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ON DIRICHLET PROBLEM FOR QUASILINEAR GENERALIZED CAUCHY–RIEMANN EQUATIONS

In the article, a series of theorems has been obtained that establishes conditions for the existence of regular solutions to the Dirichlet problem for quasilinear generalized Cauchy–Riemann equations with continuous boundary data in arbitrary bounded simply connected domains of the complex plane. Such equations model compressible fluids steady flows in anisotropic and inhomogeneous media.

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1. Introduction.

As it is well-known, the characteristic property of an analytic function $f = u+i\,v$ in the complex plane $\mathbb{C}$ is that its real and imaginary parts satisfy the **Cauchy-Riemann system**

$$\frac{\partial u}{\partial x} \;=\; \frac{\partial v}{\partial y}\,, \quad \frac{\partial u}{\partial y} \;=\; -\,\frac{\partial v}{\partial x}\,, \qquad z = x + i\,y\ . \tag{1}$$

Euler was the first who has found the connection of the system (1) to the analytic functions. A physical interpretation of (1), going back to Riemann works on function theory, is that u represents a **potential function** of incompressible fluid steady flows in homogeneous isotropic media and v is its **stream function**.

This system can be written as the one equation in the matrix form

$$\nabla\,v \;=\; \mathbb{H}\;\nabla\,u\ , \tag{2}$$

where $\nabla\,v$ and $\nabla\,u$ denotes the gradient of v and u, correspondingly, interpreted as vector-columns in $\mathbb{R}^2$, and $\mathbb{H} : \mathbb{R}^2 \to \mathbb{R}^2$ is the so-called **Hodge operator** represented as the 2×2 matrix

$$\mathbb{H} \;:=\; \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right]\ , \tag{3}$$

which carries out the counterclockwise rotation of vector columns by the angle $\pi/2$ in the real plane $\mathbb{R}^2$. Thus, (2) shows that streamlines and equipotential lines of the fluid flow are mutually orthogonal. Note also that $\mathbb{H}$ is an analog of the imaginary unit in the space $\mathbf{M^{2\times 2}}$ of all 2×2 matrices with real entries because

$$\mathbb{H}^2 \;:=\; -\,I\ , \tag{4}$$

where I is the unit 2×2 matrix.

In works [1] and [2], it was studied **generalized Cauchy–Riemann equations**:

$$\nabla v \;=\; B\,\nabla u \tag{5}$$

with matrix valued coefficients $B : D \to M^{2\times 2}$ that describe incompressible fluid steady flows in anisotropic and inhomogeneous media. On the basis of the well developed theory of the Beltrami equations, see e.g. monographs [3]– [8] and articles [9]– [11], in particular, it was given there the corresponding consequences for the Dirichlet problem with continuous data to these equations.

The **Dirichlet problem** for a generalized Cauchy-Riemann equation (5) was to find its solutions (u, v) with prescribed continuous date $\varphi : \partial D \to \mathbb{R}$ of the potential function u at the boundary

$$\lim_{z\to\zeta} u(z) \;=\; \varphi(\zeta) \qquad \forall\, \zeta \in \partial D \tag{6}$$

in arbitrary bounded simply connected domains D in $\mathbb{R}^2$.

Given a simply connected domain D in $\mathbb{R}^2$, it is said that a pair (u, v) of continuous functions $u : D \to \mathbb{R}$ and $v : D \to \mathbb{R}$ in the class $W^{1,1}_{\rm loc}$ is a **regular solution of the Dirichlet problem** (6) for the generalized Cauchy–Riemann equation (5) in D if (u, v) satisfies (5) a.e. in D and, moreover, $\nabla u \neq 0$ and $\nabla v \neq 0$ a.e. in D, and the correspondence $(x, y) \mapsto (u, v)$ is a discrete and open mapping of D into $\mathbb{R}^2$.

Recall that a mapping of a domain D in $\mathbb{R}^2$ into $\mathbb{R}^2$ is called **discrete** if the preimage of each point in $\mathbb{R}^2$ under the mapping consists of isolated points in D and **open** if the mapping maps every open set in D onto an open set in $\mathbb{R}^2$.

