $\mathbb{D}$ generating solutions of the Poisson equation $\triangle U = G$ a.e. in the class $C \cap W^{2,p}_{\mathrm{loc}}(\mathbb{D})$ with the Dirichlet boundary condition (16).

Namely, the following function Φ_G is an antiderivative for the function φ_G:

$$\Phi_G(\zeta) \ := \ \int\limits_0^{\vartheta} N_G(e^{i\theta}) \, d\theta \ - \ S(\vartheta) \ , \quad \zeta = e^{i\vartheta}, \quad \theta \, , \vartheta \in [0, 2\pi] \ , \tag{19}$$

where $S : [0, 2\pi] \to \mathbb{C}$ is either zero or a singular function of the form

$$S(\vartheta) \ := \ C(\vartheta) \int\limits_0^{2\pi} N_G(e^{i\theta}) \, d\theta \ , \quad \zeta = e^{i\vartheta}, \quad \theta \, , \vartheta \in [0, 2\pi] \ , \tag{20}$$

with a singular function $C : [0, 2\pi] \to [0, 1]$ of the Cantor ladder type, i.e., C is continuous, nondecreasing, $C(0) = 0$, $C(2\pi) = 1$ and $C' = 0$ q.e. Recall that the existence of such functions C follows from Lemma 3.1 in [10].

Setting $u_G = \mathrm{Re}\, \mathbb{S}_{\Phi_G}$, it is easy to see by (14) that

$$|\Phi_G(\zeta)| \ \leq \ 4\pi M \cdot \|G\|_p \qquad \forall \, \zeta \in \partial\mathbb{D} \tag{21}$$

and by (6) that, for constants C_r and C_r^* depending only on $r \in (0, 1)$,

$$|u_G(z)| \ \leq \ |\mathbb{S}_{\Phi_G}(z)| \ \leq \ C_r \cdot \|G\|_p \ , \quad \forall \, z \in \mathbb{D}_r \ , \tag{22}$$

$$|u_G(z_1) - u_G(z_2)| \leq |\mathbb{S}_{\Phi_G}(z_1) - \mathbb{S}_{\Phi_G}(z_2)| \leq C_r^* \|G\|_p |z_1 - z_2|, z_1, z_2 \in \mathbb{D}_r. \tag{23}$$

Consequently, the operator $u_G := \mathrm{Re}\, \mathbb{S}_{\Phi_G}$ is completely continuous on compact sets in $\mathbb{D}$ by the Arzela-Ascoli theorem, see e.g. Theorem IV.6.7 in [8]. Thus, we obtain the next conclusion.

Lemma 2. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable over logarithmic capacity. Then there is a Dirichlet operator $\mathcal{D}_G$ over $G : \mathbb{D} \to \mathbb{C}$ in $L^p(\mathbb{D})$, $p > 1$, generating continuous solutions $U : \mathbb{D} \to \mathbb{R}$ of the Poisson equation $\triangle U = G$ in the class $W^{2,p}_{\rm loc}(\mathbb{D})$ with the Dirichlet boundary condition (16) in the sense of angular limits q.e. on $\partial\mathbb{D}$, that is completely continuous over $\mathbb{D}_r$ for each $r \in (0, 1)$.*

Remark 5. Note that the nonlinear operator $\mathcal{D}_G$ constructed above is not bounded except the trivial case $\Phi \equiv 0$ because then $\mathcal{D}_0 = \mathbb{S}_\Phi \neq 0$. However, the restriction of the operator $\mathcal{D}_G$ to $\mathbb{D}_r$ under each $r \in (0, 1)$ is bounded at infinity in the sense that $\max\limits_{z\in\mathbb{D}_r} |\mathcal{D}_G(z)| \leq M \cdot \|G\|_p$ for some $M > 0$ and all G with large enough $\|G\|_p$. Note also that by Corollary 1 we are able always to choose Φ for any φ, including $\varphi \equiv 0$, which is not identically 0 in the unit disk $\mathbb{D}$.

Moreover, by the above construction $U := \mathcal{D}_G$ belongs to the class $W^{1,q}_{\rm loc}(\mathbb{D})$ for some $q > 2$, consequently, U is locally Hölder continuous. If $G \in L^p(\mathbb{D})$, $p > 2$, then $U \in C^{1,\alpha}_{\rm loc}(\mathbb{D})$ for $\alpha := (p-2)/p$, and for all $\alpha \in (0, 1)$ under $p = \infty$.

