UDC: 517.5
DOI: 10.37069/1683-4720-2023-37-4

©2023. **V. Ryazanov**[1], **E. Sevost'yanov**[1,2]

ON THE SURFACES MODULI THEORY

In this article we continue to develop the theory of several moduli of families of surfaces, in particular, strings (open surfaces) of various dimensions in Euclidean spaces. Since the surfaces in question can be extremely fractal (wild), the natural basis for studying them is the so-called Hausdorff measures. As is known, these moduli are the main geometric tool in the modern mapping theory and related topics in geometry, topology and the theory of partial differential equations with appropriate applications to the boundary-value problems of mathematical physics in anisotropic and inhomogeneous media. In addition, this theory can also find its further applications in many other fields, including mathematics itself (nonlinear dynamics, minimal surfaces), theoretical physics (conformal field theory, string theory), and engineering (mathematical models of the filtration of gases and fluids in underground mines of water, gas and oil seams, crystal growth and others). On the basis of the proof of Lemma 1 about the connections between moduli and the Lebesgue measures, we have proved the corresponding analogue of the Fubini theorem in the terms of the moduli that extends the known Väisälä theorem for families of curves to families of surfaces of arbitrary dimensions. It is necessary to note specially here that the most refined place in the proof of Lemma 1 is Proposition 1 on measurable (Borel) hulls of sets in Euclidean spaces. We also prove here the corresponding Lemma 2 and Proposition 2 on families of centered spheres. Finally, in a similar way, suitable results can be also obtained for families of several spheroids.

MSC: Primary 30C62, Secondary 30C65.

Keywords: *moduli of surfaces in Euclidean spaces, Hausdorff measures, measurable (Borel) hulls of sets.*

1. Introduction.

Later on, $\mathcal{H}^k$ denotes **k-dimensional Hausdorff measure**, $k = 1, \ldots, n$, in $\mathbb{R}^n$, $n \geqslant 2$. Namely, see, e.g., Section 2.1 in [1] or Section 2.10.2 in [2],

$$\mathcal{H}^k(A) \quad = \quad \sup_{\varepsilon>0} \mathcal{H}^k_\varepsilon(A)\,, \tag{1}$$

$$\mathcal{H}^k_\varepsilon(A) \quad = \quad \Omega_k \inf \sum_{i=1}^{\infty} \left(\frac{\delta_i}{2}\right)^k \tag{2}$$

for each set A in $\mathbb{R}^n$, where the infimum is taken over all countable collections of numbers $\delta_i \in (0, \varepsilon)$ such that some sets A_i in $\mathbb{R}^n$ with diameters δ_i cover A. Here Ω_k denotes the volume of the unit ball in $\mathbb{R}^k$. $\mathcal{H}^k$ is an **outer measure in the sense of Caratheodory**, see, e.g., Section II.4 in [9], i.e.,

(I) $\mathcal{H}^k(X) \leqslant \mathcal{H}^k(Y)$ whenever $X \subseteq Y$,
(II) $\mathcal{H}^k(\Sigma X_i) \leqslant \Sigma \mathcal{H}^k(X_i)$ for each sequence X_i of sets,
(III) $\mathcal{H}^k(X \cup Y) = \mathcal{H}^k(X) + \mathcal{H}^k(Y)$ whenever $\operatorname{dist}(X, Y) > 0$.

A set $E \subset \mathbb{R}^n$ is called **measurable** with respect to $\mathcal{H}^k$ if

(IV) $\mathcal{H}^k(X) = \mathcal{H}^k(X \cap E) + \mathcal{H}^k(X \setminus E)$ for every set $X \subset \mathbb{R}^n$.

It is well known that every Borel set is measurable with respect to any outer measure in the sense of Caratheodory, see, e.g., [9], p. 52. Moreover, $\mathcal{H}^k$ is **Borel regular**, i.e., for every set $X \subset \mathbb{R}^n$, there is a Borel set $B \subset \mathbb{R}^n$ such that $X \subset B$ and $\mathcal{H}^k(X) = \mathcal{H}^k(B)$, see, e.g., [9], p. 53, and Section 2.10.1 in [2]. The latter implies that, for every measurable set $E \subset \mathbb{R}^n$, there exist Borel sets B_* and $B^* \subset \mathbb{R}^n$ such that $B_* \subset E \subset B^*$ and $\mathcal{H}^k(B^* \setminus B_*) = 0$, see, e.g., Section 2.2.3 in [2], and, in particular, $\mathcal{H}^k(B^*) = \mathcal{H}^k(E) = \mathcal{H}^k(B_*)$.