In the following exposition, attention will be focused on quasilinear generalized Cauchy–Riemann equations of the form

$$\nabla v \;=\; a(u)\, B\, \nabla u\ , \tag{7}$$

where $B : D \to \mathbb{B}^{2\times 2}$ is a matrix-valued coefficient, and $a : \mathbb{R} \to [0, \infty]$ is a real-valued coefficient depending on u.

Such equations model compressible fluids steady flows in anisotropic and inhomogeneous media. The Dirichlet boundary condition for the equation (7) and its regular solutions are defined in the same way as above.

It is necessary to note that quasilinear generalized Cauchy–Riemann equations are closely related to quasilinear Beltrami equationns, see e.g. monograph [4] and article [12].

Recall also that a function $g : \mathbb{R} \to \mathbb{R}$ is called a **quasi-isometry** if

$$L^{-1}|t_1 - t_2| \leqslant |g(t_1) - g(t_2)| \leqslant L|t_1 - t_2|$$

for some $L \in [1, \infty)$ and for all t_1 and $t_2 \in \mathbb{R}$.

2. Preliminaries.

Let us denote by $\mathbb{B}^{2\times 2}$ space of all 2×2 matrices with real entries,

$$B \;=\; \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\,, \tag{8}$$

with $\det B = 1$, antisymmetric with respect to its auxiliary diagonal, i.e., with $b_{22} = -b_{11}$, and with the **ellipticity condition** $|\mu| < 1$, where

$$\mu \;=\; \mu_B \;:=\; \frac{b_{12}+b_{21}-2\,i\,b_{11}}{b_{12}-b_{21}-2}\,. \tag{9}$$

Note that, under the above conditions $\det B = 1$ and $b_{22} = -b_{11}$, the ellipticity condition $|\mu|^2 < 1$ is equivalent to the condition $b_{21} > b_{12}$ and, furthermore, to the conditions $b_{12} < 0$ and $b_{21} > 0$.

The quantity (9) is called a **complex characteristic** of (5). Criteria of solvability of the equation (5) in [1] and [2], were formulated in terms of its **dilatation quotient**

$$K_{\mu_B} \;:=\; \frac{1+|\mu_B|}{1-|\mu_B|}\,. \tag{10}$$

Hereafter $d\mathcal{L}(Z)$ corresponds to the Lebesgue measure in $\mathbb{R}^2$ with the notation $Z := (x,y)\in\mathbb{R}^2$.

Let us formulate the following lemma on the existence of regular solutions to the Dirichlet problem for generalized Cauchy–Riemann equations, which is a consequence of Lemma 4.1 in [1]. Here and later on we assume that K_{μ_B} is extended by1 outside D.

Lemma 1. *Let D be a bounded simply connected domain in $\mathbb{R}^2$, let $B : D \to \mathbb{B}^{2\times 2}$ be a measurable function in D and $\mu = \mu_B$ be given by the formula (9) with $K_{\mu_B}\in L^1(D)$. Suppose also that*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_{\mu_B}(z)\cdot\psi^2_{z_0,\varepsilon}(|z-z_0|)\,d\mathcal{L}(z) = o(I^2_{z_0}(\varepsilon)) \qquad \forall\ z_0\in\overline{D} \tag{11}$$

as $\varepsilon\to 0$ for $\varepsilon_0 = \varepsilon(z_0) > 0$ and a family of measurable functions $\psi_{z_0,\varepsilon} : (0,\varepsilon_0)\to(0,\infty)$ such that

$$I_{z_0}(\varepsilon)\colon\;=\int\limits_{\varepsilon}^{\varepsilon_0}\psi_{z_0,\varepsilon}(t)\,dt<\infty \qquad \forall\ \varepsilon\in(0,\varepsilon_0)\,. \tag{12}$$

Then the generalized Cauchy-Riemann equation (5) has a regular solution (u,v) of the Dirichlet problem (6) in D for each continuous inconstant boundary date $\varphi : \partial D\to\mathbb{R}$.

3. The main lemma.

The lemma below concerns the existence of regular solutions to the Dirichlet problem (6) for quasilinear generalized Cauchy–Riemann equations (7).

