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ASYMPTOTIC ESTIMATION OF THE DUFFING OSCILLATOR PARAMETERS

The problem of identification of the Duffing oscillator parameters is considered. These parameters characterize the linear dissipation, linear and nonlinear restoring force. An invariant relations method for the parameters estimation is used. Such approach is based on dynamical extension of original system and synthesis of appropriate invariant relations, from which the unknowns can be expressed as a function of the known quantities on the trajectories of extended system during the observed motion. The stability property is formally checked considering the oscillatory behavior of the system.

MSC: 34C15, 34D20.

Ключові слова: *parameter identification, Duffing oscillator, invariant relations, asymptotic estimates.*

1. Introduction.

In many practical applications of physics, biology and other sciences an approach based on the concept of model equations is used as an approximate mathematical model of complex nonlinear processes. The basis of this concept is the provision that a small number of characteristic type's movements of simple mathematical models inherent in systems give the key to understanding and exploring a huge number of different phenomena. Such problems are typical, for example, for compensating for harmonic disturbances in automatic control algorithms caused by feedback delay effects, for vibration isolation of structures, for adaptive filtering in signal processing, and so on. Many physical systems are conveniently modelled by the Duffing equation, which describes an oscillator with a cubic nonlinearity [1] and used as an approximate model of many physical systems or as a convenient mathematical model. Such equation exhibits a wide range of well-known behavioral patterns and is used by many researchers to illustrate such behavior.

These systems include mechanical as well as electrical systems with nonlinear restoring force characteristics. Therefore, a number of theoretical and engineering problems naturally arise in which determining the parameters of these vibrations becomes crucial. Typically, determining the parameters of a dynamic system does not allow for direct measurement of their values. Instead, they are filtered through the dynamic system and must be reconstructed based on its response. In principle, the least-squares method, Fourier analysis, and the Laplace transform offer potential solutions for such problems. However, these methods may prove unsuitable, for example, for control algorithms with real-time data processing. In this regard, works have appeared that use observers to determine the parameters of oscillations (see, for example [2]).

One of the objectives of this article is to use the method of invariant relations in the problem of parameters observation of mechanical systems. This method was developed [3] in analytical mechanics and was intended, in particular, to find particular solutions (dependencies between variables) in problems of the dynamics of a rigid body with a fixed point. Its modification to problems in observation and identification theory allowed for the synthesis of additional connections between known and unknown quantities of the original system that arise on the trajectories of its extended dynamic model [4,5]. The given approach turned out to be effective in the problems of observation, parameter estimation and determination of external influence for the van der Pol, Duffing, Liénard oscillator models [6, 8].

It is worth noting that a more general approach than the method of invariant ratios to observation problems was proposed in the works [10, 11] as some modification of the method of stabilization of nonlinear systems $I\&I$ (Input and Invariance).

2. The method of invariant relations in the problem of determining the Duffing equation coefficients.

The Duffing equation can be seen as describing the oscillations of a mass attached to a nonlinear spring and a linear damper:

$$\ddot{x}(t) + \delta\dot{x}(t) + \alpha x(t) + \beta x(t)^3 = 0. \tag{1}$$

Here, $x(t)$ – is the displacement of the mass point from the origin of coordinates; $\dot{x}(t)$ is the velocity of this displacement. The term $\delta\dot{x}(t))$ and function $\alpha x(t) + \beta x(t)^3$ in the equation describe, respectively, the law of dissipation and restoring forces. It is assumed that (1) simulates the process of permanent oscillations of a real object and its solution are correctly defined functions $x(t), \dot{x}(t)$ within $[0, \infty)$. Next, the property of boundedness of solutions will be used. Let us show that the phase vector of the system is bounded and its values belong $\forall t \geq 0$ to a certain finite region $(x(t), \dot{x}(t)) \in S_x \subset R^2$.

