DOI: https://doi.org/10.15407/pmach2019.04.024

UDC 539.3

INVESTIGATION OF THE STRESS STATE OF A COMPOSITE IN THE FORM OF A LAYER AND A HALF SPACE WITH A LONGITUDINAL CYLINDRICAL CAVITY AT STRESSES GIVEN ON BOUNDARY SURFACES

Vitalii Yu. Miroshnikov
m0672628781@gmail.com
ORCID: 0000-0002-9491-0181

Kharkiv National University of Construction and Architecture,
40, Sumska St., Kharkiv,
61002, Ukraine

An analytical-numerical approach to solving the spatial problem of the theory of elasticity for a half-space rigidly coupled to a layer is proposed. In the half-space, parallel to its boundaries, there is an infinite circular cylindrical cavity. Both the layer and half-space are homogeneous isotropic materials, different from each other. It is necessary to investigate the stress-strain state of the elastic bodies of both the layer and half-space. On both the surface of the cavity and upper boundary of the layer, stresses are given. On the flat surface of contact between the layer and half-space, conjugation conditions arise. The solution to the spatial problem of the elasticity theory is obtained by the generalized Fourier method with regard to both the system of Lamé equations in cylindrical coordinates associated with the cavity and Cartesian coordinates associated with both the layer and half-space. The infinite systems of linear algebraic equations obtained as a result of satisfying both the boundary and conjugation conditions are solved by a reduction method. As a result, displacements and stresses are obtained at different points of both the elastic layer and elastic half-space. The fulfillment of boundary conditions was reduced to 10^{-4} by using the selected reduction parameter for the given geometric characteristics. An analysis of the stress-strain state of both the layer and half-space with the given physical and geometric parameters has been carried out. Graphs of stresses at the boundary between the layer and half-space, on the surface of the cavity and upper boundary of the layer, as well as on the bridge between the cavity and boundary of the half-space are presented. The indicated stress graphs show that the maximum stresses are concentrated both on the surface of the cylindrical cavity and surface of the half-space. The proposed method can be used to calculate parts and components, underground structures and communications, whose design schemes coincide with the purpose of this paper. The stress analysis presented above can be used to select geometric parameters at the design stage, and the stress graphs at the boundary between the layer and half-space, to analyze the coupling strength.

Keywords: *cylindrical cavity in a half-space, composite, Lamé equation, conjugation conditions, generalized Fourier method.*

Introduction

When designing complex technical underground structures and communications, as well as parts and components made of composite materials containing cylindrical cavities, it is necessary to have an understanding of the stress state that arises in these elements. For this purpose, it is necessary to have a method of calculation that would allow us to obtain the result with the required accuracy.

The most developed topic is the one where the cavity is located transversely to the boundaries of a plate or layer [1–4]. However, the methods, that the authors use, cannot be applied to the layer with a longitudinal cavity or inclusion. For such cases, [5–7], based on the Fourier series decomposition method or the image method, consider the stationary problems of wave diffraction and stress value determination.

For problems with multiple boundary surfaces, it is necessary to use the generalized Fourier method [8], which is the basis of this paper.

On the basis of the generalized Fourier method, the problem for the half-space with a cylindrical cavity or inclusion is solved in [9–13], for the cylinder with cylindrical inclusions, in [14], for the layer with a cylindrical cavity or inclusion, in [15, 16], and for the layer with a longitudinal thick-walled tube, in [17].

The problem for the half-space with a longitudinal cylindrical cavity and a layer rigidly coupled to the half-space has not been studied before, but it can be found in the calculation schemes, and therefore it is relevant.

Problem Formulation

In a homogeneous elastic half-space, there is a circular cylindrical cavity of radius R. A layer is perfectly coupled to the half-space boundary (Fig. 1).

