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ON THE POSSIBILITY OF JOINING POINTS WITH PATHS INSIDE BALLS

The article is devoted to the possibility of joining by paths the points of two sequences that converge to some (one) boundary point of a domain of n-dimensional Euclidean space. It is well known that in domains locally connected on their boundary, it is possible to construct a system of infinitesimal neighborhoods of the boundary point, the intersection of which with a given domain is path connected. Of course, this construction is impossible in an arbitrary domain, for example, it is not true in a disk with a cut on the plane. The problem of the possibility of joining pairs of points in the intersection of a domain with arbitrarily small concentric balls seems more subtle. The problems that are solved in this manuscript are of an even more subtle nature. Let us pose the following problem: let some set be nowhere dense, and a given domain is finitely connected with respect to this set, and balls centered at a fixed point of the boundary form connected intersections with the given domain, and the intersections of these balls with the complement of the specified set consist of no more than m components. In this case, is it possible to join the points of sequences converging to some boundary point by paths that lie in the intersection of the domain with infinitesimal balls centered at this point, which contain no more than $m-1$ points of the aforementioned set? The answer to this question is positive and is given in the article. It should be noted that relatively recently we have shown the existence of infinitesimal neighborhoods with the specified property for paths that join the points of neighborhoods and intersect with a "bad" set no more than $m-1$ times. The purpose of our manuscript is to prove that under some additional conditions one can take not an abstract system of neighborhoods, but a sequence of concentric balls. To a certain extent, the proof of this result is constructive. In particular, the possibility of joining points by paths that contain at most $m-1$ points from some nowhere dense set is related to finite connectedness with respect to the complement of this set. This construction has applications to the problem of Hölder continuity of maps in a given domain. In particular, the existence of families of paths with a conditionally "minimal" intersection with some nowhere dense set is related to the study of non-closed maps, or maps that do not preserve the boundary of the domain.

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1. Introduction. Given $x_0 \in \mathbb{R}^n$, we put

$$B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}\,.$$

The following statement holds, cf. [1, Lemma 2.1].

Theorem 1. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and let $x_0 \in \partial D$. Assume that E is closed and nowhere dense in D, and D is finitely connected on E, i.e., for any $z_0 \in E$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V} \subset \widetilde{U}$ of z_0 such that $(D \cap \widetilde{V}) \setminus E$ consists of finite number of components.*

In addition, assume that the following condition holds: for any $x_0 \in \partial D$ there are $m = m(x_0) \in \mathbb{N}$, $1 \leqslant m < \infty$ and $r_0^{\prime} = r_0^{\prime}(x_0) > 0$ such that

2a) $B(x_0, r) \cap D$ is connected for any $0 < r < r_0^{\prime}$;

2b) $(B(x_0,r)\cap D)\setminus E$ *consists at most of* m *components.*

Let $x_k, y_k\in D\setminus E$, $k=1,2,\ldots,$ *be a sequences converging to* x_0 *as* $k\to\infty$. *Then there are paths* γ_k, $k=1,2,\ldots,$ *such that* γ_k *lies in* $\overline{B(x_0,r_k)}\cap D$, $r_k=\max\{|x_k-x_0|,|y_k-x_0|\}$, $\gamma_k:[0,1]\to\overline{B(x_0,r_k)}\cap D$, $\gamma_k(0)=x_k$, $\gamma_k(1)=y_k$, $\gamma_k(t)\in B(x_0,r_k)\cap D$ *for* $0<t<1$, *whenever* $|\gamma_k|:=\{x\in\mathbb{R}^n:\exists\, t\in[0,1]:\gamma_k(t)=x\}$ *contains at most* $m-1$ *points in* E.