Indeed, multiplying (1) by $\dot{x}(t)$ we get

$$\dot{x}(t)\left[\ddot{x}(t) + \delta\dot{x}(t) + \alpha x(t) + \beta x(t)^3\right] = 0.$$

From this equality it follows that

$$\frac{d}{dt}\left[\frac{1}{2}\dot{x}(t)^2 + \frac{1}{2}\alpha x(t)^2 + \frac{1}{4}\beta x(t)^4\right] = -\delta\dot{x}(t)^2.$$

Then by Lyapunov function method the functions $x(t), \dot{x}(t)$ are bounded when $\delta \geq 0$.

Let us rewrite the original equation (1) in the form of a second-order system in a state space representation. Denote

$$x_1(t) = x(t), \quad x_2(t) = \dot{x}(t)$$

and as a result we get:

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -\delta x_2(t) - \alpha x_1(t) - \beta x_1(t)^3. \tag{2}$$

We will assume that the output of the system is the phase vector $x_1(t), x_2(t)$, measured during oscillations, i.e. known as a function of time. The coefficients δ, α, β are unknown and must be determined from data on the movement of the oscillator.

Now let us briefly outline the method of invariant relations we propose. The main idea is to obtain additional relations for the unknowns based on the data about motion of the system. Such information is the output $x_1(t), x_2(t)$ and those quantities that can only be obtained from the output. In particular, based on the number of unknowns, we introduce into consideration the vector $y = (y_1, y_2, y_3)^T$, which we define as the solution of the Cauchy problem for some, not yet defined, system of differential equations and the initial conditions for its trajectory:

$$\dot{y}(t) = U(x_1(t), x_2(t), y(t)), \quad y(0) \in R^3. \tag{3}$$

Further we will consider the system (3) as a kind of dynamic extension of the original system (2).

The idea of the method of invariant relations is to determine additional connections $H(x_1, x_2, y, \delta, \alpha, \beta) = 0$ that arise between known values $x_1(t), x_2(t), y(t)$ and unknowns δ, α, β on some trajectories of the extended system of differential equations (2), (3). In this case, the selection of the appropriate function $U(x_1, x_2, y)$ must ensure that the following conditions are met:

(i) rank$\frac{\partial H(x_1, x_2, y, \delta, \alpha, \beta)}{\partial(\delta, \alpha, \beta)} = 3$;

(ii) the invariant manifold $\{(x_1, x_2, y) \in R^5 : H(x_1, x_2, y, \delta, \alpha, \beta) = 0\}$ corresponding to these additional equalities has the property of global attraction for any solutions of the extended system. In other words, on any trajectories of the differential equations (2), (3)

$$\lim_{t \to \infty} \|H(x_1(t), x_2(t), y(t), \delta, \alpha, \beta)\| = 0.$$

This method is further used to solve the

Problem. Find asymptotically accurate estimates of the coefficients δ, α, β based on the output data about phase vector $x_1(t), x_2(t)$ of the system (2).

3. Derivation of a family of invariant relations.

To satisfy property (i), we will look for additional relations on the trajectories of the system of differential equations (2), (3) in the form:

$$\begin{aligned} \delta &= \Psi_1(x_1(t), x_2(t)) + y_1(t), \\ \alpha &= \Psi_2(x_1(t), x_2(t)) + y_2(t), \\ \beta &= \Psi_3(x_1(t), x_2(t)) + y_3(t). \end{aligned} \tag{4}$$

Similar to the right side $U(x_1, x_2, y)$ of the differential equations (3), the functions $\Psi_i(x_1, x_2), i = \overline{1,3}$ are still undefined. It is assumed that these functions are continuously differentiable with respect to their arguments. Let us also introduce into consideration the vector $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^T$ that characterize deviations from relations (4):

$$\begin{aligned}\varepsilon_1 &= \delta - \Psi_1(x_1(t), x_2(t)) - y_1(t),\\ \varepsilon_2 &= \alpha - \Psi_2(x_1(t), x_2(t)) - y_2(t),\\ \varepsilon_3 &= \beta - \Psi_3(x_1(t), x_2(t)) - y_3(t).\end{aligned} \tag{5}$$

If the equalities $\varepsilon_1(t) = \varepsilon_2(t) = \varepsilon_3(t) \equiv 0$ holds on some trajectories of the extended system for certain functions $U(x_1, x_2, y), \Psi_i(x_1, x_2)$, then the relations (4) become invariant and determine coefficients δ, α, β on these trajectories. Non-zero deviations $\varepsilon_i(t), i = \overline{1,3}$ appear on other trajectories.