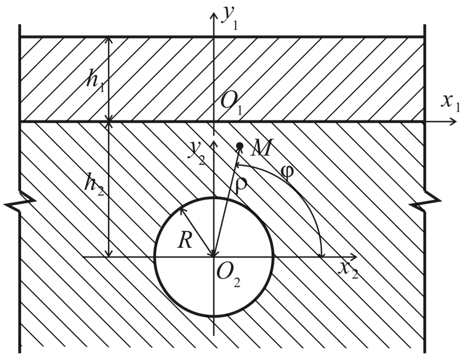


Fig. 1. A half-space with a cylindrical cavity and a layer

The cavity will be considered in the cylindrical coordinate system (ρ, φ, z), the layer, in the Cartesian coordinate system (x_1, y_1, z_1), the half-space, in the Cartesian coordinate system (x_2, y_2, z_2) combined with the cavity coordinate system. The boundaries of the layer are located at the distance $y_1=h_1$ and $y_1=0$, and the half-space, at the distance $y_2=h_2$, with $h_2>R$.

It is necessary to find a solution to the Lamé equation

$$\Delta\vec{U}_j + (1-2\sigma_j)^{-1}\nabla div\vec{U}_j = 0,$$

where j=1 corresponds to the layer and j=2, to the half-space. At the upper boundary of the layer $y_1=h_1$ and at the surface of the cavity $\rho=R$, are given the stresses

$$F_1\vec{U}_1(x,z)\big|_{y_1=h_1} = \vec{F}_h^0(x,z),\ F_2\vec{U}_2(\varphi,z)\big|_{\rho=R} = \vec{F}_R^0(\varphi,z),$$

where

$$\begin{aligned}\vec{F}_h^0(x_1,z_1) &= \tau_{yx}^{(h)}\vec{e}_1^{(1)} + \sigma_y^{(h)}\vec{e}_2^{(1)} + \tau_{yz}^{(h)}\vec{e}_3^{(1)},\\ \vec{F}_R^0(\varphi,z) &= \sigma_\rho^{(R)}\vec{e}_1^{(2)} + \tau_{\rho\varphi}^{(R)}\vec{e}_2^{(2)} + \tau_{\rho z}^{(R)}\vec{e}_3^{(2)}\end{aligned} \qquad (1)$$

are known functions; $\vec{e}_j^{(k)}$, j = 1, 2, 3 are the unit vectors of the Cartesian (k=1) and cylindrical (k=2) coordinate systems.

At the boundary between the layer and half-space, are given the conjugation conditions

$$\vec{U}_1(x_1,z_1)\Big|_{y_1=0} = \vec{U}_2(x_2,z_2)\Big|_{y_2=h_2}, \qquad (2)$$

$$F_1\vec{U}_1(x_1,z_1)\Big|_{y_1=0} = F_2\vec{U}_2(x_2,z_2)\Big|_{y_2=h_2}, \qquad (3)$$

where $\vec{U}_1$ are the displacements in the layer; $\vec{U}_2$ are the displacements in the half-space; $F_j\vec{U}_j\Big| = 2G_j[\frac{\sigma_j}{1-2\sigma_j}\vec{n}\,\mathrm{div}U_j + \frac{\partial}{\partial n}\vec{U}_j + \frac{1}{2}(\vec{n}\times rot\vec{U}_j)]$; $G_j = \frac{E_j}{2(1+\sigma_j)}$; σ_j, E_j are Poisson's ratio and the modulus of elasticity of the layer (j=1) or the half-space (j=2).

All the given vectors and functions will be considered to be rapidly decreasing to zero at long distances from the origin of coordinates along the z coordinate for the tube and along the x and z coordinates for the boundaries of the layer.

