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ON CONVERGENCE TO A LIGHT MAPPING IN METRIC SPACES

We investigate mappings satisfying the inverse Poletsky inequality in a domain of a metric space. We have proved that the uniform limit of a family of such mappings is a light mapping in the closure of a domain. By Reshetnyak theorem, quasiregular mappings of a domain of the Euclidean space are open and discrete. For more general classes of mappings, statements of a similar plan are also known. In particular, this is true for mappings with finite distortion (Manfredi, Villamor). The limit mapping of a sequence of quasiregular mappings is also a quasiregular mapping, or a constant. Therefore, the openness and discreteness preserve under the convergence of quasiregular mappings. The similar problem in more general classes of mappings is much more complicated, as is the problem of the discreteness in the closure of a domain. Vuorinen proved that closed quasiregular mappings are light in the closure of a domain under certain conditions to its boundary. Under some stronger conditions these mappings are even discrete in the closure. Cristea, Sevost'yanov and Skvortsov studied the problem of lightness and discreteness of mappings with the inverse Poletsky inequality. Such inequalities, in particular, hold for quasiregular mappings with a finite multiplicity. Martio, Ryazanov, Srebro, and Yakubov proved the inverse of Poletsky's inequality for mappings with finite length distortion for the conformal modulus. Salimov and Sevost'yanov extended the indicated result to an arbitrary order $p \geqslant 1$. Sevost'yanov proved the lightness of mappings with the inverse of Poletsky's inequality at the inner points of a domain provided that the corresponding majorant in this inequality satisfies the Lehto-Dini condition. Later, this result was extended to the boundary of a uniformly convergent sequence of such mappings. The author of this manuscript, together with Sevost'yanov, recently obtained the lightness of a limit mapping with respect to the closure of a domain in the Euclidean space. In this manuscript, a similar result is obtained in metric space. In particular, we have proved that if a sequence of open, discrete, and closed mappings with the inverse Poletsky inequality converge in the closure of a domain, and the majorant in this inequality has a finite mean oscillation, then the corresponding limit mapping has a continuous boundary extension that is light in the closure.

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1. Introduction.

Everywhere below, $(X,\, d, \mu)$ and $(X',\, d', \mu')$ are metric spaces with metrics d and d', endowed with locally finite Borel measures μ and μ', and finite Hausdorff dimensions $\alpha \geqslant 2$ and $\alpha' \geqslant 2$, respectively. In what follows, a domain D in X is an open path connected set in X. Definitions related to Ahlfors regularity and Poincare $(1;p)$-inequality may be found, for example, in [1, Section 7.22]. Let $y_0 \in X'$, $0 < r_1 < r_2 < \infty$ and

$$A(y_0, r_1, r_2) = \left\{y \in X' : r_1 < d'(y, y_0) < r_2\right\}, \qquad (1)$$
$$B(y_0, r) = \{y \in X' : d'(y, y_0) < r\}\,, S(y_0, r) = \{y \in X' : d'(y, y_0) = r\}\,.$$

Given sets $E,\, F \subset X'$ and a domain $D \subset X'$ denote by $\Gamma(E, F, D)$ the family of all paths $\gamma : [a, b] \to X'$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in [a, b]$. Given a domain $D \subset X$, a *mapping* $f : D \to X'$ is an arbitrary continuous transformation $x \mapsto f(x)$. Let $f : D \to X'$, let $y_0 \in f(D)$ and let $0 < r_1 < r_2 < d_0 = \sup\limits_{y \in f(D)} d'(y, y_0)$. Now, we denote by $\Gamma_f(y_0, r_1, r_2)$ the family of all paths γ in D such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$. Recall, for a given continuous path $\gamma : [a, b] \to X$ in a metric space (X, d), that its length is the supremum of the sums

$$\sum_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i-1}))$$

over all partitions $a = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_k = b$ of the interval $[a, b]$. The path γ is called *rectifiable* if its length is finite. Given a family of paths Γ in X, a Borel function $\rho : X \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm} \Gamma$, if

$$\int\limits_{\gamma} \rho \, ds \geqslant 1$$

for all (locally rectifiable) $\gamma \in \Gamma$. Given $p \geqslant 1$, the p-modulus of the family Γ is the number

$$M_p(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int\limits_{X} \rho^p(x) \, d\mu(x) \, .$$