Let ω be an open set in $\overline{\mathbb{R}^k} := \mathbb{R}^k \cup \{\infty\}$, $k = 1, \ldots, n-1$. A (continuous) mapping $S : \omega \to \mathbb{R}^n$ is called a k-dimensional surface S in $\mathbb{R}^n$. Sometimes we call the image $S(\omega) \subseteq \mathbb{R}^n$ the surface S, too. The number of preimages

$$N(S, y) \;=\; \text{card } S^{-1}(y) \;=\; \text{card } \{x \in \omega : \; S(x) = y\} \tag{3}$$

is said to be a **multiplicity function** of the surface S at a point $y \in \mathbb{R}^n$. In other words, $N(S, y)$ denotes the multiplicity of covering of the point y by the surface S. It is known that the multiplicity function is lower semi continuous, i.e.,

$$N(S, y) \;\geqslant\; \liminf_{m \to \infty} \; N(S, y_m)$$

for every sequence $y_m \in \mathbb{R}^n$, $m = 1, 2, \ldots$, such that $y_m \to y \in \mathbb{R}^n$ as $m \to \infty$, see, e.g., [8], p. 160. Thus, the function $N(S, y)$ is Borel measurable and hence measurable with respect to every Hausdorff measure $\mathcal{H}^k$, see, e.g., [9], p. 52.

k-dimensional Hausdorff area in $\mathbb{R}^n$ (or simply **area**) associated with a k-dimensional surface $S : \omega \to \mathbb{R}^n$ is given by

$$\mathcal{A}_S(B) \;=\; \mathcal{A}^k_S(B) \;:=\; \int\limits_B N(S, y) \; d\mathcal{H}^k y \tag{4}$$

for every Borel set $B \subseteq \mathbb{R}^n$ and, more generally, for an arbitrary set that is measurable with respect to $\mathcal{H}^k$ in $\mathbb{R}^n$. We say that the surface S is **rectifiable** if $\mathcal{A}_S(\mathbb{R}^n) < \infty$.

If $\varrho : \mathbb{R}^n \to [0, \infty]$ is a Borel function, then its **integral over a k-dimensional surface** S in $\mathbb{R}^n$, $n \geqslant 2$, is defined by the equality

$$\int\limits_S \varrho \; d\mathcal{A} \;:=\; \int\limits_{\mathbb{R}^n} \varrho(y) \; N(S, y) \; d\mathcal{H}^k y \; . \tag{5}$$

Given a family $\mathcal{S}$ of such k-dimensional surfaces S in $\mathbb{R}^n$, a Borel function $\varrho : \mathbb{R}^n \to [0, \infty]$ is called **admissible** for $\mathcal{S}$, abbr. $\varrho \in \operatorname{adm} \mathcal{S}$, if

$$\int\limits_S \varrho^k \; d\mathcal{A} \;\geqslant\; 1 \tag{6}$$

for every $S \in \mathcal{S}$. Given $p \in [k, \infty)$, the **p-modulus** of $\mathcal{S}$ is the quantity

$$M_p(\mathcal{S}) \;=\; \inf_{\varrho \in \operatorname{adm} \mathcal{S}} \int\limits_{\mathbb{R}^n} \varrho^p(x) \; d\mathcal{L}^n(x) \; , \tag{7}$$

where $d\mathcal{L}^n(x)$ corresponds to the Lebesgue measure in $\mathbb{R}^n$, see, e.g., [1], [2] and [9]. All these moduli $M_p(\mathcal{S})$, $p \in [k, \infty)$, are themselves outer measures in the sense of Caratheodory on the collection of all families $\mathcal{S}$ of k-dimensional surfaces in $\mathbb{R}^n$, see, e.g., Theorem and Remark to it in [3].

We say that $\mathcal{S}_2$ is **minorized** by $\mathcal{S}_1$ and write $\mathcal{S}_2 > \mathcal{S}_1$ if every $S \in \mathcal{S}_2$ has a subsurface that belongs to $\mathcal{S}_1$. It is known that $M_p(\mathcal{S}_1) \geqslant M_p(\mathcal{S}_2)$, see, e.g., [3, p. 176–178]. We also say that a property P holds for **p-a.e.** (almost every) k-dimensional surface in a family $\mathcal{S}$ if a subfamily of all surfaces in $\mathcal{S}$, for which P fails, has the p-modulus zero. If $0 < q < p$, then P also holds for q-a.e. S, see, e.g., Theorem 3 in [3].

Remark 1. Arguing by contradiction, we obtain immediately by the definition of these moduli M_p that, for every $p \in (0, \infty)$ and $k = 1, \ldots, n-1$,
(i) p-a.e. bounded k-dimensional surface in $\mathbb{R}^n$ is rectifiable,
(ii) given a Borel set B in $\mathbb{R}^n$ of the (Lebesgue) measure zero,

$$\mathcal{A}_S(B) \ = \ 0 \tag{8}$$

for p-a.e. k-dimensional surface S in $\mathbb{R}^n$, e.g., compare arguments of the first item in the proof of Theorem 33.1 in [11].