Problem Solution

Choose the basic solutions to the Lamé equation for the given coordinate systems in the form [8]

$$\begin{aligned}\vec{u}_k^{\pm}(x,y,z;\lambda,\mu) &= N_k^{(d)}e^{i(\lambda z+\mu x)\pm\gamma y};\\ \vec{R}_{k,m}(\rho,\varphi,z;\lambda) &= N_k^{(p)}I_m(\lambda\rho)e^{i(\lambda z+m\varphi)};\\ \vec{S}_{k,m}(\rho,\varphi,z;\lambda) &= N_k^{(p)}\left[(\mathrm{sign}\,\lambda)^m K_m(|\lambda|\rho)\cdot e^{i(\lambda z+m\varphi)}\right],\ k=1,2,3;\end{aligned} \qquad (4)$$

$$N_1^{(d)} = \frac{1}{\lambda}\nabla;\ N_2^{(d)} = \frac{4}{\lambda}(\sigma-1)\vec{e}_2^{(1)} + \frac{1}{\lambda}\nabla(y\,\cdot);\ N_3^{(d)} = \frac{i}{\lambda}\mathrm{rot}(\vec{e}_3^{(1)}\cdot);\ N_1^{(p)} = \frac{1}{\lambda}\nabla;$$

$$N_2^{(p)} = \frac{1}{\lambda}\left[\nabla\left(\rho\frac{\partial}{\partial\rho}\right) + 4(\sigma-1)\left(\nabla - \vec{e}_3^{(2)}\frac{\partial}{\partial z}\right)\right];\ N_3^{(p)} = \frac{i}{\lambda}\mathrm{rot}(\vec{e}_3^{(2)}\cdot);\ \gamma = \sqrt{\lambda^2+\mu^2},\ -\infty<\lambda,\mu<\infty,$$

where m is a member of the Fourier series; σ is Poisson's ratio; $I_m(x)$, $K_m(x)$ are the modified Bessel functions; $\vec{R}_{k,m}$, $\vec{S}_{k,m}$, k=1, 2, 3 are, respectively, the inner and outer solutions to the Lamé equation for the cylinder, $\vec{u}_k^{(-)}$, $\vec{u}_k^{(+)}$ are the solutions to the Lamé equation for the layer.

We present the solution to the problem in the form

$$\vec{U}_1 = \sum_{k=1}^{3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(H_k^{(1)}(\lambda,\mu) \cdot \vec{u}_k^{(+)}(x_1, y_1, z_1; \lambda, \mu; \sigma_1) + \tilde{H}_k^{(1)}(\lambda,\mu) \cdot \vec{u}_k^{(-)}(x_1, y_1, z_1; \lambda, \mu; \sigma_1) \right) d\mu d\lambda, \quad (5)$$

$$\vec{U}_2 = \sum_{k=1}^{3} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} B_{k,m}(\lambda) \cdot \vec{S}_{k,m}(\rho, \varphi, z; \lambda; \sigma_1) d\lambda + \sum_{k=1}^{3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(H_k^{(2)}(\lambda,\mu) \cdot \vec{u}_k^{(+)}(x_2, y_2, z_2; \lambda, \mu; \sigma_2) \right) d\mu d\lambda, \quad (6)$$

where $\vec{S}_{k,m}(\rho, \varphi, z; \lambda)$, $\vec{R}_{k,m}(\rho, \varphi, z; \lambda)$, $\vec{u}_k^{(+)}(x, y, z; \lambda, \mu)$, and $\vec{u}_k^{(-)}(x, y, z; \lambda, \mu)$ are the basic solutions given by formulas (4); and the unknown functions $H_k^{(1)}(\lambda,\mu)$, $\tilde{H}_k^{(1)}(\lambda,\mu)$, $H_k^{(2)}(\lambda,\mu)$, and $B_{k,m}(\lambda)$ must be found from boundary conditions (1) and conjugation conditions (2) and (3).

To relate the basic solutions in different coordinate systems, we use formulas [17]:

– for the transition from the solutions $\vec{S}_{k,m}$ for the cylindrical coordinate system to the solutions for the layer $\vec{u}_k^{(-)}$ (at y> 0) and $\vec{u}_k^{(+)}$ (at y <0)