2. Proof of Theorem 1. Without loss of generality, we may assume that $|x_k-x_0|\geqslant|y_k-x_0|$ for any $k=1,2,\ldots.$ From the conditions of the lemma it follows that the sets $W_k:=V_k\cap D$, $V_k:=B(x_0,r_k)$, $k=1,2,\ldots,$ is connected for sufficiently large $k\in\mathbb{N}$, and $(V_k\cap D)\setminus E$ consists at most of m components, $1\leqslant m<\infty$. We may assume that the above holds for any $k\in\mathbb{N}$. Since E is closed, there is $\varepsilon_k>0$ such that $B(x_k,\varepsilon_k)\subset D\setminus E$. Now, the segment $I_k=x_k+(x_0-x_k)t$, $t\in[0,1]$, consists some a subsegment $\widetilde{I_k}=x_k+(x_0-x_k)t$, $t\in[0,a_k]$, $a_k>0$, lying entirely in $B(x_k,\varepsilon_k)\cap\overline{B(x_0,r_k)}$ such that $\widetilde{I_k}$ lies in $B(x_0,r_k)$ excepting its start point x_k. Let $u_k:=x_k+(x_0-x_k)a_k$. Similarly, there is some a segment $J_k=x_0+(y_k-x_0)t$, $t\in[0,1]$, consists some a subsegment $\widetilde{J_k}=x_0+(y_k-x_0)t$, $t\in[b_k,1]$, $0<a_k<b_k$, lying entirely in $B(y_k,\varepsilon_k)\cap\overline{B(x_0,r_k)}$ such that $\widetilde{J_k}$ lies in $B(x_0,r_k)$ excepting its end point y_k. Let $v_k:=x_0+(y_k-x_0)b_k$. Since by the assumption $B(x_0,r_k)\cap D$ is connected, we may join the points u_k and v_k by some a path $\Delta_k:[a_k,b_k]\to B(x_0,r_k)\cap D$, $\Delta_k(a_k)=u_k$, $\Delta_k(b_k)=v_k$. We set

$$\gamma_k(t)=\begin{cases} I_k(t)\,, & t\in[0,a_k]\,,\\ \Delta_k(t)\,, & t\in[a_k,b_k]\,,\\ J_k(t)\,, & t\in[b_k,1]\,,\end{cases}\ . \tag{1}$$

Now, the path γ_k is a path satisfying "almost all" requirements we need, in particular, γ_k lies in $\overline{B(x_0,r_k)}\cap D$, $r_k=\max\{|x_k-x_0|,|y_k-x_0|\}$, $\gamma_k:[0,1]\to\overline{B(x_0,r_k)}\cap D$, $\gamma_k(0)=x_k$, $\gamma_k(1)=y_k$, $\gamma_k(t)\in B(x_0,r_k)\cap D$ for $0<t<1$. It remains to show that γ_k may be chosen so that $|\gamma_k|$ contains at most $m-1$ points in E. Obviously, it is enough to carry out the reasoning for the path Δ_k instead of γ_k, because the parts I_k and J_k of the path γ_k do not contain points in E by the construction.

We apply the approaches used under the proof of [1, Lemma 2.1]. So, we need to prove that the path Δ_k in (1) may be chosen such that $|\Delta_k|$ contains not more than $m-1$ points in E. Let K_1 be a component of $(V_k\cap D)\setminus E$ containing u_k. If $v_k\subset K_1$, the proof is complete. In the contrary case, $|\Delta_k|\cap(D\setminus K_1)\neq\varnothing$. Let us to show that, in this case,

$$|\Delta_k|\cap(D\setminus\overline{K_1})\neq\varnothing\,. \tag{2}$$

Indeed, since $v_k\in(V_k\cap D)\setminus E$, there is a component K_* of $(V_k\cap D)\setminus E$ such that $v_k\in K_*$. Observe that $v_k\not\in\overline{K_1}$. Indeed, in the contrary case there is a sequence $z_s\in K_1$, $s=1,2,\ldots,$ such that $z_s\to v_k$ as $s\to\infty$. But all components of $(V_k\cap D)\setminus E$ are closed in $(V_k\cap D)\setminus E$ and disjoint (see, e.g., [2, Theorem 1.5.III]). Thus $v_k\in K_1$, as well. It is possible only if $K_1=K_*$, that contradicts the assumption mentioned above. Therefore, the relation (2) holds, as required.

By (2), there is $t_1 \in (a_k, b_k)$ such that

$$t_1 = \sup_{t \geqslant a_k : \Delta_k(t) \in \overline{K_1}} t\,. \tag{3}$$

Set

$$z_1 = \Delta_k(t_1) \in V_k \cap D\,. \tag{4}$$

Observe that $z_1 \in E$. Indeed, in the contrary case there is a component K_{**} of $(V_k \cap D) \setminus E$ such that $z_1 \in K_{**}$. Since K_1 is closed, $K_{**} = K_1$. Since K_1 is open set (see [2, Theorem 4.6.II]), there is $t_1^* > t_1$ such that a path $\Delta_k|_{[t_1, t_1^*]}$ belongs to K_1 yet. But this contradicts with the definition of t_1 in (3).