4. The Dirichlet problem in the unit disc.

In this section we study the solvability of the Dirichlet problem for semi-linear Poisson equations of the form $\triangle U(z) = H(z) \cdot Q(U(z))$ in the unit disk $\mathbb{D}$.

Theorem 1. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to the logarithmic capacity. Suppose that $H : \mathbb{D} \to \mathbb{R}$ is a function in the class $L^p(\mathbb{D})$ for $p > 1$ with compact support in $\mathbb{D}$ and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with*

$$\lim_{t\to\infty} \frac{Q(t)}{t} \;=\; 0\ . \tag{24}$$

Then there is a function $U : \mathbb{D} \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc}(\mathbb{D})$ such that

$$\triangle U(z) \;=\; H(z) \cdot Q(U(z)) \qquad a.e.\ in\ \mathbb{D} \tag{25}$$

with the angular limits

$$\lim_{z\to\zeta} U(z) \;=\; \varphi(\zeta) \qquad q.e.\ on\ \ \partial\mathbb{D}\ . \tag{26}$$

Moreover, U belongs to the class $W^{1,q}_{\rm loc}(\mathbb{D})$ for some $q > 2$, consequently, U is locally Hölder continuous and, if $G \in L^p(\mathbb{D})$, $p > 2$, then $U \in C^{1,\alpha}_{\rm loc}(\mathbb{D})$ for $\alpha := (p-2)/p$, and for all $\alpha \in (0, 1)$ under $p = \infty$.

Proof. If $\|H\|_p = 0$ or $\|Q\|_C = 0$, then any harmonic function from Theorem 7.2 in [18] gives the desired solution of (25). Thus, we may assume that $\|H\|_p \neq 0$ and $\|Q\|_C \neq 0$. Set $Q_*(t) = \max\limits_{|\tau|\leq t} |Q(\tau)|$, $t \in \mathbb{R}^+ := [0, \infty)$. Then the function $Q_* : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing and, moreover, by (24)

$$\lim_{t\to\infty} \frac{Q_*(t)}{t} \;=\; 0\ . \tag{27}$$

By Lemma 2 and Remark 3 we obtain the family of operators $F(G;\tau) : L^p_H(\mathbb{D}) \to L^p_H(\mathbb{D})$, where $L^p_H(\mathbb{D})$ consists of functions $G \in L^p(\mathbb{D})$ with supports in the support of H,

$$F(G;\tau) \ := \ \tau H \cdot Q(\mathcal{D}_G) \qquad \forall\, \tau \in [0,1] \tag{28}$$

which satisfies hypothesis H1-H3 of Theorem 1 in [22], see Proposition 1. Indeed:

H1). First of all, by Lemma 2 the function $F(G;\tau) \in L^p_H(\mathbb{D})$ for all $\tau \in [0,1]$ and $G \in L^p_H(\mathbb{C})$ because the function $Q(\mathcal{D}_G)$ is continuous and, furthermore, the operators $F(\cdot\,;\tau)$ are completely continuous for each $\tau \in [0,1]$ and even uniformly continuous with respect to the parameter $\tau \in [0,1]$.

H2). The index of the operator $F(\cdot\,;0)$ is obviously equal to 1.

H3). Let us assume that solutions of the equations $G = F(G;\tau)$ is not bounded in $L^p_H(\mathbb{D})$, i.e., there is a sequence of functions $G_n \in L^p_H(\mathbb{D})$ with $\|G_n\|_p \to \infty$ as $n \to \infty$ such that $G_n = F(G_n;\tau_n)$ for some $\tau_n \in [0,1]$, $n = 1, 2, \ldots$. However, then by Remark 5 we have that, for some constant $M > 0$,

$$\|G_n\|_p \ \le \ \|H\|_p \; Q_*\left(M \,\|G_n\|_p\right)$$

and, consequently,

$$\frac{Q_*(M \,\|G_n\|_p)}{M\,\|G_n\|_p} \ \ge \ \frac{1}{M\,\|H\|_p} \ > \ 0 \tag{29}$$

for all large enough n. The latter is impossible in view of the condition (27). The obtained contradiction disproves the above assumption.

