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ON THE ASYMPTOTIC BEHAVIOR AT INFINITY OF RING Q-HOMEOMORPHISMS WITH RESPECT TO p-MODULUS

We study the asymptotic behavior at infinity of ring Q-homeomorphisms with respect to p-modulus for $p > n$. We obtain an analogue of the Martio–Rickman–Väisälä theorem on the growth at infinity of quasi-regular mappings. Examples of mappings are constructed that show the accuracy of the estimates obtained in the main theorem.

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1. Introduction.

Let us recall some definitions, see [37]. Let Γ be a family of paths γ in $\mathbb{R}^n$, $n \geqslant 2$. A Borel function $\rho : \mathbb{R}^n \to [0,\infty]$ is called an *admissible* for a family Γ of paths γ in $\mathbb{R}^n$, (abbr. $\rho \in \operatorname{adm}\Gamma$), if

$$\int\limits_{\gamma} \rho(x)\, ds \;\geqslant\; 1$$

holds for any locally rectifiable path $\gamma \in \Gamma$.

Let $p \in (1,\infty)$. The quantity

$$M_p(\Gamma) \;=\; \inf_{\rho\in\operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)\, dm(x)$$

is called *p–modulus* of the family Γ. Here $dm(x)$ corresponds to the Lebesgue measure in $\mathbb{R}^n$, $n \geqslant 2$.

Let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 2$, and let $Q : D \to [0,\infty]$ be a measurable function. A homeomorphism $f : D \to D'$ is called a *Q-homeomorphism with respect to p-modulus* if

$$M_p(f\Gamma) \leqslant \int\limits_{D} Q(x)\, \rho^p(x)\; dm(x) \tag{1}$$

for every family Γ of curves in D and every admissible function ρ for Γ.

This conception is a natural generalization of the geometric definition of a quasiconformal mapping: if $Q(x) \leqslant K < \infty$ a.e., then f is quasiconformal for $p = 2$ in $\mathbb{C}$, (see definition A, p. 21–22 in [1]), for $p = n$ in $\mathbb{R}^n$, $n \geqslant 2$, (see 13.1 and 34.6 in [37]), and has local Lipschitz property, for $n - 1 < p < n$, (see [5]). Note that the estimate of the type (1) was first established in the classical quasiconformal theory, (see [18, p. 221]). Next, it was obtain in [2], Lemma 2.1, for quasiconformal mappings in space $\mathbb{R}^n$, $n \geqslant 2$.

The class of Q-homeomorphisms with respect to the n-modulus was first considered in the papers [21–23], see also the monograph [25]. The main goal of the theory of Q-homeomorphisms is to clear up various interconnections between properties of the majorant $Q(x)$ and the corresponding properties of the mappings themselves. In particular, the problem of the local and boundary behavior of Q-homeomorphisms has been studied in $\mathbb{R}^n$ first in the case $Q \in BMO$ (bounded mean oscillation) in the papers [22–24] and then in the case of $Q \in FMO$ (finite mean oscillation) and other cases in the papers [10, 11, 27, 31].

For arbitrary sets E, F and G of $\mathbb{R}^n$ we denote by $\Delta(E, F, G)$ a set of all continuous curves $\gamma : [a, b] \to \mathbb{R}^n$ that connect E and F in G, i.e., such that $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in G$ for $a < t < b$.

Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, $x_0 \in D$ and $d_0 = \mathrm{dist}(x_0, \partial D)$. Set

$$\mathbb{A}(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\},$$

$$S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}, \quad i = 1, 2.$$

The following concept generalizes and localizes the concept of a Q-homeomorphism. It is motivated by the ring definition of quasiconformal mappings in the sense of Gehring (see [4]), introduced originally by V. Ryazanov, U. Srebro, and E. Yakubov on the plane, and later extended by V. Ryazanov and S. Sevost'yanov in the space $\mathbb{R}^n$, $n \geqslant 2$, (see [25, 29], Chapters VII and XI).