$$\begin{aligned} &\vec{S}_{k,m}(\rho, \varphi, z; \lambda) = \frac{(-i)^m}{2} \int_{-\infty}^{\infty} \omega_{\mp}^m \cdot \vec{u}_k^{(\mp)} \cdot \frac{d\mu}{\gamma}, \quad k = 1,\ 3; \\ &\vec{S}_{2,m}(\rho, \varphi, z; \lambda) = \frac{(-i)^m}{2} \int_{-\infty}^{\infty} \omega_{\mp}^m \cdot \left(\left(\pm m \cdot \mu - \frac{\lambda^2}{\gamma} \right) \vec{u}_1^{(\mp)} - \lambda^2 \vec{u}_2^{(\mp)} \pm 4\mu(1-\sigma) \vec{u}_3^{(\mp)} \right) \frac{d\mu}{\gamma^2}, \end{aligned} \quad (7)$$

where $\gamma = \sqrt{\lambda^2 + \mu^2}$, $\omega_{\mp}(\lambda,\mu) = \dfrac{\mu \mp \gamma}{\lambda}$, $m = 0, \pm 1, \pm 2, \ldots$;

– for the transition from the solutions $\vec{u}_k^{(+)}$ and $\vec{u}_k^{(-)}$ for the layer to the solutions $\vec{R}_{k,m}$ for the cylindrical coordinate system

$$\begin{aligned} &\vec{u}_k^{(\pm)}(x, y, z) = \sum_{m=-\infty}^{\infty} (i \cdot \omega_{\mp})^m \vec{R}_{k,m}, \quad (k = 1,\ 3); \\ &\vec{u}_2^{(\pm)}(x, y, z) = \sum_{m=-\infty}^{\infty} \left[(i \cdot \omega_{\mp})^m \cdot \lambda^{-2} \left((m \cdot \mu) \cdot \vec{R}_{1,m} \pm \gamma \cdot \vec{R}_{2,m} + 4\mu(1-\sigma) \vec{R}_{3,m} \right) \right], \end{aligned} \quad (8)$$

where $\vec{R}_{k,m} = \vec{\tilde{b}}_{k,m}(\rho, \lambda) \cdot e^{i(m\varphi + \lambda z)}$;

$$\vec{\tilde{b}}_{1,n}(\rho, \lambda) = \vec{e}_\rho \cdot I_n'(\lambda\rho) + i \cdot I_n(\lambda\rho) \cdot \left(\vec{e}_\varphi \frac{n}{\lambda\rho} + \vec{e}_z \right);$$

$$\vec{\tilde{b}}_{2,n}(\rho, \lambda) = \vec{e}_\rho \cdot \left[(4\sigma - 3) \cdot I_n'(\lambda\rho) + \lambda\rho_p I_n''(\lambda\rho) \right] + \vec{e}_\varphi i \cdot m \left(I_n'(\lambda\rho) + \frac{4(\sigma-1)}{\lambda\rho} I_n(\lambda\rho) \right) + \vec{e}_z i\lambda\rho I_n'(\lambda\rho);$$

$$\vec{\tilde{b}}_{3,n}(\rho, \lambda) = -\left[\vec{e}_\rho \cdot I_n(\lambda\rho) \frac{n}{\lambda\rho} + \vec{e}_\varphi \cdot i \cdot I_n'(\lambda\rho) \right],$$

$\vec{e}_\rho$, $\vec{e}_\varphi$, $\vec{e}_z$ are the unit vectors in the cylindrical coordinate system.

To satisfy the boundary conditions at the upper boundary of the layer y_1=h_1, we find the stress for (5) and equate it to the given $\vec{F}_h^0(x, z)$ represented as the double Fourier integral. In this way, we get three equations (one for each projection) with six unknowns $H_k^{(1)}(\lambda,\mu)$, $\tilde{H}_k^{(1)}(\lambda,\mu)$.

To satisfy the conjugation conditions, we substitute expressions (5) and (6) into (2), using the formula of transition from the basic solutions $\vec{S}_{k,m}$ for the cylinder to the basic solutions $\vec{u}_k^{(-)}$ for the layer (7). We carry out a similar operation for stresses (3).

Having obtained a system of nine infinite equations, we express from them $H_k^{(1)}(\lambda,\mu)$, $\tilde{H}_k^{(1)}(\lambda,\mu)$, and $H_k^{(2)}(\lambda,\mu)$ through $B_{k,m}(\lambda)$.