Should $\operatorname{adm} \Gamma$ be empty, we set $M_p(\Gamma) = \infty$. A family of paths Γ_1 in X is said to be *minorized* by a family of paths Γ_2 in X, abbr. $\Gamma_1 > \Gamma_2$, if, for every path $\gamma_1 \in \Gamma_1$, there is a path $\gamma_2 \in \Gamma_1$ such that γ_2 is a restriction of γ_1. In this case, $M_p(\Gamma_1) \leq M_p(\Gamma_2)$ (see [2, Theorem 1]). Let $Q : X' \to [0, \infty]$ be a Lebesgue measurable function such that $Q(y) \equiv 0$ for $y \in X' \setminus D$. Assume that D and X' have a finite Hausdorff dimensions α and $\alpha' \geqslant 1$, respectively, and $f : D \to X'$. We consider the inequality

$$M_\alpha(\Gamma(y_0, r_1, r_2)) \leqslant \int\limits_{G'} Q(y) \cdot \eta^{\alpha'}(d'(y, y_0)) \, d\mu'(y) \, , \tag{2}$$

where $\eta : (r_1, r_2) \to [0, \infty]$ is arbitrary nonnegative Lebesgue measurable function such that

$$\int\limits_{r_1}^{r_2} \eta(r) dr \geqslant 1 \, . \tag{3}$$

Let X and X' be metric spaces. A mapping $f : X \to X'$ is *discrete* if $f^{-1}(y)$ is discrete for all $y \in X'$ and f is *open* if f maps open sets onto open sets in X'. A mapping $f : G \to X'$ is *closed* in $G \subset X$ if $f(A)$ is closed in $f(G)$ whenever A closed in G. A mapping $f : X \to Y$ is called *light* if each connected component $\{f^{-1}(y)\}$ degenerates to a point for any $y \in Y$.

A domain $D \subset X$ is called *locally connected at the point* x_0, if for any neighborhood U of the point x_0 there is a neighborhood $V \subset U$ of the same point such that $V \cap D$

is connected. The domain D is called *locally connected* on ∂D, if this domain is such at each point of its boundary. The boundary of a domain D is called *weakly flat at the point* x_0, if for every number $P>0$ and for every neighborhood U of this point there is a neighborhood V of point x_0 such that $M_\alpha(\Gamma(E,F,D))>P$ for arbitrary continua E and F satisfying conditions $F\cap\partial U\neq\varnothing\neq F\cap\partial V$ and $E\cap\partial U\neq\varnothing\neq E\cap\partial V$. The boundary of a domain D is called *weakly flat* if it is such at each point of its boundary.

Let G be a domain in (X,d,μ). Following [3, Sec. 13.4], we say that a locally integrable in X function $\varphi: G\to\mathbb{R}$ has a *finite mean oscillation at* $x_0\in\overline{G}$, write $\varphi\in FMO(x_0)$, if

$$\overline{\lim_{\varepsilon\to 0}}\ \frac{1}{\mu(G(x_0,\varepsilon))}\int\limits_{G(x_0,\varepsilon)}|\varphi(x)-\overline{\varphi}_\varepsilon|\ d\mu(x)<\infty\,,$$

where $\overline{\varphi}_\varepsilon=\frac{1}{\mu(G(x_0,\varepsilon))}\int\limits_{G(x_0,\varepsilon)}\varphi(x)\ d\mu(x)$ is the integral average of the function φ over the set $G(x_0,\varepsilon)=B(x_0,\varepsilon)\cap G=\{x\in X: d(x,x_0)<\varepsilon\}\cap G$. We say that a space (X,d,μ) is *upper α-regular at a point* $x_0\in X$ if there is a constant $C>0$ such that

$$\mu(B(x_0,r))\leqslant Cr^\alpha$$

for the balls $B(x_0,r)$ centered at $x_0\in X$ with all radii $r<r_0$ for some $r_0>0$. We also say that a space (X,d,μ) is *upper α-regular* if the above condition holds at every point $x_0\in X$.

Theorem 1. *Let $(X,\,d,\mu)$ and $(X',\,d',\mu')$ be metric spaces with metrics d and d', endowed with locally finite Borel measures μ and μ', the measure μ is doubling, X' is upper α'-regular, and let D and D' be domains in X and X', having finite Hausdorff dimensions $\alpha\geqslant 2$ and $\alpha'\geqslant 2$, respectively. Assume that, the following conditions hold:*

1) D is a locally connected space which is α-regular by Ahlfors and supports $(1;\alpha)$-Poincare inequality, while $\overline{D}$ is a compactum in X;

2) $\overline{D'}$ is a compactum in X';

3) D has a weakly flat boundary;

4) $Q\in FMO(\overline{D'})$;

5) D' is locally path connected on $\overline{D'}$.