Let a function $Q : D \to [0, \infty]$ be Lebesgue measurable. We say that a homeomorphism $f : D \to \mathbb{R}^n$ is ring Q-homeomorphism with respect to p-modulus at $x_0 \in D$ if the relation

$$M_p(\Delta(fS_1, fS_2, fD)) \leqslant \int\limits_{\mathbb{A}} Q(x)\, \eta^p(|x - x_0|)\, dm(x) \tag{2}$$

holds for any ring $\mathbb{A} = \mathbb{A}(x_0, r_1, r_2)$, $0 < r_1 < r_2 < d_0$, $d_0 = \mathrm{dist}(x_0, \partial D)$, and for any measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr = 1.$$

The theory of ring Q-homeomorphisms for $p = n$ was studied in works [25, 28–30], for $1 < p < n$ in works [6–8, 32–35] and for $p > n$ in works [13–17]. In this paper, we obtain an analogue by Martio–Rickman–Väisälä's theorem on the growth at infinity of quasi-regular mappings, (see [20]).

The theory of ring Q-homeomorphisms can be applied to mappings of finite distortion belonging to the Orlicz– Sobolev classes $W^{1,\varphi}_{\mathrm{loc}}$ under the Calderon condition, and, in particular, to the Sobolev classes $W^{1,p}_{\mathrm{loc}}$ with $p > n - 1$, (see [6, 12]).

Denote by ω_{n-1} the area of the unit sphere $\mathbb{S}^{n-1}=\{x\in\mathbb{R}^n:\,|x|=1\}$ in $\mathbb{R}^n$ and by

$$q_{x_0}(r)=\frac{1}{\omega_{n-1}\,r^{n-1}}\int\limits_{S(x_0,r)}Q(x)\,d\mathcal{A}$$

the integral mean over the sphere $S(x_0,r)=\{x\in\mathbb{R}^n:\,|x-x_0|=r\}$, here $d\mathcal{A}$ is the element of the surface area.

Now we formulate a criterion which guarantees for a homeomorphism to be the ring Q-homeomorphisms with respect to p-modulus for $p>1$ in $\mathbb{R}^n$, $n\geqslant 2$.

Proposition 1. *Let D be a domain in $\mathbb{R}^n$, $n\geqslant 2$, and let $Q:D\to[0,\infty]$ be a Lebesgue measurable function such that $q_{x_0}(r)\neq\infty$ for a.e. $r\in(0,d_0)$, $d_0=\operatorname{dist}(x_0,\partial D)$. A homeomorphism $f:D\to\mathbb{R}^n$ is ring Q-homeomorphism with respect to p-modulus at a point $x_0\in D$ if and only if the quantity*

$$M_p\left(\Delta(fS_1,fS_2,f\mathbb{A})\right)\leqslant\frac{\omega_{n-1}}{\left(\int\limits_{r_1}^{r_2}\frac{dr}{r^{\frac{n-1}{p-1}}\,q_{x_0}^{\frac{1}{p-1}}(r)}\right)^{p-1}}$$

holds for any $0<r_1<r_2<d_0$ (see [33], Theorem 2.3).

Following the paper [19], a pair $\mathcal{E}=(A,C)$ where $A\subset\mathbb{R}^n$ is an open set and C is a nonempty compact set contained in A, is called *condenser*. We say that a condenser $\mathcal{E}=(A,C)$ lies in a domain D if $A\subset D$. Clearly, if $f:D\to\mathbb{R}^n$ is a homeomorphism and $\mathcal{E}=(A,C)$ is a condenser in D then (fA,fC) is also condenser in fD. Further, we denote $f\mathcal{E}=(fA,fC)$.

Let $\mathcal{E}=(A,C)$ be a condenser. Denote by $\mathcal{C}_0(A)$ a set of continuous functions $u:A\to\mathbb{R}^1$ with compact support. Let $\mathcal{W}_0(\mathcal{E})=\mathcal{W}_0(A,C)$ be a family of nonnegative functions $u:A\to\mathbb{R}^1$ such that 1) $u\in\mathcal{C}_0(A)$, 2) $u(x)\geqslant 1$ for $x\in C$ and 3) u belongs to the class ACL and

$$|\nabla u|=\left(\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2\right)^{\frac{1}{2}}.$$