Thus, by Proposition 1 there is a function $G \in L^p_H(D)$ with $F(G;1) = G$, and by Lemma 2 the function $U := \mathcal{D}_G$ gives the desired solution of (25). The rest properties of the given solution U follows from Remark 5. $\Box$

Remark 6. By the construction in the above proof, $U = \mathcal{D}_G$, where $\mathcal{D}_G$ is the completely continuous Dirichlet operator described in the last section, and the support of G is in the support of H and the upper bound of $\|G\|_p$ depends only on $\|H\|_p$ and on the function Q. Moreover, $G : \mathbb{D} \to \mathbb{R}$ is a fixed point of the nonlinear operator $\Omega_G := H \cdot Q(\mathcal{D}_G) : L^p_H(\mathbb{D}) \to L^p_H(\mathbb{D})$, where $L^p_H(\mathbb{D})$ consists of functions G in $L^p(\mathbb{D})$ with supports in the support of H.

5. On quasihyperbolic boundary condition.

Further, it is said that a domain D satisfies the **quasihyperbolic boundary condition** *by Gehring-Martio*, see [12], if there exist some constants a and b and a point $z_0 \in D$ such that

$$k_D(z, z_0) \ \le \ a \ + \ b \ \ln \frac{d(z_0, \partial D)}{d(z, \partial D)} \qquad \forall\, z \in D\,, \tag{30}$$

where

$$k_D(z, z_*) \ := \ \inf_{\gamma} \int\limits_{\gamma} \frac{ds}{d(\zeta, \partial D)} \qquad \forall\, z, z_* \in D$$

is the **quasihyperbolic distance** *by Gehring-Palka*, see [13]. Here $d(\zeta, \partial D)$ denotes the Euclidean distance from the point $\zeta \in D$ to ∂D and the infimum is taken over all rectifiable curves γ joining the points z and z_* in D.

Recall that by the discussion in [18], every smooth (or Lipschitz) domain satisfies the quasihyperbolic boundary condition but such boundaries can be even nowhere locally rectifiable.

Note that it is well–known the so-called, $(A)-$*condition by Ladyzhenskaya–Ural'tseva*, which is standard in the theory of boundary-value problems for PDE, see e.g. 1.1.3 in [20]. Recall that a domain D in $\mathbb{R}^n$, $n \geq 2$, is called satisfying **(A)-condition** if

$$\text{mes}\ D \cap B(\zeta, \rho) \ \leq\ \Theta_0 \,\text{mes}\ B(\zeta, \rho) \qquad \forall\ \zeta \in \partial D\ ,\ \rho \leq \rho_0 \tag{31}$$

for some Θ_0 and $\rho_0 \in (0, 1)$, where $B(\zeta, \rho)$ denotes the ball with the center $\zeta \in \mathbb{R}^n$ and the radius ρ.

Recall also that a domain D in $\mathbb{R}^n$, $n \geq 2$, is said to be satisfying the **outer cone condition** if there is a cone that makes possible to be touched by its top to every boundary point of D from the completion of D after its suitable rotations and shifts. It is clear that the outer cone condition implies (A)–condition.

Probably one of the simplest examples of a domain D with the quasihyperbolic boundary condition and without (A)–condition is the union of 3 open disks with the radius 1 centered at the points 0 and $1 \pm i$. It is clear that this domain has zero interior angle at its boundary point 1.

Theorem 2. *Let D be a Jordan domain in $\mathbb{C}$ with the quasihyperbolic boundary condition, ∂D have a tangent q.e. and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose that $H : D \to \mathbb{R}$ is a function in the class $L^p(D)$ for $p > 1$ with compact support in D and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with

$$\lim_{t \to \infty} \frac{Q(t)}{t} \ =\ 0\ . \tag{32}$$

Then there is a continuous function $U : D \to \mathbb{R}$ in class $W^{2,p}_{\text{loc}}(D)$ such that

$$\triangle U(\xi) \ =\ H(\xi) \cdot Q(U(\xi)) \qquad \text{a.e. in } D \tag{33}$$

with the angular limits

$$\lim_{\xi \to \omega} U\ (\xi) \ =\ \varphi(\omega) \qquad \text{q.e. on }\ \partial D\ . \tag{34}$$

Moreover, U belongs to the class $W^{1,q}_{\text{loc}}(D)$ for some $q > 2$, consequently, U is locally Hölder continuous in D and, if $G \in L^p(D)$, $p > 2$, then $U \in C^{1,\alpha}_{\text{loc}}(D)$ for $\alpha := (p-2)/p$, and for all $\alpha \in (0, 1)$ under $p = \infty$.