Let f_m, $m=1,2,\ldots$, be a sequence of open, discrete and closed mappings of D onto D' that converge uniformly in D to some (not identically constant) mapping $f: D\to D'$ and satisfy the relation (2)–(3) at each point $y_0\in\overline{D'}$. Then f is light and closed in D; moreover, f has a continuous extension $\overline{f}:\overline{D}\to\overline{D'}$, $\overline{f}(\overline{D})=\overline{D'}$ and $\overline{f}$ is light in $\overline{D}$, as well.

2. The main Lemmas. Proof of Theorem 1.

The following statement can be found in [3, Lemma 13.2].

Proposition 1. Let G be a domain in upper α-regular metric space (X,d,μ) at $\alpha\geqslant 2$. Assume that $x_0\in\overline{G}$ and $Q: G\to[0,\infty]$ belongs to $FMO(x_0)$. If

$$\mu(G\cap B(x_0,2r))\leqslant\gamma\cdot\log^{\alpha-2}\frac{1}{r}\cdot\mu(G\cap B(x_0,r))$$

for some $r_0 > 0$ and every $r \in (0, r_0)$, then

$$\int\limits_{\varepsilon<d(x,x_0)<\varepsilon_0} Q(x)\cdot\psi^{\alpha}(d(x,x_0))\,d\mu(x) = o(I^{\alpha}(\varepsilon,\varepsilon_0))\,, \qquad \varepsilon\to 0\,,$$

$I(\varepsilon,\varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\,dt$ and $\psi(t) := \frac{1}{t\log\frac{1}{t}}$.

The following result holds (see [4, Proposition 4.7]).

Proposition 2. Let X be a Q-Ahlfors regular metric measure space that supports $(1;p)$-Poincaré inequality for some $p > 1$ such that $Q-1 < p \leqslant Q$. Then there exists a constant $M > 0$ having the property that, for $x \in X$, $R > 0$ and continua E and F in $B(x,R)$,

$$M_{\alpha}(\Gamma(E,F,X)) \geqslant \frac{1}{M}\cdot\frac{\min\{d(E),d(F)\}}{R^{1+p-Q}}\,.$$

Some another versions of the following lemma may be found in [5, Lemma 1], cf. [6, Lemma 3.1], [7, Lemma 4.4].

Lemma 1. *Let (X, d, μ) and (X', d', μ') be metric spaces with metrics d and d', endowed with locally finite Borel measures μ and μ', the measure μ is doubling, X' is α' upper regular, and let D and D' be domains in X and X', having finite Hausdorff dimensions $\alpha \geqslant 2$ and $\alpha' \geqslant 2$, respectively. Assume that, the conditions 1)–2) from Theorem 1 hold. Suppose that, for each $y_0 \in \overline{D'}$ there exists $\varepsilon_0 = \varepsilon_0(y_0) > 0$ and a Lebesgue measurable function $\psi : (0,\varepsilon_0) \to [0,\infty]$ such that*

$$I(\varepsilon,\varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\,dt < \infty \quad \forall\, \varepsilon \in (0,\varepsilon_0)\,, \quad I(\varepsilon,\varepsilon_0) \to \infty \quad as \quad \varepsilon \to 0\,, \tag{4}$$

and

$$\int\limits_{A(y_0,\varepsilon,\varepsilon_0)} Q(y)\cdot\psi^{\alpha'}(d'(y,y_0))\,d\mu'(x) = o(I^{\alpha'}(\varepsilon,\varepsilon_0))\,, \tag{5}$$

as $\varepsilon \to 0$, where $A(y_0,\varepsilon,\varepsilon_0)$ is defined in (1). Suppose that f_m, $m = 1,2,\ldots$, a sequence of mappings of D onto D' that converge uniformly in D to some (not identically constant) mapping $f : D \to D'$ and satisfy the relation (2)–(3) at each point $y_0 \in \overline{D'}$. Let C_j, $j = 1,2,\ldots$, be an arbitrary sequence of continua in D such that $d(C_j) \geqslant \delta > 0$ for some $\delta > 0$ and all $j \in \mathbb{N}$. Then there exists $\delta_1 > 0$ such that $d'(f(C_j)) \geqslant \delta_1 > 0$ for all $j \in \mathbb{N}$.