Lemma 2. *Let D be a bounded simply connected domain in $\mathbb{R}^2$, $B : D \to \mathbb{B}^{2\times 2}$ be a measurable function in D, $\mu = \mu_B$ be given by formula (9) with $K_{\mu_B} \in L^1(D)$ and $a : \mathbb{R} \to [L^{-1}, L]$, $L \in [1, \infty)$, be a measurable function. Suppose also that*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_{\mu_B}(z) \cdot \psi^2_{z_0,\varepsilon}(|z-z_0|)\, d\mathcal{L}(z) = o(I^2_{z_0}(\varepsilon)) \qquad \forall\ z_0 \in \overline{D} \tag{13}$$

as $\varepsilon \to 0$ for $\varepsilon_0 = \varepsilon(z_0) > 0$ and a family of measurable functions $\psi_{z_0,\varepsilon} : (0, \varepsilon_0) \to (0, \infty)$ such that

$$I_{z_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt < \infty \qquad \forall\ \varepsilon \in (0, \varepsilon_0)\,. \tag{14}$$

Then quasilinear generalized Cauchy–Riemann equation (7) has a regular solution (u, v) of the Dirichlet problem (6) in D for each continuous inconstant date $\varphi : \partial D \to \mathbb{R}$.

Proof. Indeed, to transform equation (7), we perform the following substitution:

$$\widetilde{u} = \Psi_a(u)\,, \qquad \widetilde{v} = v\,,$$

where

$$\Psi_a(t) \ := \ \int\limits_0^t a(s)\, ds\ ,$$

and set

$$\widetilde{\varphi}(\zeta) := \Psi_a(\varphi(\zeta)) \qquad \forall \zeta \in \partial D\,.$$

Note that $a \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ is locally integrable. Consequently, the function $\Psi_a(t)$ is absolutely continuous, and, by the Lebesgue theorem on the differentiation of the indefinite integral, see e.g. Theorem IV(6.3) in [13], we have that

$$\Psi'_a(t) \ = \ a(t) \qquad \text{a.e.} \tag{15}$$

Moreover, since $a : D \to [L^{-1}, L]$, we obtain the following two-sided estimate:

$$L^{-1}\,|t_2 - t_1| \ \leqslant\ |\Psi_a(t_2) - \Psi_a(t_1)| \ \leqslant\ L\,|t_2 - t_1| \quad \forall\, t_1, t_2 \in \mathbb{R}\ ,$$

i.e., Ψ_a is a quasi-isometry. In particular, the latter means that Ψ_a is a homeomorphic mapping of $\mathbb{R}$ onto $\mathbb{R}$.

Hence the transformation

$$T \ :\ (u, v) \ \to\ (\widetilde{u}, \widetilde{v}) \ :=\ (\Psi_a(u), v)$$

is also a quasi-isometry and, consequently, it is also a homeomorphism but of $\mathbb{R}^2$ onto $\mathbb{R}^2$. Moreover, by the construction T translates functions of the Sobolev class $W^{1,1}_{\mathrm{loc}}$ into itself and vice versa, i.e., $(\widetilde{u}, \widetilde{v}) \in W^{1,1}_{\mathrm{loc}}$ if and only if $(u, v) \in W^{1,1}_{\mathrm{loc}}$.

In addition, taking into account equality (15), we have that

$$\nabla\widetilde{u} = a(u)\nabla u \quad \text{and} \quad \nabla\widetilde{v} = \nabla v\,.$$

Then, after this change of functions, the quasilinear generalized Cauchy-Riemann equation (7) with the Dirichlet boundary condition (6) is equivalent to the linear generalized Cauchy-Riemann equation

$$\nabla\,\widetilde{v} \;= B\,\nabla\widetilde{u} \tag{16}$$

with the Dirichlet boundary condition

$$\lim_{z\to\zeta}\widetilde{u}(z) \;=\; \widetilde{\varphi}(\zeta) \;:=\; \Psi_a(\varphi(\zeta)) \quad \forall\,\zeta\in\partial D\,. \tag{17}$$

Now, by Lemma 1, the generalized Cauchy-Riemann equation (16) admits a regular solution $(\widetilde{u},\widetilde{v})$ to the Dirichlet problem (17). Thus, by the equivalence of the two Dirichlet problems established above, it follows that a regular solution to the Dirichlet problem (6) for equation (5) exists that completes the proof of Lemma 2. □