For $p\geqslant 1$ the quantity

$$\operatorname{cap}_p\mathcal{E}=\operatorname{cap}_p(A,C)=\inf_{u\in\mathcal{W}_0(\mathcal{E})}\int\limits_{A}|\nabla u|^p\,dm(x)$$

is called *p-capacity* of the condenser $\mathcal{E}$. It is known that for $p>1$

$$\operatorname{cap}_p\mathcal{E}=M_p(\Delta(\partial A,\partial C;A\setminus C)),\tag{3}$$

see in [36], Theorem 1. For $p>n$ the inequality

$$\operatorname{cap}_p(A,C)\geqslant n\,\Omega_n^{\frac{p}{n}}\left(\frac{p-n}{p-1}\right)^{p-1}\left[m^{\frac{p-n}{n(p-1)}}(A)-m^{\frac{p-n}{n(p-1)}}(C)\right]^{1-p}\tag{4}$$

holds where Ω_n is a volume of the unit ball in $\mathbb{R}^n$ (see, e.g., the inequality 8.7 in [26]).

Let us recall the so-called isodiametric inequality or Bieberbach inequality (1915), see Corollary 2.10.33 in [3]. Here and in what follows, $\mathrm{diam}\,(\cdot)$ denotes the Euclidean diameter in $\mathbb{R}^n, n \geqslant 2$.

Proposition 2. *Let E be a compact set in $\mathbb{R}^n, n \geqslant 2$. Then*

$$m(E) \leqslant 2^{-n}\Omega_n\,(\mathrm{diam}\,E)^n,$$

where Ω_n is a volume of the unit ball in $\mathbb{R}^n$.

2. Main results.

Now we consider the main result of our paper on the behavior at infinity of ring Q-homeomorphisms with respect to p-modulus for $p > n$. The case $p = n$ was studied in the work [30].

Theorem 1. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a ring Q-homeomorphism with respect to p-modulus at a point x_0 with $p > n$ where x_0 is some point in $\mathbb{R}^n$ and for some numbers $r_0 > 0$, $\kappa = \kappa(x_0) > 0$ the condition*

$$q_{x_0}(t) \leqslant \kappa\, t^{\alpha} \tag{5}$$

holds for a.e. $t \in [r_0, +\infty)$. If $\alpha \in [0, p-n)$ then

$$\liminf_{R\to\infty} \frac{\mathrm{diam}\, f\,(B(x_0,R))}{R^{\frac{p-n-\alpha}{p-n}}} \geqslant 2\,\kappa^{\frac{1}{n-p}} \left(\frac{p-n}{p-n-\alpha}\right)^{\frac{p-1}{p-n}} > 0\,.$$

If $\alpha = p - n$ then

$$\liminf_{R\to\infty} \frac{\mathrm{diam}\, f\,(B(x_0,R))}{(\ln R)^{\frac{p-1}{p-n}}} \geqslant 2\,\kappa^{\frac{1}{n-p}} \left(\frac{p-n}{p-1}\right)^{\frac{p-1}{p-n}} > 0\,,$$

where $B(x_0,R) = \{x \in \mathbb{R}^n : |x - x_0| \leqslant R\}$.

Proof. Consider a condenser $\mathcal{E} = (A, C)$ in $\mathbb{R}^n$, where $A = \{x \in \mathbb{R}^n : |x - x_0| < R\}$, $C = \{x \in \mathbb{R}^n : |x - x_0| \leqslant r_0\}$, $0 < r_0 < R < \infty$. Then $f\mathcal{E} = (fA, fC)$ is a ringlike condenser in $\mathbb{R}^n$ and by (3) we have equality

$$\mathrm{cap}_p\, f\mathcal{E} = \mathrm{M}_p\left(\Delta(\partial fA, \partial fC; f(A\setminus C))\right).$$

Due to the inequality (4)

$$\mathrm{cap}_p\,(fA, fC) \geqslant n\,\Omega_n^{\frac{p}{n}} \left(\frac{p-n}{p-1}\right)^{p-1} \left[m^{\frac{p-n}{n(p-1)}}(fA) - m^{\frac{p-n}{n(p-1)}}(fC)\right]^{1-p}$$

we obtain

$$\mathrm{cap}_p\,(fA, fC) \geqslant n\,\Omega_n^{\frac{p}{n}} \left(\frac{p-n}{p-1}\right)^{p-1} \left[m(fA)\right]^{\frac{n-p}{n}}\,. \tag{6}$$