Consider formulas (5) as a substitution of parameters δ, α, β in the equation of original system (2). Then equalities $\dot{\delta} = 0, \dot{\alpha} = 0, \dot{\beta} = 0$ are transformed into differential relations (here $(.)'_q$ means the partial derivative $\frac{\partial(.)}{\partial q}$)

$$\dot{y}_i + \dot{\varepsilon}_i = -\Psi'_{i x_!} x_2 - \Psi'_{i x_2} u, \quad i = \overline{1,3}, \tag{6}$$

where

$$u = -(\Psi_1 + y_1 + \varepsilon_1)x_2 - (\Psi_2 + y_2 + \varepsilon_2)x_1 - (\Psi_3 + y_3 + \varepsilon_3)x_1^3.$$

Let us define the structure of the additional system of differential equations (3). At the first step of the proposed approach to determine the function $U(x_1, x_2, y)$. We select from the previous equalities all terms that do not depend on deviations ε. As a result we get

$$\dot{y}_i = -\Psi'_{i x_1} x_2 - \Psi'_{i x_2} v \quad i = \overline{1,3}, \tag{7}$$

where

$$v = -(\Psi_1 + y_1)x_2 - (\Psi_2 + y_2)x_1 - (\Psi_3 + y_3)x_1^3.$$

In accordance with this choice, we have that if $y_i(t), i = \overline{1,3}$ are any functions that satisfy (7), then the terms remaining in ratios (6) form the linear homogeneous differential equations with respect to deviations $\varepsilon_i(t)$

$$\dot{\varepsilon}_i = -\Psi'_{i x_2}(x_2\varepsilon_1 + x_1\varepsilon_2 + x_1^3\varepsilon_3), \quad i = \overline{1,3}. \tag{8}$$

System (8) has the trivial solution $\epsilon_1(t) = \varepsilon_2(t) = \varepsilon_3(t) \equiv 0$. So, we have obtained a family of additional systems (7), depending on arbitrary functions $\Psi_i(x_1, x_2), i = \overline{1,3}$ and their derivatives. Thus, the following lemma is proved.

Lemma. *For any functions $\Psi_i(x_1, x_2), i = \overline{1,3}$ differentiable with respect to their arguments the corresponding system (2),(7) on some trajectories turns the equalities (4) into invariant relations, from which the parameters δ, α, β can be directly found.*

4. Stabilization of Deviations.

We do not know on which trajectories of the system of differential equations (2), (7) the relations (4) become invariant. Therefore, it is natural to consider the problem of choosing functions $\Psi_i(x_1, x_2),\ i = \overline{1,3}$ in such a way that deviations in formulas (5) tend to zero. In other words, the trivial solution of equations (8) should become globally asymptotically stable. The second stage of the proposed method is related to solving this problem. There are several approaches to stabilize non-autonomous systems (see [12,13]). At this stage, we assume that the appropriate method is selected depending on the analytical form of the original dynamic system.