4. Criteria in terms of BMO, FMO and VMO.

Recall first of all that a real-valued function Φ in a domain D of $\mathbb{R}^2$ is called of **bounded mean oscillation** in D, abbr. $\Phi\in\mathrm{BMO}(D)$, if

$$\|\Phi\|_* := \sup_B \frac{1}{|B|}\int\limits_B |\Phi(Z)-\Phi_B|\,d\mathcal{L}(Z) < \infty\,, \tag{18}$$

where $\Phi\in L^1_{\rm loc}(D)$, the supremum is taken over all discs B in D and

$$\Phi_B \;:=\; \frac{1}{|B|}\int\limits_B \Phi(Z)\,d\mathcal{L}(Z)\,.$$

We also write $\Phi\in\mathrm{BMO}(\overline{D})$ if $\Phi_*\in\mathrm{BMO}(D_*)$ for some extension Φ_* of the function Φ into a domain D_* containing $\overline{D}$.

The class BMO was introduced by John and Nirenberg (1961) in the paper [17] and soon became an important concept in harmonic analysis, partial differential equations and related areas, see e.g. monographs [15] and [18].

Following [16], we say that a locally integrable function $\Phi : D\to\mathbb{R}$ has **finite mean oscillation** at a point $Z_0\in D$, abbr. $\Phi\in\mathrm{FMO}(Z_0)$, if

$$\overline{\lim_{\varepsilon\to 0}} \;\;⨍_{B(Z_0,\varepsilon)} |\Phi(Z)-\widetilde{\Phi}_\varepsilon(Z_0)|\,d\mathcal{L}(Z) < \infty\,, \tag{19}$$

where

$$\widetilde{\Phi}_\varepsilon(Z_0) = ⨍_{B(Z_0,\varepsilon)} \Phi(Z)\,d\mathcal{L}(Z) \tag{20}$$

is the mean integral value of the function $\Phi(Z)$ over disk $B(Z_0,\varepsilon) := \{Z\in\mathbb{R}^2 : |Z-Z_0|<\varepsilon\}$.

Further we always assume by definition that $K_{\mu_B} \equiv 1$ outside D.

Choosing $\psi_{z_0,\varepsilon}(t) = 1/(t\log(1/t))$ in Lemma 2, we obtain by Lemma 11.1 in [8] the following result with the FMO type criterion.

Theorem 1. *Let D be a bounded simply connected domain in $\mathbb{R}^2$ and $B : D \to \mathbb{B}^{2\times 2}$ be a measurable function in D, $\mu = \mu_B$ be given by formula (9) and $a : \mathbb{R} \to [L^{-1}, L]$, $L \in [1,\infty)$, be a measurable function. Suppose also that $K_{\mu_B}(Z) \leqslant Q_{Z_0}(Z)$ a.e. in U_{Z_0} for every point $Z_0 \in \overline{D}$, a neighborhood U_{Z_0} of Z_0 and a function $Q_{Z_0} : U_{Z_0} \to [0,\infty]$ in the class* FMO(Z_0). *Then quasilinear generalized Cauchy–Riemann equation (7) has a regular solution (u,v) of the Dirichlet problem (6) in D for each continuous inconstant boundary data $\varphi : \partial D \to \mathbb{R}$.*

Corollary 1. *In particular, if*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{B(Z_0,\varepsilon)} K_{\mu_B}(Z)\, d\mathcal{L}(Z) < \infty \qquad \forall\ Z_0 \in \overline{D}\,, \tag{21}$$

then the conclusion of Theorem 1 holds.

Corollary 2. *Let D be a bounded simply connected domain in $\mathbb{R}^2$, $B : D \to \mathbb{B}^{2\times 2}$ be a measurable function in D, $\mu = \mu_B$ be given by formula (9) and $a : \mathbb{R} \to [L^{-1}, L]$, $L \in [1,\infty)$, be a measurable function. Suppose also that $K_{\mu_B}(z) \leqslant Q(z)$ a.e. in D where Q is a function of the class* FMO$(\overline{D})$. *Then the conclusion holds.*

Theorem 2. *Let D be a bounded simply connected domain in $\mathbb{R}^2$, $B : D \to \mathbb{B}^{2\times 2}$ be a measurable function in D, $\mu = \mu_B$ be given by formula (9) and $a : \mathbb{R} \to [L^{-1}, L]$, $L \in [1,\infty)$, be a measurable function. Suppose also that K_{μ_B} has a dominant $Q : \mathbb{R}^2 \to [1,\infty)$ in the class* BMO$(\overline{D})$. *Then quasilinear generalized Cauchy–Riemann equation (7) has a regular solution (u,v) of the Dirichlet problem (6) in D for each continuous inconstant boundary data $\varphi : \partial D \to \mathbb{R}$.*

A function Φ in BMO is said to have **vanishing mean oscillation**, abbr. $\Phi \in$ VMO, if the supremum in (18) taken over all balls B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. VMO has been introduced by Sarason in [19]. There are a number of papers devoted to the study of PDEs with coefficients of the class VMO. Note that $W^{1,2}(D) \subset \mathrm{VMO}(D)$, see e.g. [14].