The determinant Δ of this system has the form

$$\Delta = -\frac{64\cdot\gamma^9\cdot\sigma^3\cdot e^{-3\gamma(h_1-h_2)}\cdot\Phi(\gamma)}{\lambda^6},$$

where $\gamma=\sqrt{\lambda^2+\mu^2}$; $\Phi(\gamma)$ is a function that has a cumbersome appearance and, as a consequence, is omitted. The study of $\Phi(\gamma)$ showed that, for $\gamma>0$, it has only positive values and does not converge to zero, which is why the equation system has a single solution.

To satisfy the boundary conditions on the surface of the cavity $\rho=R$, we, using the transition formulas from the solutions $\vec{u}_k^{(+)}$ to the solutions $\vec{R}_{k,m}$ [17, formula (8)] rewrite the right part (6) in the cylindrical coordinate system. For the obtained vector, we find the stress and equate it to the given $\vec{F}_R^0(\varphi,z)$ represented by the integral and Fourier series. We substitute $H_k^{(2)}(\lambda,\mu)$ for the obtained relations through the functions $B_{k,m}(\lambda)$. As a result, we obtain a set of three infinite systems of linear algebraic equations with respect to $B_{k,m}(\lambda)$.

These systems have the properties of equations of the second kind and, as a consequence, a reduction method can be applied to them [18]. The numerical studies have also shown that the determinant of the reduced system does not converge to zero at any m, for $0\le m\le 12$, and as a consequence, this system of equations has a single solution.

Having obtained the values of the functions $B_{k,m}(\lambda)$, we can find the values of the unknowns $H_k^{(1)}(\lambda,\mu)$, $\tilde{H}_k^{(1)}(\lambda,\mu)$, $H_k^{(2)}(\lambda,\mu)$ which we have previously represented through $B_{k,m}(\lambda)$. This will identify all unknown tasks.

Numerical Studies of the Stress State

The cylindrical cavity is located in the homogeneous isotropic half-space perfectly coupled to the homogeneous isotropic layer. The physical properties of the layer (concrete B30) are the following: Poisson's ratio σ_1 = 0.16 and the modulus of elasticity E_1=3,250 kN/cm^2. The physical properties of the half space (aerated concrete) are the following: σ_2=0.2, E_2=160 kN/cm^2. The radius of the cylindrical cavity R=10 cm. The thickness of the layer h_1=10 cm. The distance from the half-space boundary to the center of the cylindrical cavity h_2=15 cm.

At the upper boundary of the layer are given stresses from the action of technological equipment in the form of waves along the z axis and along the x axis, $\sigma_y^{(h)}(x,z)=-10^8\cdot\left(z^2+10^2\right)^{-2}\cdot\left(x^2+10^2\right)^{-2}$, $\tau_{yx}^{(h)}=\tau_{yz}^{(h)}=0$; on the surface of the cylindrical cavity, the stresses $\sigma_\rho^{(R)}=\tau_{\rho\varphi}^{(R)}=\tau_{\rho z}^{(R)}=0$.

The accuracy of boundary conditions depends on the order of the system m and the distance between the boundary surfaces. For the proposed geometric characteristics, the accuracy of satisfying the boundary conditions on the surface of the layer, depending on m, is shown in Table 1.

Table 1. Error in satisfying boundary conditions

Surface for calculation	Given	Results obtained, σ_y, kN/cm^2				
		m=4	m=6	m=8	m=10	m=12
On the layer boundary	1	0.98398	0.98855	0.99536	0.99961	0.99991
On the cavity boundary	0	0.00412	0.00062	0.00011	0.00004	10^{-6}

A finite system of equations of order m=12 was solved. The calculations of the integrals were performed using Filon's quadrature formulas (for the functions having $e^{i\lambda z}d\lambda$ and $e^{i\mu x}d\mu$) and Simpson's ones (for other functions).