On the other hand, by Proposition 1, one gets

$$\mathrm{cap}_p\,(fA, fC) \leqslant \frac{\omega_{n-1}}{\left(\int\limits_{r_0}^{R} \frac{dt}{t^{\frac{n-1}{p-1}}\, q_{x_0}^{\frac{1}{p-1}}(t)}\right)^{p-1}}\,. \tag{7}$$

Combining the inequalities (6) and (7), we obtain

$$n\,\Omega_n^{\frac{p}{n}} \left(\frac{p-n}{p-1}\right)^{p-1} [m(fA)]^{\frac{n-p}{n}} \leqslant \frac{\omega_{n-1}}{\left(\int\limits_{r_0}^{R} \frac{dt}{t^{\frac{n-1}{p-1}}\, q_{x_0}^{\frac{1}{p-1}}(t)}\right)^{p-1}}\,.$$

Due to $\omega_{n-1} = n\,\Omega_n$, the last inequality can be rewritten as

$$\Omega_n^{\frac{p}{n}-1} \left(\frac{p-n}{p-1}\right)^{p-1} [m(fA)]^{\frac{n-p}{n}} \leqslant \left(\int\limits_{r_0}^{R} \frac{dt}{t^{\frac{n-1}{p-1}}\, q_{x_0}^{\frac{1}{p-1}}(t)}\right)^{1-p}\,. \tag{8}$$

Consider a case when $\alpha \in [0, p-n)$. Then from the condition (5) the estimate

$$\Omega_n^{\frac{p}{n}-1} \left(\frac{p-n}{p-1}\right)^{p-1} [m(fA)]^{\frac{n-p}{n}} \leqslant \kappa \left(\frac{p-n-\alpha}{p-1}\right)^{p-1} \left(R^{\frac{p-n-\alpha}{p-1}} - r_0^{\frac{p-n-\alpha}{p-1}}\right)^{1-p}$$

holds. Therefore

$$m(fA) \geqslant \Omega_n\,\kappa^{\frac{n}{n-p}} \left(\frac{p-n}{p-n-\alpha}\right)^{\frac{n(p-1)}{p-n}} \left(R^{\frac{p-n-\alpha}{p-1}} - r_0^{\frac{p-n-\alpha}{p-1}}\right)^{\frac{n(p-1)}{p-n}}\,.$$

Hence, by Proposition 2, we have

$$\mathrm{diam}\, f\,(B(x_0, R)) \geqslant 2\,\kappa^{\frac{1}{n-p}} \left(\frac{p-n}{p-n-\alpha}\right)^{\frac{p-1}{p-n}} \left(R^{\frac{p-n-\alpha}{p-1}} - r_0^{\frac{p-n-\alpha}{p-1}}\right)^{\frac{p-1}{p-n}}\,.$$

Dividing the last inequality by $R^{\frac{p-n-\alpha}{p-n}}$ and taking the lower limit as $R \to \infty$, we conclude

$$\liminf_{R\to\infty} \frac{\mathrm{diam}\, f\,(B(x_0, R))}{R^{\frac{p-n-\alpha}{p-n}}} \geqslant 2\,\kappa^{\frac{1}{n-p}} \left(\frac{p-n}{p-n-\alpha}\right)^{\frac{p-1}{p-n}}\,.$$

Now we consider a case when $\alpha = p - n$. Then from (8) we get

$$\Omega_n^{\frac{p}{n}-1} \left(\frac{p-n}{p-1}\right)^{p-1} [m(fA)]^{\frac{n-p}{n}} \leqslant \kappa \left(\ln\frac{R}{r_0}\right)^{1-p}\,.$$

Therefore

$$m(fB(x_0, R)) \geqslant \Omega_n\,\kappa^{\frac{n}{n-p}} \left(\frac{p-n}{p-1}\right)^{\frac{n(p-1)}{p-n}} \left(\ln\frac{R}{r_0}\right)^{\frac{n(p-1)}{p-n}}\,.$$

Hence, by Proposition 2, we obtain

$$\operatorname{diam} f\,(B(x_0,R)) \geqslant 2\,\kappa^{\frac{1}{n-p}}\,\left(\frac{p-n}{p-1}\right)^{\frac{p-1}{p-n}}\,\left(\ln\frac{R}{r_0}\right)^{\frac{p-1}{p-n}}\,.$$