Proof. Suppose the opposite. Now, we may find an increasing sequence of numbers j_k, $k = 1,2,\ldots$, such that $d'(f(C_{j_k})) \to 0$ as $k \to \infty$. Since by the assumption 2) $\overline{D'}$ is a compactum in X', we may assume that $d'(f(C_{j_k}), y_0) \to 0$ as $k \to \infty$, where $y_0 \in \overline{D'}$ is some point. By [8, Lemma 1] f is either a light, or a constant mapping. Since by assumption f is not constant, f is light. Therefore, there exists a point $z_0 \in D$ such that $f(z_0)\neq y_0$. Since f is a continuous mapping and D is locally compact and locally connected, we can find a continuum F such that $z_0\in \operatorname{Int} F$ and

$$F\subset D\,,\qquad f(F)\cap \overline{B(y_0,\varepsilon_1)}=\varnothing\,, \tag{6}$$

where $\varepsilon_1>\varepsilon_0$ and ε_0 is a number from (4)–(5). Let $\Gamma_k:=\Gamma(F, C_{j_k}, D)$. Now, we show that there exists a neighborhood V of y_0 such that

$$f_j(F)\cap V=\varnothing\,,\qquad j=1,2,\ldots\,. \tag{7}$$

Let us assume the opposite. Then there exist sequences $i_m\in\mathbb{N}$ and $x_m\in F$, $m=1,2,\ldots$, such that $f_{i_m}(x_m)\to y_0$ as $m\to\infty$. By the triangle inequality

$$d^{\prime}(f(x_m),y_0)\leqslant d^{\prime}(f(x_m),f_{i_m}(x_m))+d^{\prime}(f_{i_m}(x_m),y_0)\to 0 \tag{8}$$

as $m\to\infty$, because $d^{\prime}(f(x_m),f_{i_m}(x_m))\to 0$ as $m\to\infty$ by the uniform convergence of f_m to f in D. The relation (8) contradicts (6), which proves (7). Reducing the number $\varepsilon_0>0$ from the condition of the lemma as necessary, we may consider that the neighborhood V in (7) is such that $\overline{B(y_0,\varepsilon_0)}\subset V$.

Since by the assumption 1) $\overline{D}$ is a compactum, D is bounded. Now, there is $x_0\in D$ and $R_0>0$ such that $D\subset B(x_0,R_0)$. Now, by Proposition 2,

$$M_{\alpha}(\Gamma_k)\geqslant \frac{1}{M}\cdot\frac{\min\{d(C_{j_k}),d(F)\}}{R}\geqslant \delta_2>0 \tag{9}$$

for all $k\in\mathbb{N}$ and some $\delta_2>0$. We prove that for each $l\in\mathbb{N}$ there exists a number $k=k_l$ such that

$$f_{j_k}(C_{j_k})\subset B(y_0,1/l)\,,\qquad k\geqslant k_l\,. \tag{10}$$

Suppose the opposite. Then there exists $l_0\in\mathbb{N}$ such that

$$f_{j_{k_m}}(C_{j_{k_m}})\cap(X^{\prime}\setminus B(y_0,1/l_0))\neq\varnothing \tag{11}$$

for some increasing sequence of numbers k_m, $m=1,2,\ldots$. In this case, there is a sequence $z_m\in f_{j_{k_m}}(C_{j_{k_m}})\cap(X^{\prime}\setminus B(y_0,1/l_0))$, $m\in\mathbb{N}$. Let $z_m=f_{j_{k_m}}(x_m)$, $x_m\in C_{j_{k_m}}$, $m=1,2,\ldots$. Since $d^{\prime}(f(C_{j_k}),y_0)\to 0$ as $k\to\infty$, we have that

$$d^{\prime}(f(C_{j_{k_m}}),y_0)\to 0\qquad \text{as}\qquad m\to\infty\,. \tag{12}$$

Since $d^{\prime}(f(C_{j_k}),y_0)=\inf\limits_{y\in f(C_{j_{k_m}})}d^{\prime}(y,y_0)$ and $f(C_{j_{k_m}})$ is a compact set, it follows that $d^{\prime}(f(C_{j_{k_m}}),y_0)=d^{\prime}(y_m,y_0)$, where $y_m\in f(C_{j_{k_m}})$. Then (12) implies that $y_m\to y_0$ as $m\to\infty$. Since by the assumption $d^{\prime}(f(C_{j_k}))=\sup\limits_{y,z\in f(C_{j_k})}d^{\prime}(y,z)\to 0$ as $k\to\infty$, we have that $d^{\prime}(y_m,f(x_m))\leqslant d^{\prime}(f(C_{j_{k_m}}))\to 0$ as $m\to\infty$. Then by the triangle inequality and the above proof, we have that:

$$\begin{gathered} d^{\prime}(z_m,y_0)=d^{\prime}(f_{j_{k_m}}(x_m),y_0)\leqslant\\ \leqslant d^{\prime}(f_{j_{k_m}}(x_m),f(x_m))+d^{\prime}(f(x_m),y_m)+d^{\prime}(y_m,y_0)\to 0\,,\qquad m\to\infty\,, \end{gathered} \tag{13}$$

because $d'(f_{j_{k_m}}(x_m), f(x_m)) \leqslant \sup\limits_{x\in D} d'(f_{j_{k_m}}(x), f(x)) \to 0$ as $m\to\infty$ by the uniform convergence of f_j to f as $j\to\infty$; the last two terms in (13) also tend to zero by the construction and proving above. The relation (13) implies that $z_m\to y_0$ as $m\to\infty$, which contradicts the definition of z_m. Now, (11) fails, which proves (10).

From (10) it follows that

$$f_{j_{k_l}}(C_{j_{k_l}})\subset B(y_0, 1/l)\,, \tag{14}$$

where k_l, $l=1,2,\ldots$, may be chosen the increasing sequence of numbers. Using the renumbering in (14), if required, we may consider j_k, $k=1,2,\ldots$, instead of j_{k_l}. Now,

$$f_{j_k}(C_{j_k})\subset B(y_0, 1/k)\,,\qquad k=1,2,\ldots\,. \tag{15}$$

Furthermore, let $k_0\in\mathbb{N}$ be such that $B(y_0,1/k)\subset B(y_0,\varepsilon_0)$ for all $k\geqslant k_0$. Let us to show that

$$f_{j_k}(\Gamma_{j_k}) > \Gamma(S(y_0,1/k), S(y_0,\varepsilon_0), A(y_0,1/k,\varepsilon_0))\,. \tag{16}$$

Indeed, let $\widetilde{\gamma}\in f_{j_k}(\Gamma_{j_k})$. Then $\widetilde{\gamma}(t)=f_{j_k}(\gamma(t))$, where $\gamma\in\Gamma_{j_k}$, $\gamma:[0,1]\to D$, $\gamma(0)\in F$, $\gamma(1)\in C_{j_k}$. By the relation (7) we have that $f_{j_k}(\gamma(0))\in f(F)\subset X'\setminus\overline{B(y_0,\varepsilon_0)}$, moreover, by the relation (15) we obtain that $f_{j_k}(\gamma(1))\in f_{j_k}(C_{j_k})\subset B(y_0,\varepsilon_0)$. Therefore, $|f_{j_k}(\gamma(t))|\cap B(y_0,\varepsilon_0)\neq\varnothing\neq|f_{j_k}(\gamma(t))|\cap(X'\setminus B(y_0,\varepsilon_0))$. By [9, Theorem 1.I.5.46], there exists $0<t_1<1$ such that $f_{j_k}(\gamma(t_1))\in S(y_0,\varepsilon_0)$. Let $\gamma_1:=\gamma|_{[t_1,1]}$. We may assume that $f_{j_k}(\gamma(t))\in B(y_0,\varepsilon_0)$ for all $t\geqslant t_1$. Reasoning similarly, we obtain a point $t_2\in(t_1,1]$ such that $f_{j_k}(\gamma(t_2))\in S(y_0,1/k)$. Let $\gamma_2:=\gamma|_{[t_1,t_2]}$. We may assume that $f_{j_k}(\gamma(t))\notin B(y_0,1/k)$ for all $t\in[t_1,t_2]$. Now, $f_{j_k}(\gamma_2)$ is a subpath of the path $f_{j_k}(\gamma)=\widetilde{\gamma}$, which belongs to $\Gamma(S(y_0,1/k), S(y_0,\varepsilon_0), A(y_0,1/k,\varepsilon_0))$. The relation (16) is established.

From (16) it follows that

$$\Gamma_{j_k} > \Gamma_{f_{j_k}}(y_0,1/k,\varepsilon_0)\,. \tag{17}$$

Let

$$\eta_k(t)=\begin{cases}\psi(t)/I(1/k,\varepsilon_0), & t\in(1/k,\varepsilon_0)\,,\\ 0, & t\notin(1/k,\varepsilon_0)\,,\end{cases}$$

where $I(1/k,\varepsilon_0)=\int\limits_{1/k}^{\varepsilon_0}\psi(t)\,dt$. Observe that $\int\limits_{1/k}^{\varepsilon_0}\eta_k(t)\,dt=1$. Then, by the relations (5) and (17), taking into account the definition of the mapping f_{j_k} in (2), we obtain that

$$M_\alpha(\Gamma_{j_k})\leqslant M_\alpha(\Gamma_{f_{j_k}}(y_0,1/k,\varepsilon_0))\leqslant$$
$$\leqslant\frac{1}{I^{\alpha'}(1/k,\varepsilon_0)}\int\limits_{A(y_0,1/k,\varepsilon_0)} Q(y)\cdot\psi^{\alpha'}(d'(y,y_0))\,d\mu'(y)\to 0\quad\text{as}\quad k\to\infty\,.$$

The latter contradicts with (9). The obtained contradiction proves the lemma. $\Box$

The following lemma generalizes Corollary 4.5 in [7] to the case of maps with unbounded characteristic, cf. [5, Lemma 2] and Lemma 3.2 in [6].