Fig. 2 shows the stresses (in kN/cm^2) in the body of the half-space at z=0. Figure 2, a shows the stresses on the surface of the cylindrical cavity from φ=0 to φ=2π, and Fig. 2, b, on the bridge between the cylindrical cavity and the half-space boundary along the y axis.

The maximum stresses σ_φ occur on the cavity surface at φ=0.63 and φ=2.51 (Fig. 2, a).

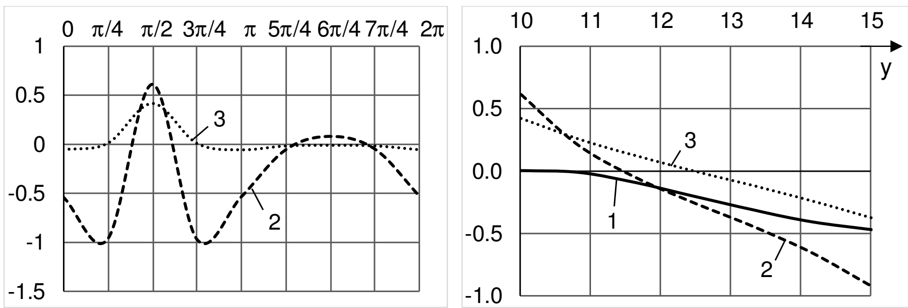
As far as the bridge is concerned (Fig. 2, b), the stresses σ_φ and σ_z on the surface of the cavity are positive, and at the boundary of the half-space they converge to negative ones.

Fig. 3 shows the stresses (in kN/cm^2) at the upper (Fig. 3, a) and lower (Fig. 3, b) boundaries of the layer along the z axis in the plane x=0.

With the given stresses σ_y (Fig. 3, a, line 1), the stresses σ_x and σ_z at the upper boundary of the half-space, at z=0, acquire almost identical values. Further along the z axis, the stresses gradually decrease. In addition, the stresses σ_z at $|z| \geq 13$ cm have positive values (Fig. 3, a, line 3).

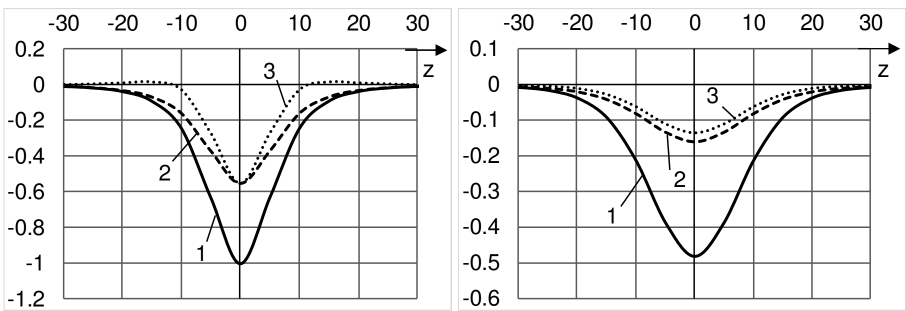
At the lower boundary of the layer (Fig. 3, b), the stresses σ_x and σ_z gradually decrease, acquiring only negative values.

The stresses σ_z along the x-axis have almost the same appearance as along the z axis (Fig. 3, b, line 3). The stresses along the x axis, unlike the stresses along the z axis (Fig. 3, b, line 2), decrease somewhat faster. Because of the slight deviations of the stress graph along the z axis (Fig. 3, b) from the stress graph along the x axis, the latter is omitted.



a b

Fig. 2. Stresses in the half-space body:
a – on the cavity surface; b – on the bridge between the cavity and half-space boundary
1 – σ_ρ ; 2 – σ_φ ; 3 – σ_z



a b

Fig. 3. Stresses in the layer body along the z-axis:
a – at the upper boundary; b – at the lower boundary; 1 – σ_y ; 2 – σ_x ; 3 – σ_z

Conclusions

An analytical-numerical method for solving the spatial problem of the elasticity theory for the half-space with a longitudinal cavity and a coupled elastic layer has been developed using the generalized Fourier method. The problem is reduced to a set of infinite systems of linear algebraic equations, which are solved by the method of reduction to a finite system.