Let us to show that, there is another component K_2 of $(V_k \cap D) \setminus E$ such that $z_1 \in \partial K_2$. Indeed, the points $\Delta_k(t)$, $t > t_1$, do not belong to $\overline{K_1}$, so that there is a sequence $z_1^l \in (V_k \cap D) \setminus K_1$, $l = 1, 2 \ldots$, such that $z_1^l \to z_1$ as $l \to \infty$. Since E is nowhere dense in D, $V_k \cap D$ is open and $z_1 \in V_k \cap D$, we may consider that $z_1^l \in (V_k \cap D) \setminus E$ for any $l \in \mathbb{N}$. Since there are m components of $(V_k \cap D) \setminus E$, we may choose a component K_2 of them which contains infinitely many elements of the sequence z_1^l. Without loss of generality, passing to a subsequence, if need, we may consider that all elements $z_1^l \in K_2$, $l = 1, 2, \ldots$. Thus, $z_1 \in \partial K_2$, as required.

Observe that, any component K of $(V_k \cap D) \setminus E$ is finitely connected at any point $z_0 \in \partial K$. Indeed, since D is finitely connected on E, for any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V} \subset \widetilde{U}$ of z_0 such that $(D \cap \widetilde{V}) \setminus E$ consists of finite number of components. Now, $(K \cap \widetilde{V}) \setminus E$ consists of finite number of components, as well.

Now, due to Lemma 3.10 in [3] we may replace $\Delta_k|_{[a_k, t_1]}$ by a path belonging to K_1 for any $t \in [a_k, t_1)$ and tending to z_1 as $t \to t_1 - 0$. (Here z_1 is defined in (4)). In order to simplify the notation, without limiting the generality of the reasoning, we may assume that the path $\Delta_k|_{[a_k, t_1]}$ already has the indicated property. Now, there are two cases:

a) $v_k \in K_2$. Then we join the points v_k and z_1 by a path $\alpha_1 : (t_1, b_k] \to K_2$ belonging to K_2 and tending to z_1 as $t \to t_1 + 0$. This is possible due to the finitely connectedness of K_2 proved above and by Lemma 3.10 in [3]. Uniting paths $\Delta_k|_{[a_k, t_1]}$ and α_1, we obtain the desired path. In particular, only one point z_1 belonging to this path also belongs to E.

b) $v_k \notin K_2$. Arguing similarly to mentioned above, we may prove that there is $t_2 \in (t_1, b_k)$ such that

$$t_2 = \sup_{t \geqslant t_1 : \Delta_k(t) \in \overline{K_2}} t\,. \tag{5}$$

Reasoning similarly to the case with point t_1, we may show that:

$$(b_1) \qquad z_2 = \Delta_k(t_2) \in (V_k \cap D) \cap E\,,$$

(b_2) there is another component K_3 of $(V_k \cap D) \setminus E$, $K_1 \neq K_3 \neq K_2$ such that $z_2 \in \partial K_3$,

(b_3) due to Lemma 3.10 in [3] we may replace $\Delta_k|_{[t_1,t_2]}$ by a path belonging to K_2 for any $t \in (t_1, t_2)$, tending to z_1 as $t \to t_1 + 0$ and tending to z_2 as $t \to t_2 - 0$. In order to simplify the notation, without loss of the generality of the reasoning, we may assume that the path $\Delta_k|_{[t_1,t_2]}$ already has the indicated property.

Now, there are two cases:

a) $v_k \in K_3$. Then we join the points v_k and z_3 by a path $\alpha_2 : (t_2, b_k] \to K_3$ belonging to K_3 and tending to z_2 as $t \to t_2 + 0$. This is possible due to finitely connectedness of K_3 proved above and by Lemma 3.10 in [3]. Uniting paths $\Delta_k|_{[a_k,t_2]}$ and α_2, we obtain the desired path. In particular, precisely two points z_1 and z_2 belonging to this path also belong to E.