Corollary 3. *In particular, the conclusion of Theorem 1 on existence of a regular solution for the Dirichlet problem (6) to quasilinear generalized Cauchy–Riemann equation (7) holds if the dominant Q of K_{μ_B} belongs to the class $W^{1,2}(\overline{D})$.*

5. Criteria of the Calderon–Zygmund type.

Choosing in Lemma 2 the functional parameter $\psi_{z_0,\varepsilon}(t) = 1/t$, we come to the next statement with the Calderon–Zygmund type criterion.

Theorem 3. *Let D be a bounded simply connected domain in $\mathbb{R}^2$, $B : D \to \mathbb{B}^{2\times 2}$ be a measurable function in D, $\mu = \mu_B$ be given by formula (9) with $K_{\mu_B} \in L^1(D)$ and $a : \mathbb{R} \to [L^{-1}, L]$, $L \in [1,\infty)$, be a measurable function. Suppose also that*

$$\int\limits_{\varepsilon<|Z-Z_0|<\varepsilon_0} K_{\mu_B}(Z)\, \frac{d\mathcal{L}(Z)}{|Z-Z_0|^2} = o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \qquad \forall\ Z_0 \in \overline{D} \tag{22}$$

as $\varepsilon \to 0$ *for some* $\varepsilon_0 = \varepsilon(Z_0) > 0$. *Then quasilinear generalized Cauchy–Riemann equation (7) has a regular solution* (u, v) *of the Dirichlet problem (6) in* D *for each continuous inconstant boundary data* $\varphi : \partial D \to \mathbb{R}$.

Remark 1. We are also able here to replace (22) by

$$\int\limits_{\varepsilon<|Z-Z_0|<\varepsilon_0} \frac{K_{\mu_B}(Z)\, d\mathcal{L}(Z)}{\left(|Z-Z_0| \log \frac{1}{|Z-Z_0|}\right)^2} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \tag{23}$$

In general, we are able to give here the whole scale of the corresponding conditions in terms of iterated logarithms, i.e., in terms of functions of the form $1/(t \log 1/t \cdot \log\log 1/t \cdot \ldots \cdot \log \ldots \log 1/t)$.

6. Criteria of the Lehto type.

Further $k_{\mu_B}(Z_0, r)$ denotes the integral mean of $K_{\mu_B}(Z)$ over the circle $S(Z_0, r) := \{Z \in \mathbb{R}^2 : |Z - Z_0| = r\}$.

Choosing in Lemma 2 the functional parameter $\psi_{z_0}(t) := 1/[tk_{\mu_B}(Z_0, t)]$, we obtain the Lehto type criterion.

Theorem 4. *Let* D *be a bounded simply connected domain in* $\mathbb{R}^2$, $B : D \to \mathbb{B}^{2\times 2}$ *be a measurable function in* D, $\mu = \mu_B$ *be given by formula (9) with* $K_{\mu_B} \in L^1(D)$ *and* $a : \mathbb{R} \to [L^{-1}, L]$, $L \in [1, \infty)$, *be a measurable function. Suppose also that for some* $\varepsilon_0 = \varepsilon(Z_0) > 0$,

$$\int\limits_0^{\varepsilon_0} \frac{dr}{rk_{\mu_B}(Z_0, r)} = \infty \qquad \forall\ Z_0 \in \overline{D}\ . \tag{24}$$

Then quasilinear generalized Cauchy-Riemann equation (7) has a regular solution (u, v) *of the Dirichlet problem (6) in* D *for each continuous inconstant boundary data* $\varphi : \partial D \to \mathbb{R}$.