To stabilize trivial solution of the system (8) we use partial derivatives ${\Psi_i}'_{x_2}, i = \overline{1,3}$ of uncertain functions as a control. The purpose of control is to fulfill the condition

$$\lim_{t\to\infty} \sum_{i=1}^{3} \varepsilon_i^2(t) = 0. \tag{9}$$

Let us consider the positive definite function $V = \frac{1}{2}\sum_{i=1}^{3} \varepsilon_i^2(t)$ and write its derivative with respect to system (8):

$$\dot V = -({\Psi_1}'_{x_2}\varepsilon_1 + {\Psi_2}'_{x_2}\varepsilon_2 + {\Psi_3}'_{x_2}\varepsilon_3)(x_2\varepsilon_1 + x_1\varepsilon_2 + x_1^3\varepsilon_3).$$

Let us choose the remaining undefined functions by setting:

$$\Psi_1(x_1,x_2) = \lambda\frac{x_2^2}{2},\ \Psi_2(x_1,x_2) = \lambda x_1x_2,\ \Psi_3(x_1,x_2) = \lambda x_1^3x_2, \tag{10}$$

where λ is a positive constant. Under such functions, the derivative

$$\dot V = -\lambda(x_2\varepsilon_1 + x_1\varepsilon_2 + x_1^3\varepsilon_3)^2 \le 0.$$

That is, the function V becomes a Lyapunov function with a negative semidefinite derivative. In this case, the zero solution of the equations in deviations is uniformly stable. Let us show that it is asymptotically stable. To do this, we will use the LaSalle's invariance principle [14] for autonomous systems.

Taking into account (10), this system has the form

$$\begin{aligned}\dot\varepsilon_1 &= -\lambda x_2(x_2\varepsilon_1 + x_1\varepsilon_2 + x_1^3\varepsilon_3),\\ \dot\varepsilon_2 &= -\lambda x_1(x_2\varepsilon_1 + x_1\varepsilon_2 + x_1^3\varepsilon_3),\\ \dot\varepsilon_3 &= -\lambda x_1^3(x_2\varepsilon_1 + x_1\varepsilon_2 + x_1^3\varepsilon_3).\end{aligned} \tag{11}$$

Proposition 1. *If the functions $\Psi_i(x_1,x_2), i = \overline{1,3}$ are chosen according to formulas (10), then the trivial solution of the system of differential equations for the deviations (8) is globally exponentially stable.*

Proof. The phase vector of system (2),(11) is $(x_1,x_2,\varepsilon_1,\varepsilon_2,\varepsilon_3) \in S_x \times S_\varepsilon = \Omega \subset R^5$, where

$$S_\varepsilon = \{(\varepsilon_1,\varepsilon_2,\varepsilon_3) : \sum_{i=1}^{3} \varepsilon_i^2(t) \le \sum_{1}^{3} \varepsilon_i^2(0)\}$$

is a ball in R^3 Thus, the set $\overline{S_x} \times S_\varepsilon = \Omega$ is compact and positive invariant to (11).

Let E be the set of all points $(x_1, x_2, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ where the derivative of the Lyapunov function is identically equal to zero:

$$\dot{V} = \lambda(x_2\varepsilon_1 + x_1\varepsilon_2 + x_1^3\varepsilon_3)^2 \equiv 0.$$

It is clear that the set $M = \{(x_1(t), x_2(t).\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = 0), \forall t \geq 0\}$ is invariant in E. If M is the largest invariant set, then according to LaSalle's invariant principle all solutions with initial values in E tend to M, i.e. condition (9) is satisfied.

Let us show that M is the largest invariant set in E. To this end, let us assume the opposite, i.e., the set E contains trajectories with non-zero deviations $\varepsilon_i(t)$. From the differential equations (11) it follows that these non-zero deviations are constant $\varepsilon_i(t) = \varepsilon_i^*, i = \overline{1,3}$. Hence, our assumption leads to the existence of the relation

$$x_2(t)\varepsilon_1^* + x_1(t)\varepsilon_2^* + x_1(t)^3\varepsilon_3^* \equiv 0.$$

It is easy to check that the last relation is not a first integral of the system (2) for any constants $\varepsilon_i^*, i = \overline{1,3}$. The resulting contradiction proves the statement that the set M is the largest invariant set in E, and therefore, according to LaSalle's invariant principle, conditions (9) are satisfied. Hence, the Proposition 1 is proved

