UDC 517.5
DOI: 10.37069/1683-4720-2023-37-8

 O. Rovenska

ASYMPTOTIC ESTIMATES FOR DEVIATIONS OF FEJER MEANS ON CLASSES OF POISSON INTEGRALS

The work is devoted to the investigation of problem of approximation of continuous periodic functions of a real variable by trigonometric polynomials generated by linear methods of summation of Fourier series. The simplest example of the process of linear approximation of periodic functions is approximation of functions by partial sums of their Fourier series. However, sequences of partial Fourier sums are not uniformly convergent on the whole class of continuous periodic functions. Therefore, numerous studies are devoted to the study of approximation properties of approximating methods, which are generated by different transformations of sequences of partial sums of the Fourier series and allow obtaining sequences of trigonometric polynomials that are uniformly convergent for the entire class of continuous functions. In particular, Fejer means have been intensively studied in recent decades. One of the important tasks in this direction is study of the asymptotic behavior of the upper bounds of the deviations of trigonometric polynomials for a fixed class of periodic functions. The aim of the work is to systematize the known results concerning the approximation of classes of periodic functions of high smoothness by arithmetic means of Fourier sums, and to present new facts obtained for a more general case. In paper we investigate the asymptotic behavior of upper bounds on classes of Poisson integrals of periodic functions of real variable of deviations of linear means of Fourier series, which are constructed using the Fejer summation method. We study the classes consist of analytic functions of a real variable, which can be regularly extended into the corresponding strip of the complex plane. In the work, asymptotic inequalities for the upper bounds of the deviations of the Fejer means on the class of Poisson integrals were found. In certain case of parameters defining the class, the obtained formulas coincide with previously known asymptotic equalities.

MSC: 42A10.

Key words: *Fejer mean, Poisson integral, asymptotic formula.*

1. Introduction.

Let $\mathrm{L}(\mathbb{T})$, $\mathbb{T}=[-\pi;\pi]$ be the space of summable 2π-periodic functions f with the norm

$$\|f\|_{\mathrm{L}}=\int\limits_{\mathbb{T}} \mid f(x) \mid \, dx.$$

Let $f\in \mathrm{L}(\mathbb{T})$, let

$$s[f]=\frac{a_0[f]}{2}+\sum_{k=1}^{\infty}\left(a_k[f]\cos kx+b_k[f]\sin kx\right),$$

be the Fourier series of the function f, where

$$a_0[f]=\frac{1}{\pi}\int\limits_{\mathbb{T}} f(x)\,dx, \quad a_k[f]=\frac{1}{\pi}\int\limits_{\mathbb{T}} f(x)\cos kx\,dx,$$

$$b_k[f]=\frac{1}{\pi}\int\limits_{\mathbb{T}} f(x)\sin kx\,dx,\quad k\in\mathbb{N},$$

are the Fourier coefficients of the function f and let

$$S_n[f](x)=\frac{a_0[f]}{2}+\sum_{k=1}^{n}(a_k[f]\cos kx+b_k[f]\sin kx)$$

be the n-partial sum of the Fourier series of the function f.

Let $\mathrm{C}(\mathbb{T})$ be the space of continuous 2π-periodic functions f with the norm

$$\|f\|_{\mathrm{C}}=\max_{x\in\mathbb{T}}\mid f(x)\mid .$$

We denote by $\mathrm{G}(q;\beta)$, $q\in(0;1)$, $\beta\in\mathbb{R}$ the class of functions $f\in\mathrm{C}(\mathbb{T})$, given by the convolutions

$$f(x)=A_0+\frac{1}{\pi}\int\limits_{\mathbb{T}}\varphi(x+t)P_\beta^q(t)\,dt,$$

where A_0 is a fixed constant,

$$P_\beta^q(t)=\sum_{k=1}^{\infty}q^k\cos\left(kt+\frac{\beta\pi}{2}\right)$$

is the Poisson kernel, the summable 2π-periodic function φ satisfies the conditions

$$\operatorname*{ess\,sup}_{t\in\mathbb{T}}\mid\varphi(t)\mid\le 1,\qquad \int\limits_{\mathbb{T}}\varphi(t)\,dt=0.$$