The following lemma was first proved in [4], see also Lemma 9.1 in [6].

Lemma 1. *Let $k = 1, \ldots, n-1$, $p \in [k, \infty)$, and let C be an open cube in $\mathbb{R}^n$, $n \geqslant 2$, whose edges are parallel to coordinate axes. Suppose that a property P holds for p-a.e. intersections with C of k-dimensional planes $\mathcal{P}$ that is parallel to a k-dimensional coordinate plane H. Then the set $\mathcal{E}$ of all projections of such $\mathcal{P}$ with the false P for $S := \mathcal{P} \cap C$ into the corresponding $(n-k)$-dimensional coordinate plane $H^{\perp}$ (perpendicular to H) has the Lebesgue measure zero.*

In its proof there was a gap related to the measurability of the set $\mathcal{E}$. In view of the importance of this lemma, we fill this gap here.

On the basis of Lemma 1, the following analogue of the Fubini theorem, see, e.g., Theorem III(8.1) in [9], in the terms of the moduli was first obtained in [4], see also Theorem 9.1 in [6]. It extends the Väisälä theorem with $k = 1$, see, e.g., Theorem 33.1 in [11], cf. Theorem 3 in [3] and Lemma 2.13 in [7], too.

Theorem 1. *Let $k = 1, \ldots, n-1$, $p \in [k, \infty)$, and let E be a subset in an open set $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$. Then E is measurable by Lebesgue in $\mathbb{R}^n$ if and only if E is measurable with respect to area on p-a.e. k-dimensional surface S in Ω. Moreover, $\mathcal{L}^n(E) = 0$ if and only if on p-a.e. k-dimensional surface S in Ω*

$$\mathcal{A}_S(E) \ = \ 0\ . \tag{9}$$

2. On measurable (Borel) hulls of sets.

Let $\mathcal{L}^m$ be the outer Lebesgue measure in $\mathbb{R}^m$, $m \geqslant 1$. Recall that a set $B \subset \mathbb{R}^m$ is called **measurable** if

(V) $\mathcal{L}^m(X) = \mathcal{L}^m(X \cap B) + \mathcal{L}^m(X \setminus B)$ for every set $X \subseteq \mathbb{R}^m$.

It is said that a set $B \subseteq \mathbb{R}^m$ is a **measurable hull of a set** $A \subseteq \mathbb{R}^m$ if B is measurable, $A \subseteq B$ and

(VI) $\mathcal{L}^m(X \cap A) = \mathcal{L}^m(X \cap B)$ for every set $X \subseteq \mathbb{R}^m$.

If, in addition, B is a Borel set, then B is called a **Borel hull of the set** A.

In particular, choosing $X = \mathbb{R}^m$ in (VI), we obtain that

(VII) $\mathcal{L}^m(A) = \mathcal{L}^m(B)$.

As is known, see, e.g., 2.1.4 in [2], if $\mathcal{L}^m(A) < \infty$ and a measurable (Borel) set $B \subset \mathbb{R}^m$ includes A and satisfies (VII), then B is a measurable (Borel) hull of A.

Note also that $\mathcal{L}^m$ in $\mathbb{R}^m$ coincides with the Hausdorff measure $\mathcal{H}^m$, see, e.g., Theorem 2.2.2 in [1]. Consequently, by Introduction $\mathcal{L}^m$ is Borel regular. Thus, by 2.1.5 in [2] we have the following conclusion:

Remark 2. Every set A in $\mathbb{R}^m$ with $\mathcal{L}^m(A) < \infty$ has a Borel hull B.

Proposition 1. Let C be an open cube in $\mathbb{R}^n$, $n \geqslant 2$, whose edges are parallel to coordinate axes. Suppose that A is a set in the open $m-$dimensional cube C^m, which is a projection of C into an $m-$dimensional coordinate plane H^m, $m = 1, \ldots, n-1$. Then, for all $p \in [k, \infty)$, $k := n - m$, and for every Borel hull B of the set A in C^m,

$$M_p(\mathcal{P}_A) \ = \ M_p(\mathcal{P}_B) \ , \tag{10}$$

where $\mathcal{P}_A$ and $\mathcal{P}_B$ denote the families of all $k-$dimensional surfaces consisting of intersections with C of $k-$dimensional planes $\mathcal{P}$ that are perpendicular to H^m and have projections in A and B, correspondingly.

Proof. For every fixed $p \in [k, \infty)$, $M_p(\mathcal{P}_A) \ \leqslant \ M_p(\mathcal{P}_B)$ by the monotonicity of these moduli because by definition $A \subseteq B$ and, correspondingly, $\mathcal{P}_A \subseteq \mathcal{P}_B$.