Proof. Let c be a conformal mapping of D onto $\mathbb{D}$ that exists by the Riemann mapping theorem, see e.g. Theorem II.2.1 in [14]. Now, by the Caratheodory theorem, see e.g. Theorem II.3.4 in [14], c is extended to a homeomorphism $\tilde{c}$ of $\overline{D}$ onto $\overline{\mathbb{D}}$. Furthermore, by Corollary of Theorem 1 in [4], $c_* := \tilde{c}|_{\partial D} : \partial D \to \partial\mathbb{D}$ and its inverse function are Hölder continuous. Then $\tilde{\varphi} := \varphi \circ c_*^{-1}$ is measurable with respect to the logarithmic capacity by Remark 4.

Now, set $\tilde{H} = |C'|^2 \cdot H \circ C$, where C is the inverse conformal mapping $C := c^{-1} : \mathbb{D} \to D$. Then it is clear by the hypothesis of Theorem 2 that $\tilde{H}$ has compact support in $\mathbb{D}$ and belongs to the class $L^p(\mathbb{D})$. Consequently, by Theorem 1 there is a continuous function $\tilde{U} : \mathbb{D} \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc}(\mathbb{D})$ such that

$$\triangle\tilde{U}(z) \ = \ \tilde{H}(z) \cdot Q(\tilde{U}(z)) \qquad \text{a.e. in } \mathbb{D} \tag{35}$$

with the angular limits

$$\lim_{z\to\zeta} \tilde{U}(z) \ = \ \tilde{\varphi}(\zeta) \qquad \text{q.e. on } \ \partial\mathbb{D}\ , \tag{36}$$

moreover, $\tilde{U} = \mathcal{D}_{\tilde{G}}$, where $\mathcal{D}_{\tilde{G}}$ is the completely continuous Dirichlet operator described in Section 3, and the support of $\tilde{G}$ is in the support of $\tilde{H}$ and the upper bound of $\|\tilde{G}\|_p$ depends only on $\|\tilde{H}\|_p$ and on the function Q.

Next, setting $U = \tilde{U} \circ c$, by simple calculations, see e.g. Section 1.C in [1], we obtain that $\triangle U = |c'|^2 \cdot \triangle\tilde{U} \circ c$ and, consequently, the continuous function $U : D \to \mathbb{C}$ is in the class $W^{1,p}_{\rm loc}(D)$ that satisfies equation (33) a.e. and, moreover, $U(\xi) = \mathcal{D}_{\tilde{G}}(c(\xi))$, where $\mathcal{D}_{\tilde{G}}$ is the completely continuous Dirichlet operator from Section 3. Hence, by Remark 5, U belongs to the class $W^{1,q}_{\rm loc}(D)$ for some $q > 2$, consequently, U is locally Hölder continuous in D and, if $G \in L^p(D)$, $p > 2$, then $U \in C^{1,\alpha}_{\rm loc}(D)$ for $\alpha := (p-2)/p$, and for all $\alpha \in (0,1)$ under $p = \infty$.

It remains to show that (36) implies (34). Indeed, by the Lindelöf theorem, see e.g. Theorem II.C.2 in [19], if ∂D has a tangent at a point ω, then arg $[c_*(\omega) - c(\xi)]$ − arg $[\omega - \xi] \to$ const as $\xi \to \omega$. In other words, the images under the conformal mapping c of sectors in D with a vertex at $\omega \in \partial D$ is asymptotically the same as sectors in $\mathbb{D}$ with a vertex at $\zeta = c_*(\omega) \in \partial\mathbb{D}$. Consequently, nontangential paths in D are transformed under c into nontangential paths in $\mathbb{D}$ and inversely q.e. on ∂D and $\partial\mathbb{D}$, respectively, because ∂D has a tangent q.e. and c_* and c_*^{-1} keep sets of logarithmic capacity zero. □

Remark 7. By the construction in the above proof, $U(\xi) = \mathcal{D}_{\tilde{G}}(c(\xi))$, where c is a conformal mapping of D onto $\mathbb{D}$, $\mathcal{D}_{\tilde{G}}$ is the completely continuous Dirichlet operator described in Section 3 and $\tilde{G} : \mathbb{D} \to \mathbb{R}$ is a fixed point of the nonlinear operator $\tilde{\Omega}_{G_*} := \tilde{H} \cdot Q(\mathcal{D}_{G_*}) : L^p_{\tilde{H}}(\mathbb{D}) \to L^p_{\tilde{H}}(\mathbb{D})$, where $L^p_{\tilde{H}}(\mathbb{D})$ consists of functions G_* in $L^p(\mathbb{D})$ with supports in the support of $\tilde{H} := |C'|^2 \cdot H \circ C$, where C is the inverse conformal mapping $C := c^{-1} : \mathbb{D} \to D$.