The set $\mathrm{G}(q;\beta)$ consist of 2π–periodic functions $f(x)$ that admit analytic extension to the functions $F(z)=F(x+iy)$ in the strip $\mid\Im z\mid<\ln q^{-1}$. Denote $\mathrm{G}(q;0):=\mathrm{G}(q)$, $\mathrm{G}(q;1):=\tilde{\mathrm{G}}(q)$.

In [13] was established the asymptotic equality

$$\sup_{f\in\mathrm{G}(q;\beta)}\|f-S_{n-1}[f]\|_{\mathrm{C}}=\frac{8q^n}{\pi^2}K(q)+O(1)\frac{q^n}{n},\quad n\to\infty,$$

where

$$K(q)=\int\limits_0^{\frac{\pi}{2}}\frac{du}{(1-q^2\sin^2u)^{\frac{1}{2}}}$$

is the complete elliptic integral of the first kind, $O(1)$ is a quantity uniformly bounded with respect to n.

Let $f\in\mathrm{L}(\mathbb{T})$. The polynomials that are given by relationship [7]

$$\sigma_n[f](x)=\frac{1}{n}\sum_{k=0}^{n-1}S_k[f](x)$$

are called Fejer means of $s[f]$. Fejer means, as well as their generalizations, have been intensively studied over the past decades [3–5, 8–10, 17, 18]. It is difficult to mention all the relevant published research papers in this area. Recently, we have seen the appearance of several important works [1, 2, 19].

Asymptotically exact equality for the upper bounds of deviations of the Fejer means over classes of Poisson integrals $\mathrm{G}(q)$, were obtained in [11, 12, 14]. In [15] were considered the problems of approximation of more general classes of Poisson integrals by Fejer summation method. Also asymptotic relations for the upper bounds of deviations of the Fejer means over classes $\mathrm{G}(q)$ were obtained in [6].

The aim of this work is to investigate the asymptotic behavior of values

$$\sup_{f\in \mathrm{G}(q;\beta)} \|f-\sigma_n[f]\|_{\mathrm{C}}, \quad q\in(0;1), \quad \beta\in\mathbb{R}.$$

In this work we clarify the result of work [16].

2. Results.

Our main result is contained in the following theorem.

Theorem. *Let $q\in(0;1)$, $\beta\in\mathbb{R}$, q_0 is the only root of the equation $q^4-2q^3-2q^2-2q+1=0$, that belongs to the interval $(0;1)$, $q_0=0.346\ldots$.*

1. If $q\in(0;q_0]$ then

$$\frac{4q}{\pi n(1+q^2)}\left|\cos^3\frac{\beta\pi}{2}\right|+\frac{4q}{\pi n(1-q^2)}\left|\sin^3\frac{\beta\pi}{2}\right|+O(1)\frac{q^n}{n}$$
$$\le \sup_{f\in \mathrm{G}(q;\beta)} \|f-\sigma_n[f]\|_{\mathrm{C}}$$
$$\le \frac{4q}{\pi n(1+q^2)}\left|\cos\frac{\beta\pi}{2}\right|+\frac{4q}{\pi n(1-q^2)}\left|\sin\frac{\beta\pi}{2}\right|+O(1)\frac{q^n}{n}, \tag{1}$$

where $O(1)$ is a quantity, uniformly bounded with respect to n.