Lemma 2. *Let (X, d, μ) and (X', d', μ') be metric spaces with metrics d and d', endowed with locally finite Borel measures μ and μ', the measure μ is doubling, X' is upper α'-regular, and let D and D' be domains in X and X', having finite Hausdorff dimensions $\alpha \geqslant 2$ and $\alpha' \geqslant 2$, respectively. Assume that, the conditions 1)–3), 5) of Theorem 1 hold. Suppose that, for each $y_0 \in \overline{D'}$ there exists $\varepsilon_0 = \varepsilon_0(y_0) > 0$ and a Lebesgue measurable function $\psi : (0, \varepsilon_0) \to [0, \infty]$ such that the relations (4)–(5) hold, where $A(y_0, \varepsilon, \varepsilon_0)$ is defined in (1). Let f_m, $m = 1, 2, \ldots$, be a sequence of open, discrete and closed mappings of D onto D' that converge uniformly in D to some mapping $f : D \to X'$ satisfying the relations (2)–(3) at each point $y_0 \in \overline{D'}$. If $Q \in L^1(D')$, then the mapping f is light and closed in D, moreover, f has a continuous extension $\overline{f} : \overline{D} \to \overline{D'}$, $\overline{f}(\overline{D}) = \overline{D'}$ and $\overline{f}$ is light with respect to $\overline{D}$.*

Proof. By [8, Lemma 1] f is light in D. By [10, Theorem 1.1], every f_m, $m = 1, 2, \ldots$, has a continuous extension $\overline{f_m} : \overline{D} \to \overline{D'}$ to $\overline{D}$ such that $f_m(\overline{D}) = \overline{D'}$.

I. Let us show that there exists $\lim\limits_{x \to x_0} f(x)$ for every $x_0 \in \partial D$. Indeed, by the triangle inequality $d'(f(x), f(x_0)) \leqslant d'(f(x), f_m(x)) + d'(f_m(x), f_m(x_0)) + d'(f_m(x_0), f(x_0))$. In this inequality, the first and third terms tend to zero due to the uniform convergence of f_m in D. Therefore, for any $\varepsilon > 0$ there is a number $M = M(\varepsilon)$ such that

$$d'(f(x), f(x_0)) \leqslant \frac{2\varepsilon}{3} + d'(f_m(x), f_m(x_0)) \qquad (18)$$

for $m \geqslant M(\varepsilon)$. For $m = M(\varepsilon)$, f_m has a continuous extension to x_0. Thus, there is $\delta = \delta(\varepsilon, x_0) > 0$ such that $d'(f_M(x), f_M(x_0)) < \varepsilon/3$ for $0 < |x - x_0| < \delta$. Then, by (18), $d'(f(x), f(x_0)) < \varepsilon$ for the same δ.

II. Let us show that f is a closed set in D. Let E be closed in D. We must show that $f(E)$ is closed in D'. Let z_m, $m = 1, 2, \ldots$, be a sequence in $f(E)$ is such that $z_m \to z_0 \in D'$ as $m \to \infty$. Let us show that $z_0 \in f(E)$. Since $z_m \in f(E)$, there exists $x_m \in E$ such that $f(x_m) = z_m$, $m = 1, 2, \ldots$. Due to the compactness of $\overline{D}$, we may assume that the sequence x_m converges to some $x_0 \in \overline{D}$ as $m \to \infty$. If $x_0 \in D$, then $x_0 \in E$, consequently, $f(x_0) = z_0 \in E$ by the continuity of f in D, as required. Let us show that the case $x_0 \in \partial D$ is impossible. Indeed, since f_m has a continuous extension to x_0, there is a sequence $w_m \in D$, $w_m \to x_0$ as $m \to \infty$, such that $d'(f_m(w_m), f_m(x_0)) < \frac{1}{m}$. Then by the triangle inequality