Thus, it remains us to derive the inverse inequality $M_p(\mathcal{P}_A) \ \geqslant \ M_p(\mathcal{P}_B)$. For this purpose, note that, since B is Borel,

$$M_p(\mathcal{P}_A) \ := \inf_{\varrho \,\in\, \mathrm{adm}\, \mathcal{P}_A} \int\limits_{\mathbb{R}^n} \varrho^p(x) \ d\mathcal{L}^n(x) \ = \inf_{\varrho \,\in\, \mathrm{adm}_*\, \mathcal{P}_A} \int\limits_{\mathbb{R}^n} \varrho^p(x) \ d\mathcal{L}^n(x) \ , \tag{11}$$

$$M_p(\mathcal{P}_B) \ := \inf_{\varrho \,\in\, \mathrm{adm}\, \mathcal{P}_B} \int\limits_{\mathbb{R}^n} \varrho^p(x) \ d\mathcal{L}^n(x) \ = \inf_{\varrho \,\in\, \mathrm{adm}_*\, \mathcal{P}_B} \int\limits_{\mathbb{R}^n} \varrho^p(x) \ d\mathcal{L}^n(x) \ , \tag{12}$$

with $\mathrm{adm}_* \mathcal{P}_A$ and $\mathrm{adm}_* \mathcal{P}_B$ that denote the subclasses of $\mathrm{adm}\, \mathcal{P}_A$ and $\mathrm{adm}\, \mathcal{P}_B$, correspondingly, of ρ vanished outside the Cartesian product $B \times C_k$, where C_k is the open $k-$dimensional cube, which is a projection of C into the $k-$dimensional coordinate plane H_k that is perpendicular to H^m. Moreover, by the construction

$$\int\limits_{\{z\} \times C_k} \varrho^k(y) \ d\mathcal{L}^k(y) \ \geqslant \ 1 \qquad \forall \ z \in A \ . \tag{13}$$

Note that if $M_p(\mathcal{P}_A) = \infty$, then we have nothing to prove. Therefore, we may further assume that $M_p(\mathcal{P}_A) < \infty$. Let ϱ be an arbitrary function of the class $\mathrm{adm}_* \, \mathcal{P}_A$ such that

$$\int\limits_C \varrho^p(x) \ d\mathcal{L}^n(x) \ < \ \infty \ . \tag{14}$$

Then also

$$\int\limits_C \varrho^k(x) \ d\mathcal{L}^n(x) \ < \ \infty \tag{15}$$

because $k \leqslant p$. Since $\mathcal{L}^n = \mathcal{L}^m \times \mathcal{L}^k$, we have by the Fubini theorem, see, e.g., Theorem III(8.1) in [9], that the function

$$R(z) \ := \int\limits_{\{z\}\times C_k} \varrho^k(y) \ d\mathcal{L}^k(y) \ , \quad z \in B \ , \tag{16}$$

is measurable and finite a.e. with respect to the Lebesgue measure $\mathcal{L}^m$ in C^m.

In particular, then the set $E := \{z \in B : R(z) < 1\}$ is measurable and by the Borel regularity of $\mathcal{L}^m$, there exist Borel sets B_* and B^* in B such that $B_* \subseteq E \subseteq B^*$ and $\mathcal{L}^m(B_*) = \mathcal{L}^m(E) = \mathcal{L}^m(B^*)$, see, e.g., Theorem III(6.6) in [9]. Note that $E \cap A = \emptyset$ by the condition (13) and, consequently, $B_* \cap A = \emptyset$, hence, $A \subseteq B \setminus B_*$ and by (V)

$$\mathcal{L}^m(B) = \mathcal{L}^m(B_*) + \mathcal{L}^m(B \setminus B_*) \geqslant \mathcal{L}^m(B_*) + \mathcal{L}^m(A) = \mathcal{L}^m(B^*) + \mathcal{L}^m(B), \tag{17}$$

i.e.,

$$\mathcal{L}^m(B^*) \ = \ 0 \ . \tag{18}$$

Finally, setting $\rho_*(x) = \infty$ on the Borel set $B^* \times C_k$ and $\rho_*(x) = \rho(z)$ otherwise, we see that $\rho_* \in \mathrm{adm}_* \, \mathcal{P}_B$ and that

$$\int\limits_C \varrho_*^p(x) \ d\mathcal{L}^n(x) \ = \ \int\limits_C \varrho^p(x) \ d\mathcal{L}^n(x) \ . \tag{19}$$