Corollary 4. *Let* D *be a bounded simply connected domain in* $\mathbb{R}^2$, $B : D \to \mathbb{B}^{2\times 2}$ *be a measurable function in* D, $\mu = \mu_B$ *be given by formula (9) with* $K_{\mu_B} \in L^1(D)$ *and* $a : \mathbb{R} \to [L^{-1}, L]$, $L \in [1, \infty)$, *be a measurable function. Suppose also that*

$$k_{\mu_B}(Z_0, \varepsilon) = O\left(\log\frac{1}{\varepsilon}\right) \qquad \text{as } \varepsilon \to 0 \qquad \forall\ Z_0 \in \overline{D}\ . \tag{25}$$

Then the conclusion of Theorem 4 holds.

Remark 2. In particular, the conclusion of Theorem 4 holds if

$$K_{\mu_B}(Z)\ =\ O\left(\log\frac{1}{|Z-Z_0|}\right) \qquad \text{as} \quad Z \to Z_0 \quad \forall\ Z_0 \in \overline{D}. \tag{26}$$

Moreover, the condition (25) can be replaced by the whole series of more weak conditions

$$k_{\mu_B}(Z_0, \varepsilon) = O\left(\left[\log\frac{1}{\varepsilon} \cdot \log\log\frac{1}{\varepsilon} \cdot \ldots \cdot \log \ldots \log\frac{1}{\varepsilon}\right]\right) \qquad \forall\ Z_0 \in \overline{D}\ . \tag{27}$$

7. Criteria of the Orlicz type.

By combining Theorem 4 with Theorems 2.4 and 2.3 from [6], we obtain the following result.

Theorem 5. *Let D be a bounded simply connected domain in $\mathbb{R}^2$, $B : D \to \mathbb{B}^{2\times 2}$ be a measurable function in D, $\mu = \mu_B$ be given by formula (9) and $a : \mathbb{R} \to [L^{-1}, L]$, $L \in [1, \infty)$, be a measurable function. Suppose also that*

$$\int\limits_{D} \Phi\left(K_{\mu_B}(Z)\right)\, d\mathcal{L}(Z) \ < \ \infty \tag{28}$$

for a convex non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$ such that, for some $\Delta > 0$,

$$\int\limits_{\Delta}^{\infty} \log\, \Phi(t)\, \frac{dt}{t^2} \ = \ +\infty\ . \tag{29}$$

Then quasilinear generalized Cauchy-Riemann equation (7) has a regular solution (u, v) of the Dirichlet problem (6) in D for each continuous inconstant boundary data $\varphi : \partial D \to \mathbb{R}$.

Remark 3. Note that the condition (29) is not only sufficient but also necessary to have regular solutions (u, v) of the Dirichlet problem (6) in D to quasilinear generalized Cauchy-Riemann equation (7) with the integral constraints (28) for all continuous inconstant data $\varphi : \partial D \to \mathbb{R}$.

Corollary 5. *Let D be a bounded simply connected domain in $\mathbb{R}^2$, $B : D \to \mathbb{B}^{2\times 2}$ be a measurable function in D, $\mu = \mu_B$ be given by formula (9) and $a : \mathbb{R} \to [L^{-1}, L]$, $L \in [1, \infty)$, be a measurable function. Suppose also that for $\lambda > 0$,*

$$\int\limits_{D} \exp\, \left(\lambda K_{\mu_B}(Z)\right)\ d\mathcal{L}(Z) \ < \ \infty\,. \tag{30}$$

Then the conclusion of Theorem 5 holds.

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А. Салімов

Про задачу Діріхле для квазілінійних узагальнених рівнянь Коші–Рімана.

У статті досліджується задача Діріхле з неперервними граничними даними для квазілінійних узагальнених рівнянь Коші — Рімана з матричними коефіцієнтами, які пов'язують функцію потенціалу і функцію потоку і моделюють стаціонарні потоки стисливих рідин в анізотропних і неоднорідних середовищах. У статті отримано ряд ефективних інтегральних критеріїв існування регулярних розв'язків задачі Діріхле з неперервними граничними даними потенціалу в довільних обмежених однозв'язних областях. Запропоновані критерії дозволяють встановити достатні умови регулярної розв'язності задачі для широкого класу таких рівнянь.

Ключові слова: *система Коші–Рімана, узагальнене рівняння Коші–Рімана, задача Діріхле.*

Institute of Applied Mathematics and Mechanics
of the NAS of Ukraine, Sloviansk

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salimov.artem@gmail.com