6. Dirichlet problem in physical applications.

Theorem 2 on the Dirichlet problem for quasilinear Poisson equations with arbitrary measurable boundary data over the logarithmic capacity can be applied to mathematical problems appearing under modeling various types of physical and chemical absorption with diffusion, plasma states, stationary burning etc.

The first circle of such applications is relevant to reaction-diffusion problems. Problems of this type are discussed in [7], p. 4, and, in detail, in [2]. A nonlinear system is obtained for the density U and the temperature T of the reactant. Upon eliminating T the system can be reduced to equations of the type (33),

$$\triangle U \;=\; \sigma \cdot Q(U) \tag{37}$$

with $\sigma > 0$ and, for isothermal reactions, $Q(U) = U^{\beta}$ where $\beta > 0$ that is called the order of the reaction. It turns out that the density of the reactant U may be zero in a subdomain called a **dead core.** A particularization of results in Chapter 1 of [7] shows that a dead core may exist just if and only if $\beta \in (0, 1)$ and σ is large enough, see also the corresponding examples in [15]. In this connection, the following statements may be of independent interest.

Corollary 2. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e. and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose that $H : D \to \mathbb{R}$ is a function in the class $L^p(D)$ for $p > 1$ with compact support in D.

Then there is a solution $U : D \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc}(D) \cap W^{1,q}_{\rm loc}(D)$ for some $q > 2$, consequently, locally Hölder continuous of the equation

$$\triangle U(\xi) \;=\; H(\xi) \cdot U^{\beta}(\xi)\ , \quad 0 \;<\; \beta \;<\; 1\ , \qquad a.e.\ in\ D \tag{38}$$

satisfying the Dirichlet boundary condition

$$\lim_{\xi \to \omega} U(\xi) \;=\; \varphi(\omega) \qquad q.e.\ on\ \ \partial D \tag{39}$$

in the sense of the angular limits, i.e., along all nontangent paths.

In addition, if $G \in L^p(D)$, $p > 2$, then $U \in C^{1,\alpha}_{\rm loc}(D)$ for $\alpha := (p-2)/p$, and for all $\alpha \in (0, 1)$ under $p = \infty$.

Note also that certain mathematical models of a thermal evolution of a heated plasma lead to nonlinear equations of the type (37). Indeed, it is known that some of them have the form $\triangle \psi(u) = f(u)$ with $\psi'(0) = \infty$ and $\psi'(u) > 0$ if $u \neq 0$ as, for instance, $\psi(u) = |u|^{q-1}u$ under $0 < q < 1$, see e.g. [7]. With the replacement of the function $U = \psi(u) = |u|^q \cdot \operatorname{sign} u$, we have that $u = |U|^Q \cdot \operatorname{sign} U$, $Q = 1/q$, and, with the choice $f(u) = |u|^{q^2} \cdot \operatorname{sign} u$, we come to the equation $\triangle U = |U|^q \cdot \operatorname{sign} U = \psi(U)$.

Corollary 3. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e. and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose also that $H : D \to \mathbb{R}$ is a function in the class $L^p(D)$ for $p > 1$ with compact support in D.

Then there is a solution $U : D \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc}(D) \cap W^{1,q}_{\rm loc}(D)$ for some $q > 2$, consequently, locally Hölder continuous of the equation

$$\triangle U(\xi) \ = \ H(\xi) \cdot |U(\xi)|^{\beta-1} U(\xi) \ , \quad 0 \ < \ \beta \ < \ 1 \ , \qquad a.e. \ in \ D \tag{40}$$

with the Dirichlet boundary condition (39) in the sense of the angular limits.

In addition, if $G \in L^p(D)$, $p > 2$, then $U \in C^{1,\alpha}_{\rm loc}(D)$ for $\alpha := (p-2)/p$, and for all $\alpha \in (0,1)$ under $p = \infty$.