Thus, we come to the desired conclusion. □

3. The proof of Lemma 1.

Proof. Let us prove Lemma 1 by contradiction. Namely, let us assume that its conclusion is not true. Then, by Borel regularity of the Lebesgue measure $\mathcal{L}^m$, $m = n - k$, in $H^m := H^\perp$, there is a Borel hull B of the set $\mathcal{E}$ with $\mathcal{L}^m(B) > 0$ in the open cube $C^m := C \cap H^m$, see Remark 2. Note that by Proposition 1 it should be that

$$M_p(\mathcal{P}_B) \ = \ 0 \ . \tag{20}$$

However, if a Borel function $\varrho : \mathbb{R}^n \to [0, \infty]$ is admissible for $\mathcal{P}_B$ such that $\varrho \equiv 0$ outside the Borel set $C_k \times B$, where C_k is the open $k-$dimensional cube, which is the perpendicular projection of C into H, then, by the Hölder inequality,

$$\int\limits_{C_k\times B} \varrho^k(x)\, d\mathcal{L}^n(x) \ \leqslant\ \left(\int\limits_{C_k\times B} \varrho^p(x)\, d\mathcal{L}^n(x)\right)^{\frac{k}{p}} \left(\int\limits_{C_k\times B} d\mathcal{L}^n(x)\right)^{\frac{p-k}{p}}$$

and, consequently, by the Fubini theorem, see, e.g., Theorem III(8.1) in [9],

$$\int\limits_{\mathbb{R}^n} \varrho^p(x)\, d\mathcal{L}^n(x) \ \geqslant\ \frac{\left(\int\limits_{C_k\times B} \varrho^k(x)\, d\mathcal{L}^n(x)\right)^{\frac{p}{k}}}{\left(\int\limits_{C_k\times B} d\mathcal{L}^n(x)\right)^{\frac{p-k}{k}}} \ \geqslant\ \frac{(\mathcal{L}^m(B))^{\frac{p}{k}}}{(h^k\cdot\mathcal{L}^m(B))^{\frac{p-k}{k}}}\ ,$$

where h is the length of the edge of cube C, i.e.,

$$M_p(\mathcal{P}_B)\ \geqslant\ \frac{\mathcal{L}^m(B)}{h^{p-k}}\ >\ 0\ . \tag{21}$$

Thus, the obtained contradiction disproves the above assumption. □

4. The proof of Theorem 1.

Proof. By the Lindelöf theorem, see, e.g., Theorem I.5.XI in [5], applied to $\mathbb{R}^n$, every cover of the set Ω contains its countable subcover. Hence, by the minorant property of the modulus M_p and subadditivity of the Lebesgue measure $\mathcal{L}^n$, we may assume without loss of generality that Ω is an open cube C in $\mathbb{R}^n$ whose edges are parallel to the coordinate axes.

Suppose first that E is Lebesgue measurable in $\mathbb{R}^n$. Then, by the Borel regularity of the Lebesgue measure $\mathcal{L}^n$, there exist Borel sets B_* and B^* in $\mathbb{R}^n$ such that $B_*\subset E\subset B^*$ and $\mathcal{L}^n(B^*\backslash B_*)=0$. Thus, by (ii) in Remark 1, $\mathcal{A}_S(B^*\backslash B_*)=0$ and hence E is measurable by area on p-a.e. k-dimensional surface S in C. Conversely, if the latter is true, then E is measurable by area on a.e. k-dimensional plane H in C that is parallel to a k-dimensional coordinate plane, see Lemma 1. Thus, E is measurable by the Fubini theorem applied to its characteristic function.

Now, suppose that $\mathcal{L}^n(E)=0$. Then by Remark 2 there is a Borel set B such that $E\subset B$ and $\mathcal{L}^n(B)=0$. Then, by (ii) in Remark 1, the relation (9) holds for p-a.e. k-dimensional surface S in C. Conversely, if the latter is true, then, in particular, $\mathcal{A}_S(E)=0$ on a.e. k-dimensional plane H in C, which is parallel to a k-dimensional coordinate plane, see Lemma 1. Thus, $\mathcal{L}^n(E)=0$ again by the Fubini theorem. □

Remark 3. Say by the Lusin theorem, see, e.g., Section 2.3.5 in [2], for every measurable function $\varrho:\mathbb{R}^n\to[0,\infty]$, there is a Borel function $\varrho^*:\mathbb{R}^n\to[0,\infty]$ such that $\varrho^*=\varrho$ a.e. in $\mathbb{R}^n$. Thus, by Theorem 1, for every $k=1,\ldots,n-1$ and $p\in[k,\infty)$, ϱ is measurable on p-a.e. k-dimensional surface S in $\mathbb{R}^n$.