The graphs give a picture of the stress distribution at the upper surface of the layer, at the boundary between the layer and half-space, as well as on the cavity surface.

The stresses σ_x and σ_z found at the boundary between the layer and half-space can be used to calculate the coupling strength.

Compared with [5–7], the proposed method allows us to obtain an exact solution to the problem in the spatial variant and, in comparison with [9–17], take into account the new boundary surfaces by adding to the boundary conditions the conditions of the layer and half-space coupling.

The numerical analysis of the stress-strain state of the given composite shows:

– the maximum stresses are concentrated both on the surface of the cylindrical cavity (Fig. 2, a) and on the surface of the half-space (Fig. 2, b);

– the stresses σ_z on the surface of the layer have both positive and negative values.

The numerical studies of the algebraic system for the given composite make it possible to state that its solution can be found with any degree of accuracy by the reduction method (Table 1). This is confirmed by the high accuracy of satisfying the boundary conditions. For the geometric parameters of the problem solved at m=12, the boundary conditions are satisfied with an accuracy of 10^{-4}. As the order m of the system increases, the accuracy of the calculations will increase.

To check for validation, the materials of the layer and half-space were considered to be the same. The combined material and geometric characteristics were adopted as in [10] and, further, in [11].

The comparative analysis of the stress state of the bridge between the cavity and half-space boundary, at m=10, with the results in [10] is summarized in Table 2.

Table 2. Comparative analysis of the results obtained with the results in [10]

Data for comparison	Results obtained, σ_y, kN/cm^2				
	y=10	y=13.3	y=15	y=16,7	y=20
Results from [10]	-0.00047	-0.31152	-0.52902	-0.75062	-1.00080
Results from this paper	-0.00047	-0.31153	-0.52889	-0.75060	-1.00077

Table 2 shows that the error in the results is at the level of calculation accuracy (table 1). The comparative analysis with [11] has similar results.

Convergence with the known results and the high accuracy of boundary conditions testify to the reliability of both the method and results obtained.

One of the disadvantages is that the method does not allow us to solve problems when the boundaries of a body touch or intersect.

A further development of this area of research is possible with increasing the number of cylindrical cavities or calculating the problem with other boundary conditions.

References

1. Vaysfeld, N., Popov, G., & Reut, V. (2015). The axisymmetric contact interaction of an infinite elastic plate with an absolutely rigid inclusion. *Acta Mechanica*, vol. 226, iss. 3, pp. 797–810. https://doi.org/10.1007/s00707-014-1229-7.
2. Popov, G. Ya. & Vaysfeld, N. D. (2014). Solving an axisymmetric problem of elasticity for an infinite plate with a cylindrical inclusion with allowance for its specific weight. *International Applied Mechanics*, vol. 50, iss. 6, pp. 627–636. https://doi.org/10.1007/s10778-014-0661-7.
3. Grinchenko, V. T. & Ulitko, A. F. (1968). An exact solution of the problem of stress distribution close to a circular hole in an elastic layer. *Soviet Applied Mechanics*, vol. 4, iss. 10, pp. 31–37. https://doi.org/10.1007/BF00886618.
4. Grinchenko, V. T. & Ulitko, A. F. (1985). *Prostranstvennyye zadachi teorii uprugosti i plastichnosti. Ravnovesiye uprugikh tel kanonicheskoy formy* [Spatial problems of the theory of elasticity and plasticity. Equilibrium of elastic bodies of canonical form]. Kiyev: Naukova Dumka, 280 p. (in Russian).
5. Guz, A. N., Kubenko, V. D., & Cherevko, M. A. (1978). *Difraktsiya uprugikh voln* [Diffraction of elastic waves]. Kiyev: Naukova Dumka, 307 p. (in Russian).