2. If $q\in[q_0;1)$ then

$$\frac{2}{\pi n}\frac{(1+q^2)^2}{(1-q^2)\left(1-q^2+\sqrt{2(1+q^4)}\right)}\left|\cos^3\frac{\beta\pi}{2}\right|+\frac{4q}{\pi n(1-q^2)}\left|\sin^3\frac{\beta\pi}{2}\right|$$
$$+O(1)\frac{q^n}{n(1-q)^3}\le \sup_{f\in \mathrm{G}(q;\beta)} \|f-\sigma_n[f]\|_{\mathrm{C}}$$
$$\le \frac{2}{\pi n}\frac{(1+q^2)^2}{(1-q^2)\left(1-q^2+\sqrt{2(1+q^4)}\right)}\left|\cos\frac{\beta\pi}{2}\right|+\frac{4q}{\pi n(1-q^2)}\left|\sin\frac{\beta\pi}{2}\right|$$
$$+O(1)\frac{q^n}{n(1-q)^3}, \tag{2}$$

where $O(1)$ is a quantity, uniformly bounded with respect to q, n.

Proof. Denote
$$\delta_n[f](x) := f(x) - \sigma_n[f](x).$$

We have
$$\delta_n[f](x) = f(x) - \frac{1}{n}\sum_{k=0}^{n-1} S_k[f](x) = \frac{1}{n}\sum_{k=0}^{n-1} \rho_k[f](x), \tag{3}$$
where
$$\rho_k[f](x) := f(x) - S_k[f](x), \quad k \in \mathbb{N}.$$

For any $f \in \mathrm{G}(q;\beta)$, the quantity $\rho_m[f](x)$ can be represented in the following form
$$\rho_m[f](x) = \frac{1}{\pi}\int\limits_{\mathbb{T}} \varphi(x+t) \sum_{k=m+1}^{\infty} q^k \cos\left(kt + \frac{\beta\pi}{2}\right) dt$$
$$= \frac{1}{\pi}\int\limits_{\mathbb{T}} \varphi(x+t) \sum_{k=0}^{\infty} q^{k+m+1} \cos\left((k+m+1)t + \frac{\beta\pi}{2}\right) dt$$
$$= \frac{q^{m+1}}{\pi}\int\limits_{\mathbb{T}} \varphi(x+t) \left(\cos(m+1)t \sum_{k=0}^{\infty} q^k \cos kt \cos\frac{\beta\pi}{2} - \sin(m+1)t \sum_{k=0}^{\infty} q^k \sin kt \cos\frac{\beta\pi}{2}\right.$$
$$\left. - \sin(m+1)t \sum_{k=0}^{\infty} q^k \cos kt \sin\frac{\beta\pi}{2} - \cos(m+1)t \sum_{k=0}^{\infty} q^k \sin kt \sin\frac{\beta\pi}{2}\right) dt.$$

Since
$$\sum_{k=0}^{\infty} q^k \cos kt = \frac{1 - q\cos t}{1 - 2q\cos t + q^2}, \quad \sum_{k=0}^{\infty} q^k \sin kt = \frac{q \sin t}{1 - 2q\cos t + q^2},$$
we have
$$\rho_m[f](x) = \frac{q^{m+1}}{\pi}\int\limits_{\mathbb{T}} \varphi(x+t) \left(\left(\frac{1 - q\cos t}{1 - 2q\cos t + q^2}\cos\frac{\beta\pi}{2}\right.\right.$$
$$\left. -\frac{q\sin t}{1 - 2q\cos t + q^2}\sin\frac{\beta\pi}{2}\right)\cos(m+1)t - \left(\frac{q\sin t}{1 - 2q\cos t + q^2}\cos\frac{\beta\pi}{2}\right.$$
$$\left.\left. + \frac{1 - q\cos t}{1 - 2q\cos t + q^2}\sin\frac{\beta\pi}{2}\right)\sin(m+1)t\right) dt. \tag{4}$$