$$d'(f_m(x_0), f(x_0)) \leqslant d'(f_m(x_0), f_m(w_m)) + d'(f_m(w_m), f(w_m)) + d'(f(w_m), f(x_0)) <$$
$$< \frac{1}{m} + d'(f_m(w_m), f(w_m)) + d'(f(w_m), f(x_0)) \to 0\,, \qquad m \to \infty\,, \qquad (19)$$

because f_m converges to f uniformly in D, and f is continuous at x_0 by the proving above. Observe that, $f_m(x_0) \in \partial D$ for $m \in \mathbb{N}$ (see [11, Proposition 2.1], cf. [7, Theorem 3.3]). Now, by the closeness of the boundary ∂D, we have: $f(x_0) \in \partial D'$. Then by the continuity of f in $\overline{D}$ we have that $f(x_m) = z_m \to z_0$ and $f(x_0) = z_0 \in \partial D'$.

This contradicts the choice of $z_0 \in E \subset D'$. Finally, $f(E)$ is closed in D', because in the case $x_0 \in D$ is considered, and the case $x_0 \in \partial D$ is impossible.

III. Let us show that $f(\partial D) \subset \partial D'$. Let $x_0 \in \partial D$. We must prove that $f(x_0) \in \partial D'$. Since f_m has a continuous extension to x_0, there is a sequence $w_m \in D$, $w_m \to x_0$ as $m \to \infty$, such that $d'(f_m(w_m), f_m(x_0)) < \frac{1}{m}$. Repeating the reasoning in (19) and follow, we come to the conclusion $f(x_0) \in \partial D'$, which should have been established.

IV. Let us show that $\overline{f}(\overline{D}) = \overline{D'}$. Obviously, $\overline{f}(\overline{D}) \subset \overline{D'}$. Let us show that $\overline{D'} \subset \overline{f}(\overline{D})$. Indeed, let $y_0 \in \overline{D'}$, then either $y_0 \in D'$, or $y_0 \in \partial D'$. a) If $y_0 \in D'$, then $y_0 = f_j(x_j)$, $x_j \in D$. Due to the compactness of $\overline{D}$, we may assume that $x_j \to x_0$, $j \to \infty$, $x_0 \in \overline{D}$. By the continuity of f in $\overline{D}$ we have that $f(x_0) = \lim\limits_{j\to\infty} f(x_j)$. However, by the uniform convergence of f_j to f we have: $d'(f_j(x_j), f(x_j)) = d'(y_0, f(x_j)) \to 0$, $j \to \infty$. Then $y_0 = f(x_0)$. Since by the proving above $f(\partial D) \subset \partial D'$, we have that $x_0 \in D$. Thus, $y_0 = f(x_0)$, $x_0 \in D$. b) Finally, let $y_0 \in \partial D'$. Then there is a sequence $y_k \in D'$ such that $y_k = f(x_k) \to y_0$ as $k \to \infty$ and $x_k \in D$. Due to the compactness of $\overline{D}$ we may assume that $x_k \to x_0$, where $x_0 \in \overline{D}$. Then $\overline{f}(x_0) = y_0 \in \overline{f}(\overline{D})$. Thus, $\overline{D'} \subset \overline{f}(\overline{D})$.

V. It remains to show that $\overline{f}$ is light in $\overline{D'}$. Let us assume the opposite, namely, let there exist a point $y_0 \in \partial D'$ such that $f^{-1}(y_0) \supset K_0$, where $K_0 \subset \overline{D}$ is some nondegenerate continuum. Since f is light in D, we may assume that $K_0 \subset \partial D$.

Since $\overline{D}$ is a compactum in X and $\overline{f}$ is continuous in $\overline{D}$, it is uniformly continuous in $\overline{D}$. In this case, for each $j \in \mathbb{N}$ there exists $\delta_j < 1/j$ such that

$$d'(\overline{f}(x), \overline{f}(x_0)) = d'(\overline{f}(x), y_0) < 1/j \quad \forall\, x, x_0 \in \overline{D}, \quad d'(x, x_0) < \delta_j\,, \quad \delta_j < 1/j\,. \tag{20}$$

Given $j \in \mathbb{N}$, we set $B_j := \bigcup\limits_{x_0 \in K_0} B(x_0, \delta_j)$, $j \in \mathbb{N}$. Since the set B_j is a neighborhood of the continuum K_0, by [8, Lemma 2(III)], cf. [12, Lemma 2.2], there exists a neighborhood U_j of the set K_0 such that $U_j \subset B_j$ and the set $U_j \cap D$ is path connected. Since K_0 is a compactum, there are $z_0, w_0 \in K_0$ such that $d'(K_0) = d'(z_0, w_0)$. From this it follows that, there are $z_j \in U_j \cap D$ and $w_j \in U_j \cap D$ such that $z_j \to z_0$ and $w_j \to w_0$ as $j \to \infty$. We may assume that

$$d'(z_j, w_j) > d'(K_0)/2 \quad \forall\, j \in \mathbb{N}\,. \tag{21}$$

Since the set $U_j \cap D$ is path connected, the points z_j and w_j may be joined by some path γ_j, $|\gamma_j| \subset U_j \cap D$. Let $C_j := |\gamma_j|$.