b) $v_k \notin K_2$. Arguing similarly to mentioned above, we may prove that there is $t_3 \in (t_2, b_k)$ such that

$$t_3 = \sup_{t \geqslant t_2 : \Delta_k(t) \in \overline{K_3}} t\,. \tag{6}$$

And so on. Continuing this process, we will obtain a certain number of points $z_1 = \Delta_k(t_1), z_2 = \Delta_k(t_2), z_3 = \Delta_k(t_3), \ldots, z_{k_{\widetilde{p}-1}} = \Delta_k(t_{\widetilde{p}-1})$ in E and a certain number of components $K_1, K_2, \ldots, K_{\widetilde{p}}$ in $(V_k \cap D) \setminus E$, $K_1 \ne K_2 \ne \ldots \ne K_{\widetilde{p}}$. The corresponding path $\Delta_k|_{[a_k,t_{\widetilde{p}-1}]}$ is a part of a path Δ_k which joins the point u_k and $z_{k_{\widetilde{p}-1}} \in \partial K_{\widetilde{p}}$ in $V_k \cap D$ and such that

$$|\Delta_k|_{[a_k,t_{k_{\widetilde{p}-1}}]}| \cap E = \{z_1, z_2, z_3, \ldots, z_{k_{\widetilde{p}-1}}\}\,.$$

Since at each step the remaining part of the path does not belong to the union of the closures of the previous components $K_1, K_2, \ldots, K_{\widetilde{p}-1}$, and there are only a finite number of these components does not exceed m, then $v_k \in K_{\widetilde{p}}$ for some $1 \leqslant \widetilde{p} \leqslant m$. Then we join the points v_k and $z_{\widetilde{p}-1}$ by a path $\alpha_{\widetilde{p}-1} : (t_{\widetilde{p}-1}, b_k] \to K_{\widetilde{p}}$ belonging to $K_{\widetilde{p}}$ and tending to $z_{\widetilde{p}-1}$ as $t \to t_{\widetilde{p}-1} + 0$. This is possible due to finitely connectedness of $K_{\widetilde{p}}$ proved above and by Lemma 3.10 in [3]. Uniting paths $\Delta_k|_{[a_k,t_{\widetilde{p}-1}]}$ and $\alpha_{\widetilde{p}-1}$, we obtain the desired path. In particular, only the points $z_1, z_2, \ldots, z_{\widetilde{p}-1}$ belonging to this path also belongs to E. The number of these points does not exceed $m-1$. Theorem is proved. $\Box$

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В. Десятка

Про можливість з'єднання точок кривими всередині куль.

Стаття присвячена можливості поєднання кривими точок двох послідовностей, які збігаються до деякої (однієї) межової точки області n-вимірного евклідового простору. Добре відомо, що в областях, локально зв'язних на своїй межі, можно побудувати систему інфінітезимальних околів межової точки, перетин яких з даною областю є лінійно зв'язним. Звісно, ця конструкція неможлива у довільній області, наприклад, це не є вірним у крузі з розрізом на площині. Більш тонким виглядає питання про можливість з'єднання пар точок в перетині області з як завгодно малими концентричними кулями. Проблема, яка розв'язана в даному рукописі, має ще більш тонку природу. Поставимо таку задачу: нехай деяка множина є ніде не щільною, а задана область скінченно зв'язна відносно цієї множини, причому кулі з центром у фіксованій точці межі формують зв'язні перетини з даною областю, а перетини цих куль з доповненням до вказаної множини складається не більше, ніж з m компонент. Чи можна в такому випадку з'єднати точки послідовностей, що збігаються до деякої межової точки, кривими, які лежать в перетині області з інфінітезимальними кулями з центром у цій точці, які містять у собі не більше $m-1$ точок вищезгаданої множини? Відповідь на це питання є позитивною і вона надана у статті. Слід зауважити, що відносно нещодавно нами було показано існування інфінітезимальних околів з вказаною властивістю щодо кривих, які з'єднують точки околів та перетинаються з «поганою» множиною не більше $m-1$ разів. Мета нашого рукопису – довести, що за деяких додаткових умов можна взяти не абстрактну систему околів, а саме послідовність концентричних куль. У певній мірі доведення цього результату є конструктивним. Зокрема, можливість з'єднання точок кривими, які містять у собі не більше $m-1$ точки з деякої ніде не щільної множини, пов'язана зі скінченною зв'язністю відносно доповнення до цієї множини. Дана конструкція має застосування до проблеми неперервності відображень за Гельдером в даній області. Зокрема, побудова сімей кривих з умовно «мінімальним» перетином з деякою ніде не щільною множиною пов'язана з вивченням не замкнених відображень, або відображень, що не зберігають межу області.

Ключові слова: *криві в евклідовому просторі, багатовимірна локальна зв'язність.*

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