Combining (3), (4), we obtain
$$\delta_n[f](x) = \frac{1}{\pi n}\sum_{m=1}^{n} q^m \int\limits_{\mathbb{T}} \varphi(x+t)$$
$$\times \left(\left(\frac{1 - q\cos t}{1 - 2q\cos t + q^2}\cos mt - \frac{q\sin t}{1 - 2q\cos t + q^2}\sin mt\right)\cos\frac{\beta\pi}{2}\right.$$

$$-\left(\frac{1-q\cos t}{1-2q\cos t+q^2}\sin mt+\frac{q\sin t}{1-2q\cos t+q^2}\cos mt\right)\sin\frac{\beta\pi}{2}\right)\,dt$$

$$=\frac{1}{\pi n}\int\limits_{\mathbb{T}}\frac{\varphi(x+t)}{1-2q\cos t+q^2}\sum_{m=1}^{n}q^m\left((\cos mt-q\cos(m-1)t)\cos\frac{\beta\pi}{2}\right.$$

$$\left.-(\sin mt-q\sin(m-1)t)\sin\frac{\beta\pi}{2}\right)\,dt$$

$$=\frac{1}{\pi n}\int\limits_{\mathbb{T}}\frac{\varphi(x+t)}{1-2q\cos t+q^2}\left(\Sigma_1(q;t)\cos\frac{\beta\pi}{2}+\Sigma_2(q;t)\sin\frac{\beta\pi}{2}\right)\,dt, \tag{5}$$

where

$$\Sigma_1(q;t):=\sum_{k=1}^{n}q^k(\cos kt-q\cos(k-1)t),\quad \Sigma_2(q;t):=\sum_{k=1}^{n}q^k(q\sin(k-1)t-\sin kt).$$

Further we find

$$\Sigma_1(q;t)=\sum_{k=1}^{n}q^k\left(\frac{e^{ikt}+e^{-ikt}}{2}-q\frac{e^{i(k-1)t}+e^{-i(k-1)t}}{2}\right)$$

$$=\frac{1}{2}\sum_{k=1}^{n}\left\{\left[\left(qe^{it}\right)^k+\left(qe^{-it}\right)^k\right]-q^2\left[\left(qe^{it}\right)^{k-1}+\left(qe^{-it}\right)^{k-1}\right]\right\}$$

$$=\frac{q((1+q^2)\cos t-2q)+O(1)q^{n+1}}{1-2q\cos t+q^2}, \tag{6}$$

$$\Sigma_2(q;t)=\sum_{k=1}^{n}q^k\left(\frac{e^{ikt}-e^{-ikt}}{2i}-q\frac{e^{i(k-1)t}-e^{-i(k-1)t}}{2i}\right)$$

$$=\frac{1}{2i}\sum_{k=1}^{n}\left(\left[\left(qe^{it}\right)^k-\left(qe^{-it}\right)^k\right]-q^2\left[\left(qe^{it}\right)^{k-1}-\left(qe^{-it}\right)^{k-1}\right]\right)$$

$$=\frac{q(1-q^2)\sin t+O(1)q^{n+1}}{1-2q\cos t+q^2}, \tag{7}$$

where $O(1)$ is a quantity uniformly bounded with respect to n, q.

Since

$$\int\limits_{\mathbb{T}}\frac{1}{(1-2q\cos t+q^2)^2}\,dt=O(1)\frac{1}{(1-q)^3}, \tag{8}$$

based on (5)–(8), we have

$$\delta_n[f](x)=\frac{q}{\pi n}\int\limits_{\mathbb{T}}\frac{\varphi(x+t)}{(1-2q\cos t+q^2)^2}\left(((1+q^2)\cos t-2q)\cos\frac{\beta\pi}{2}\right.$$

$$-(1-q^2)\sin t\sin\frac{\beta\pi}{2}\Big)\ dt+O(1)\frac{q^n}{n(1-q)^3}. \tag{9}$$

Denote

$$J_q^1(t):=\frac{(1+q^2)\cos t-2q}{(1-2q\cos t+q^2)^2},\quad J_q^2(t):=\frac{(1-q^2)\sin t}{(1-2q\cos t+q^2)^2},$$

$$\xi(t):=\operatorname{sign}\left(J_q^1(t)+I(q)\right),\quad \zeta(t):=\operatorname{sign}J_q^2(t),$$

where the value $I(q)$ such that

$$\operatorname{mes}\left(J_q^1(t)+I(q)\le 0\right)=\operatorname{mes}\left(J_q^1(t)+I(q)\ge 0\right),\quad t\in\mathbb{T}.$$