We say that a Lebesgue measurable function $\varrho : \mathbb{R}^n \to [0,\infty]$ is **p-extensively admissible** for a family $\mathcal{S}$ of k-dimensional surfaces S in $\mathbb{R}^n$, abbr. $\varrho \in \mathrm{ext}_p \,\mathrm{adm}\, \mathcal{S}$, if

$$\int\limits_{S} \varrho^k \, d\mathcal{A} \;\geqslant\; 1 \tag{22}$$

for p-a.e. $S \in \mathcal{S}$. The p**-extensive modulus** $\overline{M}_p(\mathcal{S})$ of $\mathcal{S}$ is the quantity

$$\overline{M}_p(\mathcal{S}) \;=\; \inf \int\limits_{\mathbb{R}^n} \varrho^p(x)\; dm(x), \tag{23}$$

where the infimum is taken over all $\varrho \in \mathrm{ext}_p \,\mathrm{adm}\, \mathcal{S}$. In the case $p = n$, we use the notations $\overline{M}(\mathcal{S})$ and $\varrho \in \mathrm{ext}\,\mathrm{adm}\, \mathcal{S}$, respectively. For every $p \in (0,\infty)$, $k = 1,\ldots,n-1$, and every family $\mathcal{S}$ of k-dimensional surfaces in $\mathbb{R}^n$,

$$\overline{M}_p(\mathcal{S}) \;=\; M_p(\mathcal{S})\,. \tag{24}$$

5. Borel hulls and centered spheres.

Proposition 2. Let R be a ring $\{x \in \mathbb{R}^n : \; r_1 < |x| < r_2\}$, $0 < r_1 < r_2 < \infty$, in $\mathbb{R}^n$, $n \geqslant 2$. Suppose that A is a set in (r_1, r_2). Then

$$M_p(\mathcal{S}_A) \;=\; M_p(\mathcal{S}_B) \qquad \forall\; p \geqslant n-1 \tag{25}$$

for every Borel hull B in (r_1, r_2) of the set A, where $\mathcal{S}_A$ and $\mathcal{S}_B$ denote the families of $(n-1)$–dimensional spheres $S(r) := \{x \in \mathbb{R}^n : \; |x| = r\}$ with all r in A and B, correspondingly.

Proof. For every fixed $p \in [n-1,\infty)$, $M_p(\mathcal{S}_A) \;\leqslant\; M_p(\mathcal{S}_B)$ by the monotonicity of these moduli because by definition $A \subseteq B$ and, correspondingly, $\mathcal{S}_A \subseteq \mathcal{S}_B$.

Thus, it remains us to derive the inverse inequality $M_p(\mathcal{S}_A) \;\geqslant\; M_p(\mathcal{S}_B)$. For this purpose, note that, since B is Borel,

$$M_p(\mathcal{S}_A) \;:=\; \inf_{\varrho \,\in\, \mathrm{adm}\, \mathcal{S}_A} \int\limits_{\mathbb{R}^n} \varrho^p(x)\; d\mathcal{L}^n(x) \;=\; \inf_{\varrho \,\in\, \mathrm{adm}_*\, \mathcal{S}_A} \int\limits_{\mathbb{R}^n} \varrho^p(x)\; d\mathcal{L}^n(x)\;, \tag{26}$$

$$M_p(\mathcal{S}_B) \;:=\; \inf_{\varrho \,\in\, \mathrm{adm}\, \mathcal{S}_B} \int\limits_{\mathbb{R}^n} \varrho^p(x)\; d\mathcal{L}^n(x) \;=\; \inf_{\varrho \,\in\, \mathrm{adm}_*\, \mathcal{S}_B} \int\limits_{\mathbb{R}^n} \varrho^p(x)\; d\mathcal{L}^n(x)\;, \tag{27}$$

with $\mathrm{adm}_*\, \mathcal{S}_A$ and $\mathrm{adm}_*\, \mathcal{S}_B$ that denote the subclasses of $\mathrm{adm}\, \mathcal{S}_A$ and $\mathrm{adm}\, \mathcal{S}_B$, correspondingly, of ρ vanished outside the Cartesian product $B \times \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ centered at the origin. By the construction

$$\int\limits_{\{r\}\times\mathbb{S}^{n-1}} \varrho^{n-1}(y)\; d\mathcal{H}^{n-1}(y) \;\geqslant\; 1 \qquad \forall\; r \in A\;. \tag{28}$$