Finally, dividing the last inequality by $(\ln R)^{\frac{p-1}{p-n}}$ and taking the lower limit for $R\to\infty$, we conclude

$$\liminf_{R\to\infty}\frac{\operatorname{diam} f\,(B(x_0,R))}{(\ln R)^{\frac{p-1}{p-n}}} \geqslant 2\,\kappa^{\frac{1}{n-p}}\,\left(\frac{p-n}{p-1}\right)^{\frac{p-1}{p-n}}\,.$$

This completes the proof of Main Theorem. □

Corollary 1. *Suppose that $f:\mathbb{R}^n\to\mathbb{R}^n$ is a ring Q-homeomorphism with respect to p-modulus at a point x_0 with $p>n$ where x_0 is some point in $\mathbb{R}^n$ and for some numbers $r_0>0$, $\kappa=\kappa(x_0)>0$ the condition $q_{x_0}(t)\leqslant\kappa$ holds for a.e. $t\in[r_0,+\infty)$. Then*

$$\liminf_{R\to\infty}\frac{\operatorname{diam} f\,(B(x_0,R))}{R} \geqslant 2\,\kappa^{\frac{1}{n-p}}\,>0\,,$$

where $B(x_0,R)=\{x\in\mathbb{R}^n : |x-x_0|\leqslant R\}$.

Let us consider some examples.

Example 1. Let $f_1:\mathbb{R}^n\to\mathbb{R}^n$, where

$$f_1(x)=\begin{cases}\kappa^{\frac{1}{n-p}}\left(\frac{p-n}{p-n-\alpha}\right)^{\frac{p-1}{p-n}}\,|x-x_0|^{\frac{p-n-\alpha}{p-n}}\,\frac{x-x_0}{|x-x_0|}\,, & x\neq x_0\\ 0, & x=x_0\,.\end{cases}$$

Note that the mapping f_1 maps the ball $B(x_0,R)$ onto the ball $B(0,\widetilde{R})$, where

$$\widetilde{R}=\kappa^{\frac{1}{n-p}}\,\left(\frac{p-n}{p-n-\alpha}\right)^{\frac{p-1}{p-n}}\,R^{\frac{p-n-\alpha}{p-n}}\,.$$

It can be easily seen that

$$\lim_{R\to\infty}\frac{\operatorname{diam} f_1\,(B(x_0,R))}{R^{\frac{p-n-\alpha}{p-n}}}=\lim_{R\to\infty}\frac{\operatorname{diam} B(0,\widetilde{R})}{R^{\frac{p-n-\alpha}{p-n}}}$$

$$=\lim_{R\to\infty}\frac{2\widetilde{R}}{R^{\frac{p-n-\alpha}{p-n}}}=2\kappa^{\frac{1}{n-p}}\,\left(\frac{p-n}{p-n-\alpha}\right)^{\frac{p-1}{p-n}}\,.$$

Let us show that the mapping f_1 is a ring Q-homeomorphism with respect to p-modulus with the function $Q(x)=\kappa\,|x-x_0|^{\alpha}$ at the point x_0. Clearly, $q_{x_0}(t)=\kappa\,t^{\alpha}$. Consider a ring $\mathbb{A}(x_0,r_1,r_2)$, $0<r_1<r_2<\infty$. Note that the mapping f_1 maps the ring $\mathbb{A}(x_0,r_1,r_2)$ onto the ring $\widetilde{\mathbb{A}}(0,\widetilde{r}_1,\widetilde{r}_2)$, where

$$\widetilde{r}_i=\kappa^{\frac{1}{n-p}}\,\left(\frac{p-n}{p-n-\alpha}\right)^{\frac{p-1}{p-n}}\,r_i^{\frac{p-n-\alpha}{p-n}}\,,\quad i=1,\,2.$$

Denote by Γ a set of all curves that join the spheres $S(x_0, r_1)$ and $S(x_0, r_2)$ in the ring $\mathbb{A}(x_0, r_1, r_2)$. Then one can calculate p-modulus of the family of curves $f_1\Gamma$ in implicit form:

$$\mathrm{M}_p(f_1\Gamma) = \omega_{n-1} \left(\frac{p-n}{p-1}\right)^{p-1} \left(\widetilde{r}_2^{\frac{p-n}{p-1}} - \widetilde{r}_1^{\frac{p-n}{p-1}}\right)^{1-p}$$