Note that $d'(f(C_j)) \to 0$ as $j \to \infty$. Indeed, since $f(C_j)$ is a continuum in X', there exist points $y_j, y_j' \in f(C_j)$ such that $d'(f(C_j)) = d'(y_j, y_j')$. Then there exist $x_j, x_j' \in C_j$ such that $y_j = f(x_j)$ and $y_j' = f(x_j')$, while the points x_j and x_j' belong to $U_j \subset B_j$. This means that there exist x_1^j and $x_2^j \in K_0$ such that $x_j \in B(x_1^j, \delta_j)$ and $x_j' \in B(x_2^j, \delta_j)$. In such case, by the relation (20) and taking into account the triangle inequality, we obtain that

$$\begin{gathered} d'(f(C_j)) = d'(y_j, y_j') = d'(f(x_j), f(x_j')) \leqslant d'(f(x_j), f(x_1^j)) + \\ \leqslant +d'(f(x_1^j), f(x_2^j)) + d'(f(x_2^j), f(x_j')) < 2/j \to 0 \quad \text{as} \quad j \to \infty\,. \end{gathered} \tag{22}$$

It follows from the relation (21) that the continua C_j, $j = 1, 2, \ldots$, satisfy the conditions of Lemma 1. By this lemma $d'(f(C_j)) \geqslant \delta_1 > 0$ for all $j \in \mathbb{N}$ and some $\delta_1 > 0$, which contradicts (22). The obtained contradiction disproves that $\overline{f}$ is not light in $\overline{D}$. The lemma is proved. □

Proof of Theorem 1 follows directly from Lemma 2 and Proposition 1. □

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Валерій Таргонський

Про деякі оцінки неконформного модуля сімей кривих.

Досліджені відображення, які задовольняють обернену нерівність типу Полецького в області метричного простору. Доведено, що рівномірна границя сім'ї таких відображень є нульвимірним відображенням у замиканні області. Добре відомо, що квазірегулярні відображення області евклідового простору є відкритими та дискретними з огляду на відповідний результат Решетняка. Для більш загальних класів відображень також відомі твердження подібного плану. Зокрема, це є вірним для відображень зі скінченним спотворенням (Манфреді, Вілламор). Граничним відображенням послідовності квазірегулярних відображень є також квазірегулярне відображення, або стала. Отже, при збіжності квазірегулярних відображень відкритість та дискретність зберігаються. Аналогічне питання в більш загальних класах відображень значно складніше, так само, як і питання щодо дискретності у замиканні області. Вуорінен довів, що замкнені квазрегулярні відображення є нульвимірними у замиканні області за певних умов на її межу. За деяких більш сильних умов зазначене відображення є навіть дискретним у замиканні. Крістєа, Севостьянов та Скворцов вивчали проблему нульвимірності та дискретності відображень з оберненою нерівністю Полецького. Такі нерівності, зокрема, задовольняють усі скінченнократні квазірегулярні відображення. Мартіо, Рязанов, Сребро та Якубов довели обернену нерівність Полецького для відображень зі скінченним спотворенням довжини у випадку конформного модуля. Вказаний результат Салімов та Севостьянов розповсюдили на довільний порядок модуля сімей кривих. Севостьяновим доведено нульвимірність відображень з оберненою нерівністю Полецького у внутрішніх точках області за умови, що відповідна мажоранта у цій нерівності задовольняє умову Лехто-Діні. Пізніше цей результат було розповсюджено на границю рівномірно збіжної послідовності таких відображень. Автором даного рукопису разом з Севостьяновим нещодавно було отримано нульвимірність граничного відображення у замиканні області евклідового простору. У даному рукопису аналогічний результат отриманий у метричному просторі. Зокрема доведено, що якщо послідовність відкритих, дискретних та замкнених відображень з оберненою нерівністю Полецького збігається у замиканні області, а мажоранта у цій нерівності має скінченне середнє коливання, то відповідне граничне відображення має неперервне межове продовження, яке є нульвимірним у замиканні.

Ключові слова: *модулі сімей кривих, нульвимірні відображення, метричні простори.*

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