We investigate the function $J_q^1(t)$, $t\in[0;\pi]$. We have

$$\frac{d}{dt}J_q^1(t)=\frac{(6q^2-q^4-1-2q(1+q^2)\cos t)\sin t}{(1-2q\cos t+q^2)^3}.$$

Performing elementary transformations, we get $\frac{d}{dt}J_q^1(t)\neq 0$, $t\in(0;\pi)$ for $q\in\left(0;2-\sqrt{3}\right]$. For $q\in\left(2-\sqrt{3};1\right)$ the equation $\frac{d}{dt}J_q^1(t)=0$ has unique solution on interval $t\in(0;\pi)$. Therefore, for $q\in(0;2-\sqrt{3}]$ the function $J_q^1(t)$ is monotone decreasing on $(0;\pi)$. For $q\in(2-\sqrt{3};1)$ the function $J_q^1(t)$ has one extremum on $(0;\pi)$. It was also shown in [11,12].

Next we find q for which the condition $J_q^1(\pi)-J_q^1\left(\frac{\pi}{2}\right)<0$ is fulfilled. Considering the inequality

$$q^4-2q^3-2q^2-2q+1\ge 0,\quad q\in(0;1),$$

we get $q\in(0;q_0]$, where

$$q_0=\left(2+\sqrt{5}-2\sqrt{2+\sqrt{5}}\right)^{1/2}=0.346\ldots.$$

Making calculations, we obtain [11,14]

$$\int\limits_{\mathbb{T}}\mid J_q^1(t)+I(q)\mid\ dt=\begin{cases}\frac{4}{1+q^2}, & q\in(0;q_0],\\ \frac{2(1+q^2)^2}{q(1-q^2)\left(1-q^2+\sqrt{2(1+q^4)}\right)}, & q\in[q_0;1).\end{cases} \tag{10}$$

Also

$$\int\limits_{\mathbb{T}}\mid J_q^2(t)\mid\ dt=\frac{4}{1-q^2}. \tag{11}$$

Taking into account the condition $\operatorname*{ess\,sup}\limits_{t\in\mathbb{T}}\mid\varphi(t)\mid\le 1$ and formulas (10), (11), we obtain the upper estimates in formulas (1), (2).

Further let $\varphi^*(t)$ be the 2π–periodic expansion of the function

$$\tilde{\varphi}(t) := \xi(t)\cos^2\frac{\beta\pi}{2} - \zeta(t)\sin^2\frac{\beta\pi}{2} = \begin{cases} -\cos^2\frac{\beta\pi}{2} + \sin^2\frac{\beta\pi}{2}, & t \in \left[-\pi; -\frac{\pi}{2}\right], \\ \cos^2\frac{\beta\pi}{2} + \sin^2\frac{\beta\pi}{2}, & t \in \left[-\frac{\pi}{2}; 0\right], \\ \cos^2\frac{\beta\pi}{2} - \sin^2\frac{\beta\pi}{2}, & t \in \left[0; \frac{\pi}{2}\right], \\ -\cos^2\frac{\beta\pi}{2} - \sin^2\frac{\beta\pi}{2}, & t \in \left[\frac{\pi}{2}; \pi\right], \end{cases}$$

and

$$f^*(x) := A_0 + \frac{1}{\pi}\int\limits_{\mathbb{T}} \varphi^*(x+t)\sum_{k=1}^{\infty} q^k \cos\left(kt + \frac{\beta\pi}{2}\right) dt.$$

Note that $f^*(x) \in \mathrm{G}(q;\beta)$.