5. Final evaluation of unknowns.

In the proposed algorithm, all degrees of freedom have been fixed. Specifically, the structure of additional differential equations is defined by equation (7). The choice of functions $\Psi_i(x_1, x_2),\ i = \overline{1,3}$ according to formulas (10) determines the final form of these equations:

$$\dot{y}_1 = \lambda x_2 w,\ \dot{y}_2 = \lambda(-x_2 + x_1 w),\ \dot{y}_3 = \lambda(3x_1^2 x_2 - x_1^3 w), \tag{12}$$

where

$$w = (\lambda\frac{x_2^2}{2} + y_1)x_2 + (\lambda x_1 x_1 + y_2)x_1 + (\lambda x_1^3 x_2 + y_3)x_1^3.$$

According to this scheme, the terms in formulas (5) become known and the deviations $\varepsilon_i(t)$ asymptotically tend to zero, which allows us to directly obtain asymptotic estimates of the parameters δ, α, β. Thus, the statement formulated below has been proved.

Proposition 2. *The formulas*

$$\hat{\delta} = \lambda\frac{x_2(t)^2}{2} + y_1(t),\ \hat{\alpha} = \lambda x_1(t) + y_2(t),\ \hat{\beta} = \lambda x_1(t)^3 x_2(t) + y_3(t), \tag{13}$$

where $y(t) = (y_1(t), y_2(t), y_3(t))^T$ are the solutions of any Cauchy problem for system of differential equations (12) give asymptotic estimates of the coefficients δ, α, β.

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Асимптотична оцінка параметрів осцилятора Дуффінга.

Відомо, що нелінійний коливальний рух складних систем може бути приблизно промодельовано рухом системи, яка складається з одного або декількох пов'язаних між собою осциляторів. З цієї причини осцилятори широко вивчаються як спосіб моделювання, аналізу або навіть контролю у різних сферах, таких як електроніка, керування, медико-біологічні дослідження, геологія та ін. Природньо, що при цьому на практиці виникають проблеми визначення стану та параметрів таких моделей за результатами вимірювання вихідних сигналів у режимі реального часу. Одну з таких проблем, а саме проблему визначення коефіцієнтів рівняння осцилятора Дуффінга розглянуто у цій статті. Це рівняння демонструє широкий спектр поведінки відомих моделей та використовується багатьма дослідниками для ілюстрації особливостей динаміки об'єкта. Ці моделі включають механічні, а також електричні системи з нелінійними характеристиками відновлювальної сили. Тому природньо виникає низка теоретичних та інженерних проблем, у яких визначення параметрів цих коливань стає вирішальним. Як правило, визначення параметрів динамічної системи не дозволяє безпосередньо виміряти їх значення. Натомість вони фільтруються через динамічну систему та повинні бути реконструйовані на основі її реакції. В принципі, метод

найменших квадратів, аналіз Фур'є та перетворення Лапласа пропонують потенційні рішення для таких задач. Однак ці методи можуть виявитися непридатними, наприклад, для онлайн алгоритмів керування з обробкою даних у реальному часі. В роботі запропонована схема визначення асимптотичних оцінок цих невідомих за даними про рух осцилятора. При цьому однією з цілей цієї статті є використання методу інваріантних співвідношень у задачі онлайн визначення параметрів коливальних систем. Цей метод був розроблений [3] в аналітичній механіці та був призначений, зокрема, для знаходження частинних рішень (залежностей між змінними) у задачах динаміки твердого тіла з нерухомою точкою. Його модифікація для задач теорії спостереження та ідентифікації дозволила синтезувати додаткові зв'язки між відомими та невідомими величинами вихідної системи, що виникають на траєкторіях її розширеної динамічної моделі [4, 5]. Зазначений підхід виявився ефективним у задачах спостереження, оцінки параметрів та визначення зовнішнього впливу для моделей осциляторів Ван-дер-Поля, Даффінга, Лієнара [6, 8].

Keywords: *ідентифікація параметрів, осцилятор Даффінга, інваріантні співвідношення, асимптотичні оцінки.*

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