Note that if $M_p(\mathcal{S}_A) = \infty$, then we have nothing to prove. Therefore, we may further assume that $M_p(\mathcal{S}_A) < \infty$. Let ϱ be an arbitrary function of the class $\mathrm{adm}_* \mathcal{S}_A$ such that

$$\int\limits_R \varrho^p(x)\ d\mathcal{L}^n(x)\ <\ \infty\ . \tag{29}$$

Then also

$$\int\limits_R \varrho^{n-1}(x)\ d\mathcal{L}^n(x)\ <\ \infty \tag{30}$$

because $p \geqslant n-1$. Since $\mathcal{L}^n = \mathcal{H}^n = \mathcal{H}^1 \times \mathcal{H}^{n-1} = \mathcal{L}^1 \times \mathcal{H}^{n-1}$ in R, we have by the Fubini theorem, see, e.g., Theorem III(8.1) in [9], that the function

$$R(r)\ :=\ \int\limits_{\{r\}\times\mathbb{S}^{n-1}} \varrho^{n-1}(y)\ d\mathcal{H}^{n-1}(y)\ ,\quad r \in B\ , \tag{31}$$

is measurable and finite a.e. with respect to the Lebesgue measure $\mathcal{L}^1$ in (r_1, r_2).

In particular, then the set $E := \{r \in B : R(r) < 1\}$ is measurable and by the Borel regularity of $\mathcal{L}^1$, there exist Borel sets B_* and B^* in B such that $B_* \subseteq E \subseteq B^*$ and $\mathcal{L}^1(B_*) = \mathcal{L}^1(E) = \mathcal{L}^1(B^*)$, see, e.g., Theorem III(6.6) in [9]. Note that $E \cap A = \emptyset$ by the condition (28) and, consequently, $B_* \cap A = \emptyset$, hence, $A \subseteq B \setminus B_*$ and by (V)

$$\mathcal{L}^1(B)\ =\ \mathcal{L}^1(B_*) + \mathcal{L}^1(B \setminus B_*)\ \geqslant\ \mathcal{L}^1(B_*) + \mathcal{L}^1(A)\ =\ \mathcal{L}^1(B^*) + \mathcal{L}^1(B)\ , \tag{32}$$

i.e.,

$$\mathcal{L}^1(B^*)\ =\ 0\ . \tag{33}$$

Finally, setting $\rho_*(x) = \infty$ on the Borel set $B^* \times \mathbb{S}^{n-1}$ and $\rho_*(x) = \rho(z)$ otherwise, we see that $\rho_* \in \mathrm{adm}_* \mathcal{S}_B$ and that

$$\int\limits_R \varrho_*^p(x)\ d\mathcal{L}^n(x)\ =\ \int\limits_R \varrho^p(x)\ d\mathcal{L}^n(x)\ . \tag{34}$$

Thus, we come to the desired conclusion. □

6. Moduli of spheres and Lebesgue measure.

Lemma 2. *Let R be a ring $\{x \in \mathbb{R}^n :\ r_1 < |x| < r_2\}$, $0 \leqslant r_1 < r_2 < \infty$, in $\mathbb{R}^n$, $n \geqslant 2$, and let $\mathcal{S}_A$ the collection of all $(n-1)-$dimensional spheres $S(r) := \{x \in \mathbb{R}^n : |x| = r\}$ with r in A. If $M_p(\mathcal{S}_A) = 0$ for some $p \geqslant n-1$, then $\mathcal{L}^1(A) = 0$.*

Proof. By subadditivity of measures M_p and $\mathcal{L}^1$, we may assume further that $r_1 > 0$. By Remark 2 there is a Borel hull B of A, for which by Proposition 2

$$M_p(\mathcal{S}_B)\ =\ 0\ . \tag{35}$$

Let us assume that $\mathcal{L}^m(A) > 0$. Then also should be $\mathcal{L}^m(B) > 0$.

However, if a Borel function $\varrho : \mathbb{R}^n \to [0, \infty]$ is admissible for $\mathcal{S}_B$ such that $\varrho \equiv 0$ outside the Borel set $B \times \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ centered at the origin, then, by the Hölder inequality,

$$\int\limits_{B\times\mathbb{S}^{n-1}} \varrho^{n-1}(x)\ d\mathcal{L}^n(x) \ \leqslant \ \left(\ \int\limits_{B\times\mathbb{S}^{n-1}} \varrho^p(x)\ d\mathcal{L}^n(x)\right)^{\frac{n-1}{p}} \left(\ \int\limits_{B\times\mathbb{S}^{n-1}} d\mathcal{L}^n(x)\right)^{\frac{p-(n-1)}{p}}$$

and, consequently, by the Fubini theorem, see, e.g., Theorem III(8.1) in [9],

$$\int\limits_{\mathbb{R}^n} \varrho^p(x)\ d\mathcal{L}^n(x) \ \geqslant \ \frac{\left(\int\limits_{B\times\mathbb{S}^{n-1}} \varrho^{n-1}(x)\ d\mathcal{L}^n(x)\right)^{\frac{p}{n-1}}}{\left(\int\limits_{B\times\mathbb{S}^{n-1}} d\mathcal{L}^n(x)\right)^{\frac{p-(n-1)}{n-1}}} \ \geqslant \ \frac{(\mathcal{L}^1(B))^{\frac{p}{n-1}}}{(\alpha_{n-1}{r_1}^{n-1}\cdot\mathcal{L}^1(B))^{\frac{p-(n-1)}{n-1}}}\ ,$$