Then

$$\delta_n[f^*](0) = \frac{q}{\pi n}\left(\int\limits_{-\pi}^{-\pi/2} (-\cos^2\frac{\beta\pi}{2} + \sin^2\frac{\beta\pi}{2})\left(J_q^1(t)\cos\frac{\beta\pi}{2} + J_q^2(t)\sin\frac{\beta\pi}{2}\right) dt\right.$$

$$+ \int\limits_{-\pi/2}^{0} (\cos^2\frac{\beta\pi}{2} + \sin^2\frac{\beta\pi}{2})\left(J_q^1(t)\cos\frac{\beta\pi}{2} + J_q^2(t)\sin\frac{\beta\pi}{2}\right) dt$$

$$+ \int\limits_{0}^{\pi/2} (\cos^2\frac{\beta\pi}{2} - \sin^2\frac{\beta\pi}{2})\left(J_q^1(t)\cos\frac{\beta\pi}{2} + J_q^2(t)\sin\frac{\beta\pi}{2}\right) dt$$

$$\left.+ \int\limits_{\pi/2}^{\pi} (-\cos^2\frac{\beta\pi}{2} - \sin^2\frac{\beta\pi}{2})\left(J_q^1(t)\cos\frac{\beta\pi}{2} + J_q^2(t)\sin\frac{\beta\pi}{2}\right) dt\right) + O(1)\frac{q^n}{n(1-q)^3}$$

$$= \left(\cos^3\frac{\beta\pi}{2}\left(\int\limits_{-\pi}^{-\pi/2} (-1)J_q^1(t)\,dt + \int\limits_{-\pi/2}^{0} J_q^1(t)\,dt + \int\limits_{0}^{\pi/2} J_q^1(t)\,dt + \int\limits_{\pi/2}^{\pi} (-1)J_q^1(t)\,dt\right)\right.$$

$$+ \sin^3\frac{\beta\pi}{2}\left(\int\limits_{-\pi}^{-\pi/2} J_q^2(t)\,dt + \int\limits_{-\pi/2}^{0} J_q^2(t)\,dt + \int\limits_{0}^{\pi/2} (-1)J_q^2(t)\,dt + \int\limits_{\pi/2}^{\pi} (-1)J_q^2(t)\,dt\right)$$

$$+ \sin^2\frac{\beta\pi}{2}\cos\frac{\beta\pi}{2}\left(\int\limits_{-\pi}^{-\pi/2} J_q^1(t)\,dt + \int\limits_{-\pi/2}^{0} J_q^1(t)\,dt + \int\limits_{0}^{\pi/2} (-1)J_q^1(t)\,dt + \int\limits_{\pi/2}^{\pi} (-1)J_q^1(t)\,dt\right)$$

$$+ \cos^2\frac{\beta\pi}{2}\sin\frac{\beta\pi}{2}\left(\int\limits_{-\pi}^{-\pi/2} (-1)J_q^2(t)\,dt + \int\limits_{-\pi/2}^{0} J_q^2(t)\,dt + \int\limits_{0}^{\pi/2} J_q^2(t)\,dt\right.$$

$$+\int\limits_{\pi/2}^{\pi}(-1)J_q^2(t)\,dt\Bigg)\Bigg)+O(1)\frac{q^n}{n(1-q)^3}. \tag{12}$$

Since $J_q^1(t)$ is even function and $J_q^2(t)$ is odd function, we may drop the third and fourth line in (12).