(see, e.g., the relation (2) in [5]). Substituting in the above equality the values $\widetilde{r}_1$ and $\widetilde{r}_2$, defined above, one gets

$$\mathrm{M}_p(f_1\Gamma) = \omega_{n-1}\, \kappa \left(\frac{p-n-\alpha}{p-1}\right)^{p-1} \left({r_2}^{\frac{p-n-\alpha}{p-1}} - {r_1}^{\frac{p-n-\alpha}{p-1}}\right)^{1-p}.$$

Note that the last equality can be written by

$$\mathrm{M}_p(f_1\Gamma) = \frac{\omega_{n-1}}{\left(\int\limits_{r_1}^{r_2} \frac{dt}{t^{\frac{n-1}{p-1}}\, q_{x_0}^{\frac{1}{p-1}}(t)}\right)^{p-1}},$$

where $q_{x_0}(t) = \kappa\, t^{\alpha}$.

Hence, by Proposition 1, the homeomorphism f_1 is a ring Q-homeomorphism with respect to p-modulus for $p > n$ with the function $Q(x) = \kappa\, |x - x_0|^{\alpha}$.

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Про асимптотичну поведінку на нескінченності кільцевих Q-гомеоморфізмів відносно p-модуля.

Інтегральні оцінки вигляду (1) зустрічаються в роботах Л. Альфорса (див., напр., теорему 3, п. D, розд. I в [1]), О. Лехто і К. Віртанена (див. нерівність (6.6), п. 6.3, розд. V в [18]) для квазіконформних відображень на площині. У роботі В.Я. Гутлянського, К. Бішопа, О. Мартіо і М. Вуорінена (2003), див. [2], при вивченні локальних властивостей просторових квазіконформних відображень доведено нерівність вигляду (1), що стало безпосереднім поштовхом для введення поняття Q-гомеоморфізму. Цей термін був запропонований професором О. Мартіо (2001), див. [21–23]. По суті, Q-гомеоморфізми – це відображення зі скінченним спотворенням, оскільки функція Q не передбачається обмеженою. Означення Q-гомеоморфізму має геометричний характер і є аналогічним означенню Ю. Вяйсяля для квазіконформних відображень, див 13.1 і 34.6 в [37]. Поняття кільцевого Q-гомеоморфізму узагальнює поняття квазіконформного відображення за Герінгом, див. [4], і вперше зустрічається у роботі В.І. Рязанова, У. Сребро і Е. Якубова (2005) при $p = n = 2$ на комплексній площині при дослідженні вироджених рівнянь Бельтрамі, див. [28]. У просторі $\mathbb{R}^n$, $n \geqslant 2$, кільцеві Q-гомеоморфізми вперше зустрічаються у статті В.І. Рязанова, Є.О. Севостьянова (2007), див. [29]. У роботі В.Я. Гутлянського і А. Гольберга (2009) оцінка вигляду (2) при $p = n$ з деякою функцією Q була встановлена для просторових квазіконформних відображень, див. [9]. Також теорія кільцевих Q-гомеоморфізмів була застосована до дослідження локальних та граничних властивостей відображень зі скінченним спотворенням класів Орліча–Соболєва $W^{1,\varphi}_{loc}$ за умови типу Кальдерона на функцію φ та, зокрема, до класів Соболєва $W^{1,p}_{loc}$ при $p > n - 1$, див. [6, 12]. У даній роботі досліджується асимптотична поведінка на нескінченності кільцевих Q-гомеоморфізмів відносно p-модуля при $p > n$. Випадок $p = n$ вивчався у роботі [30]. Отримано аналог результату Мартіо–Рікмана–Вяйсяля (1970) про оцінку швидкості зростання відображень з обмеженим спотворенням на нескінченності, див. [20]. Знайдені достатні умови на інтегральне середнє значення функції Q по сферах, при яких відображення мають степеневий та логарифмічний порядок зростання, див. теорему 1. Також у

роботі побудовані приклади кільцевих Q-гомеоморфізмів відносно p-модуля при $p > n$, які показують точність отриманих оцінок, див. приклад 1.

Ключові слова: *кільцеві Q-гомеоморфізми, p-модуль сім'ї кривих, квазіконформні відображення, конденсатор, p-ємність конденсатора.*

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