where $\alpha_{n-1} = n\pi^{n/2}/\Gamma(1+n/2)$ is the area of $\mathbb{S}^{n-1}$ calculated through the Euler gamma function Γ, i.e., with the constant $C_{n,p} := (\alpha_{n-1})^{(n-1-p)/(n-1)}$,

$$M_p(\mathcal{S}_B) \ \geqslant \ \mathcal{L}^1(B)\cdot C_{n,p}/r_1^{p-(n-1)} \ > \ 0\ . \tag{36}$$

Thus, the obtained contradiction disproves the above assumption. $\Box$

These results will find interesting applications in modern mapping theory, see, e.g., [4, 6, 7] and [10].

The corresponding results can be also obtained for families of several spheroids (cylinders, ellipsoids, etc.) that will be published elsewhere.

References

1. Evans, L.C., Gariepy, R.F. (1992). *Measure theory and fine properties of functions.* Studies in Advanced Mathematics. Boca Raton, FL, CRC Press.
2. Federer, H. (1969). *Geometric Measure Theory.* Berlin, Springer-Verlag.
3. Fuglede, B. (1957). Extremal length and functional completion. *Acta Math., 98,* 171–219.
4. Kovtonyk, D., Ryazanov, V. (2008). On the theory of mappings with finite area distortion. *J. Anal. Math., 104,* 291—306.
5. Kuratowski, K. (1968). *Topology, v. 1.* New York–London, Academic Press.
6. Martio, O., Ryazanov, V., Srebro, U., Yakubov, E. (2009). *Moduli in Modern Mapping Theory.* Springer Monographs in Mathematics. New York etc., Springer.
7. Martio, O., Ryazanov, V., Srebro, U., Yakubov, E. (2004). Mappings with finite length distortion. *J. d'Anal. Math., 93,* 215–236.
8. Rado, T., Reichelderfer, P.V. (1955). *Continuous Trasformations in Analysis.* Berlin, Springer-Verlag.
9. Saks, S. (1964). *Theory of the Integral.* New York, Dover.
10. Sevost'yanov, E.A. (2022). On the Boundary and Global Behavior of Mappings of Riemannian Surfaces. *Filomat, 36,* 1295–1327.
11. Väisälä, J. (1971). Lectures on n-Dimensional Quasiconformal Mappings. *Lecture Notes in Math., 229.* Berlin etc., Springer–Verlag.

В. Рязанов, Є. Севостьянов

Про теорію модулів поверхонь.

У цій статті ми продовжуємо розвивати теорію модулів сімей деяких поверхонь, зокрема, струн (відкритих поверхонь) різної розмірності в евклідових просторах. Оскільки розглянуті поверхні можуть бути надзвичайно фрактальними, природною основою для їх вивчення є так звані міри Хаусдорфа. Як відомо, модуль є основним геометричним інструментом у сучасній теорії відображень та суміжних областях у геометрії, топології та теорії диференціальних рівнянь у частинних похідних з відповідним застосуванням до крайових задач математичної фізики в анізотропних і неоднорідних середовищах. Крім того, ця теорія також може знайти своє подальше застосування в багатьох інших областях, включаючи саму математику (нелінійна динаміка, мінімальні поверхні), теоретичну фізику (конформна теорія поля, теорія струн), інженерна справа (математичні моделі фільтрації газів і рідин у підземних шахтах води, газу і нафтових пластів, ріст кристалів та інших). На основі доведення леми 1 про зв'язки між модулями та мірами Лебега доведено відповідний аналог теореми Фубіні в термінах модулів, що поширює відому теорему Väisälä з сімей кривих на сім'ї поверхонь довільних розмірностей. Слід особливо відзначити, що найбільш витонченим місцем у доведенні леми 1 є твердження 1 про вимірні (борелівські) оболонки множин в евклідових просторах. Нами також доведено відповідну лему 2 і твердження 2 про сім'ї концентричних сфер. Нарешті, аналогічні результати можна отримати для різних сфероїдів.

Ключові слова: *модулі сімей поверхонь у евклідових просторах, хаусдорфові міри, вимірні (борелеві) оболонки множин.*

[1]*Institute of Applied Mathematics and Mechanics of NAS of Ukraine, Slov'yans'k, Ukraine,*
[2]*Zhytomyr Ivan Franko State University, Zhytomyr, Ukraine*
vl.ryazanov1@gmail.com,
esevostyanov2009@gmail.com

Received 20.05.2023