Combining (10), (11), (12), we have the following equalities

$$\delta_n[f^*](0)=\frac{4q}{\pi n(1+q^2)}\cos^3\frac{\beta\pi}{2}+\frac{4q}{\pi n(1-q^2)}\sin^3\frac{\beta\pi}{2}+O(1)\frac{q^n}{n},\quad q\in(0;q_0], \tag{13}$$

$$\delta_n[f^*](0)=\frac{2}{\pi n}\frac{(1+q^2)^2}{(1-q^2)\left(1-q^2+\sqrt{2(1+q^4)}\right)}\cos^3\frac{\beta\pi}{2}$$
$$+\frac{4q}{\pi n(1-q^2)}\sin^3\frac{\beta\pi}{2}+O(1)\frac{q^n}{n(1-q)^3},\quad q\in[q_0;1). \tag{14}$$

From (13), (14) we obtain the lower estimates in (1), (2). The theorem is proved. □

Corollary 1. *Let $q\in(0;1)$.*

1. *If $q\in(0;q_0]$ then*

$$\sup_{f\in \mathrm{G}(q)}\|f-\sigma_n[f]\|_{\mathrm{C}}=\frac{4q}{\pi n(1+q^2)}+O(1)\frac{q^n}{n},$$

where $O(1)$ is a quantity, uniformly bounded with respect to n.

2. *If $q\in[q_0;1)$ then*

$$\sup_{f\in \mathrm{G}(q)}\|f-\sigma_n[f]\|_{\mathrm{C}}=\frac{2}{\pi n}\frac{(1+q^2)^2}{(1-q^2)\left(1-q^2+\sqrt{2(1+q^4)}\right)}+O(1)\frac{q^n}{n(1-q)^3},$$

where $O(1)$ is a quantity, uniformly bounded with respect to q, n.

Corollary 2. *Let $q\in(0;1)$. Then*

$$\sup_{f\in \tilde{\mathrm{G}}(q)}\|f-\sigma_n[f]\|_{\mathrm{C}}=\frac{4q}{\pi n(1-q^2)}+O(1)\frac{q^n}{n(1-q)^3},$$

where $O(1)$ is a quantity, uniformly bounded with respect to q, n.

Corollaries 1 and 2 coincide with the results of works [11, 14].

Remark 1. Inequalities (1) and (2) are asymptotically exact without additional conditions. For $q=q_0$ formulas (1) and (2) are the same.

Remark 2. The upper and lower estimates in formulas (1) and (2) differ in degrees of $\cos\frac{\beta\pi}{2}$ and $\sin\frac{\beta\pi}{2}$. The difference is essential, but in case $\beta\in\mathbb{Z}$, are exact equalities.