Finally, we recall that in the combustion theory, see e.g. [3, 28] and the references therein, the following model equation

$$\frac{\partial u(z,t)}{\partial t} \ = \ \frac{1}{\delta} \cdot \triangle u \ + \ e^u \ , \quad t \geq 0, \ z \in D, \tag{41}$$

takes a special place. Here $u \geq 0$ is the temperature of the medium and δ is a certain positive parameter. We restrict ourselves here by the stationary case, although our approach makes it possible to study the parabolic equation (41), see [15]. Namely, the corresponding equation of the type (33) is appeared here after the replacement of the function u by $-u$ with the function $Q(u) = e^{-u}$ that is bounded at all.

Corollary 4. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e. and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose also that $H : D \to \mathbb{R}$ is a function in the class $L^p(D)$ for $p > 1$ with compact support in D.

Then there is a solution $U : D \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc}(D) \cap W^{1,q}_{\rm loc}(D)$ for some $q > 2$, consequently, locally Hölder continuous of the equation

$$\triangle U(\xi) \ = \ H(\xi) \cdot e^{U(\xi)} \ , \qquad a.e. \ in \ D \tag{42}$$

with the Dirichlet boundary condition (39) in the sense of the angular limits.

In addition, if $G \in L^p(D)$, $p > 2$, then $U \in C^{1,\alpha}_{\rm loc}(D)$ for $\alpha := (p-2)/p$, and for all $\alpha \in (0,1)$ under $p = \infty$.

Finally, due to the factorization theorem in [15], we are able by the quasiconformal replacements of variables to extend the above results to semi–linear equations of the Poisson type describing the corresponding physical phenomena in anisotropic and inhomogeneous media, too, that shall be published elsewhere.

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Задача Діріхле з вимірюваними даними для квазілінійних рівнянь Пуассона.

Дослідження задачі Діріхле з довільними вимірними граничними даними для гармонічних функцій в одиничному крузі $\mathbb{D}$ починається з відомої дисертації Лузіна, див., напр., [25]. Його результат був сформульований в термінах кутових границь (тобто уздовж недотичних шляхів до точок межи $\mathbb{D}$), які є традиційним інструментом для дослідження граничної поведінки в геометричній теорії функцій. Слідуючи цим шляхом, ми довели в [16] Теорему 7 про розв'язність задачі Діріхле для рівняня Пуассона $\triangle U = G$ з джерелами в класах GL^p, $p > 1$, в жорданових областях з довільними граничними даними, вимірними відносно логарифмічної ємності. Тоді ми припускали, що області задовольняють квазігіперболічну граничну умову Геринга–Мартіо, взагалі кажучи, без відомої $(A)-$умови Ладиженської–Уральцевої і, зокрема, без умови зовнішнього конуса, які були стандартними для крайових задач в теорії рівнянь з частинними похідними. Звертаємо увагу, що межи таких жорданових областей можуть бути навіть локально не спрямлювані. З метою подальшого розвитку теорії крайових задач для напівлінійних рівнянь, ця стаття присвячена задачі Діріхле з довільними вимірними (відносно логарифмічної ємності) граничними даними для квазілінійних рівнянь Пуассона в таких жорданових областях. Для цього ми спочатку будуємо цілком неперервні оператори, які генерують некласичні розв'язки задачі Діріхле з довільними вимірними даними для рівняння Пуассона $\triangle U = G$ з джерелами $G \in L^p$, $p > 1$. Останнє дозволяє застосувати підхід Лере–Шаудера до довендення теорем про існування регулярних некласичних розв'язків вимірної задачі Діріхле для квазілінійних рівнянь Пуассона вигляду $\triangle U(z) = H(z) \cdot Q(U(z))$ з множниками $h \in L^p$ з $p > 1$ і неперервними функціями $q : \mathbb{R} \to \mathbb{R}$ з $q(t)/t \to 0$ як $t \to \infty$. Наприкінці наведемо застосування до деяких конкретних квазілінейних рівнянь математичної фізики, що виникають при моделюванні різних фізичних процесів, таких як дифузія з абсорбцією, стан плазми, стаціонарне горіння і т.д. Ці результати також можуть бути застосовані до напівлінійних рівнянь математичної фізики в анізотропних і неоднорідних середовищах.

Ключові слова: *логарифмічна ємність, квазілінійне рівняння Пуассона, нелінійні джерела, задача Діріхле, вимірні граничні дані, кутові границі, недотичні шляхи.*

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