References

1. Alzer, H., Kwong, M.K. (2020). Inequalities for trigonometric sums and applications. *Aequationes mathematicae, 94*, 235–251.
2. Bauschke, H.H., Krishan, Lal M., Wang, X. (2023). Directional asymptotics of Fejer monotone sequences. *Optimization Letters, 17*, 531–544.
3. Berens, H., Xu, Y. (1996). Fejer means for multivariate Fourier series. *Mathematische Zeitschrift, 221*, 449–465.
4. Berriochoa, E., Cachafeiro, A. (2015). On the asymptotic constant for the rate of Hermite–Fejer convergence on Chebyshev nodes. *Acta Mathematica Hungarica, 147*, 32–45.
5. Brandolini, L., Travaglini, G. (1997). Pointwise convergence of Fejer type means. *Tohoku Mathematical Journal. Second Series, 49* (3), 323–336.
6. Bushev, D.N. (1984). Approximation of classes of continuous periodic functions by Zygmund sums. *Institute of Mathematics of Academy of Sciences of USSR, Kyiv*, , 62 p. (Preprint / Academy of Sciences of USSR, Institute of Mathematics; 84.56) (In Russian).
7. Fejer, L. (1903). Untersuchungen uber Fouriersche Reihen. *Mathematische Annalen, 58*, 51–69.
8. Knoop, H., Locher, F. (1990). Hermite-Fejer type interpolation and Korovkin's theorem. *Bulletin of the Australian Mathematical Society, 42* (3), 383–390.
9. Locher, F. (1985). On Hermite-Fejer interpolation at Jacobi zeros. *Journal of Approximation Theory, 44* (2).
10. Lopez-Bonilla, J., Vidal Beltran, S., Yalja Montiel, J. (2007). A note on Dirichlet and Fejer kernels. *Revista De Matematica: Teoria Y Aplicaciones, 14* (1), 101–104.
11. Novikov, O. O., Rovenska, O. G. (2017). Approximation of periodic analytic functions by Fejer sums. *Matematchni Studii, 47* (2), 196–201.
12. Novikov, O., Rovenska, O., Kozachenko, Yu. (2018). Approximation of classes of Poisson integrals by Fejer sums. *Visnyk of V.N. Karazin Kharkiv National University. Ser. Mathematics, Applied Mathematics and Mechanics, 87*, 4–12.
13. Nikolskiy, S.M. (1946). Approximation of the functions by trigonometric polynomials in the mean. *(Dokl.) Acad. Sci. URSS, 10* (3), 207–256. (In Russian).
14. Rovenska, O. (2023). Approximation of classes of Poisson integrals by Fejer means. *Matematychni Studii, 59* (2), 201–204.
15. Rovenska, O. G. (2023). Approximation of classes of Poisson integrals by Fejer means. *Visnyk of V.N. Karazin Kharkiv National University. Ser. Mathematics, Applied Mathematics and Mechanics, 97*, 4–12.
16. Rovenska, O. (2021). The Lower Estimate of Deviations of Fejer Sums on Classes of Poisson Integrals. *Lobachevskii Journal of Mathematics, 42*, 2936–2941.
17. Shi, Y.G. (1999). On the Rough Theory of Hermite-Fejer Type Interpolation of Higher Order. *Acta Mathematica Hungarica, 84*, 81–96.
18. Weisz, F. (2011). Triangular Cesaro summability of two dimensional Fourier series. *Acta Mathematica Hungarica 132*, 27–41.
19. Weisz, F. (2022). Cesaro summability and Lebesgue points of higher dimensional Fourier series. *Mathematical Foundations of Computing, 5* (3), 241–257.

О. Ровенська

Асимптотичні оцінки відхилень середніх Фейєра на класах інтегралів Пуассона.

Роботу присвячено дослідженню питань наближення неперервних періодичних функцій дійсної змінної тригонометричними поліномами, що порджуються лінійними методами підсумовування рядів Фур'є. Найпростішим прикладом процесу лінійної апроксимації періодичних функцій є наближення функцій частинними сумами їх рядів Фур'є. Однак послідовності частинних сум Фур'є не є рівномірно збіжними на цілому класі неперервних періодичних функцій. Тому численні дослідження присвячуються вивченню апроксимаційних властивостей наближуючих методів, які породжуються перетвореннями частинних сум ряду Фур'є і дозволяють отримати послідовності тригонометричних поліномів, які є рівномірно збіжними для всього класу неперервних функцій. Зокрема, середні Фейєра інтенсивно вивчаються протягом останніх десятиліть. Однією з важливих задач в цьому напрямку є вивчення асимптотичної поведінки точних верхніх меж відхилень тригонометричних поліномів по фіксованому класу періодичних функцій. Метою роботи є систематизація відомих результатів, що стосуються наближення класів періодичних функцій високої гладкості середніми арифметичними сум Фур'є, та представлення нових фактів, отриманих для більш загального випадку. У роботі досліджено асимптотичну поведінку точних верхніх меж по класам інтегралів Пуассона періодичних функцій дійсної змінної відхилень лінійних середніх рядів Фур'є, які будуються за допомогою методу підсумовування Фейєра. Зазначені класи складаються з аналітичних функцій дійсної змінної, які можуть бути регулярно подовжені у відповідну смугу комплексної площини. В роботі знайдено асимптотичні нерівності для точних верхніх меж відхилень середніх Фейєра по класу інтегралів Пуассона. При певному виборі параметрів, що визначають клас, одержані формули збігаються з раніше відомими асимптотичними рівностями.

Ключові слова: *середнє Фейєра, інтеграл Пуассона, асимптотична формула.*

Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine
`rovenskaya.olga.math@gmail.com`

Received 20.05.2023