6. Grinchenko, V. T. & Meleshko, V. V. (1981). *Garmonicheskiye kolebaniya i volny v uprugikh telakh* [Harmonic oscillations and waves in elastic bodies]. Kiyev: Naukova Dumka, 284 p. (in Russian).
7. Volchkov, V. V., Vukolov, D. S., & Storozhev, V. I. (2016). *Difraktsiya voln sdviga na vnutrennikh tunnel'nykh tsilindricheskikh neodnorodnostyakh v vide polosti i vklyucheniya v uprugom sloye so svobodnymi granyami* [Diffraction of shear waves on internal tunnel cylindrical inhomogeneities in the form of a cavity and inclusion in the elastic layer with free faces]. *Mekhanika tverdogo tela – Mechanics of Rigid Bodies*, vol. 46, pp. 119–133 (in Russian).
8. Nikolayev, A. G. & Protsenko, V. S. (2011). *Obobshchennyy metod Furye v prostranstvennykh zadachakh teorii uprugosti* [The generalized Fourier method in spatial problems of the theory of elasticity]. Kharkov: Nats. aerokosm. universitet im. N. Ye. Zhukovskogo «KHAI», 344 p. (in Russian).
9. Nikolaev, A. G. & Orlov, Ye. M. (2012). *Resheniye pervoy osesimmetrichnoy termouprugoy krayevoy zadachi dlya transversalno-izotropnogo poluprostranstva so sferoidalnoy polostyu* [Solution of the first axisymmetric thermoelastic boundary value problem for a transversely isotropic half-space with a spheroidal cavity]. *Problemy obchyslyuvalnoyi mekhaniky i mitsnosti konstruktsiy – Problems of computational mechanics and strength of structures*, vol. 20, pp. 253–259 (in Russian).
10. Miroshnikov, V. Yu. (2018). First basic elasticity theory problem in a half-space with several parallel round cylindrical cavities. *Journal of Mechanical Engineering*, vol. 21, no. 2, pp. 12–18. https://doi.org/10.15407/pmach2018.02.012.
11. Protsenko, V. & Miroshnikov, V. (2018). Investigating a problem from the theory of elasticity for a half-space with cylindrical cavities for which boundary conditions of contact type are assigned. *Eastern-European Journal of Enterprise Technologies*, vol. 4, no. 7, pp. 43–50. https://doi.org/10.15587/1729-4061.2018.139567.
12. Nikolaev, A. G., Shcherbakova, A. Yu., & Yukhno, A. I. (2006). *Deystviye sosredotochennoy sily na transversalno-izotropnoye poluprostranstvo s paraboloidalnym vklyucheniyem* [Action of a concentrated force on a transversely-isotropic half-space with paraboloidal inclusion]. *Voprosy proyektirovaniya i proizvodstva konstruktsiy letatelnykh apparatov – Questions of design and production of aircraft structures*, vol. 2, pp. 47–51 (in Russian).
13. Miroshnikov, V. Yu. (2018). Evaluation of the stress-strain state of half-space with cylindrical cavities. *Visnyk Dniprovskoho universytetu. Seriya: Mekhanika – Bulletin of the Dnipro University. Series: Mechanics*, vol. 26, no. 5, pp. 109–118.
14. Nikolaev, A. G. & Tanchik, Ye. A. (2013). *Raspredeleniye napryazheniy v yacheyke odnonapravlennogo kompozitsionnogo materiala, obrazovannogo chetyrmya tsilindricheskimi voloknami* [Stress distribution in a cell of a unidirectional composite material formed by four cylindrical fibers]. *Visnyk Odeskoho natsionalnoho universytetu. Matematyka. Mekhanika – Odesa National University Herald. Mathematics and Mechanics*, vol. 4, pp. 101–111 (in Russian).
15. Miroshnikov, V. Yu. (2019). *Doslidzhennia druhoi osnovnoi zadachi teorii pruzhnosti dlia sharu z tsylindrychnoiu porozhnynoiu* [Investigation of the second main problem of the theory of elasticity for a layer with a cylindrical cavity]. *Opir materialiv i teoriia sporud – Strength of Materials and Theory of Structures*, no. 102, pp. 77–90 (in Ukrainian).
16. Miroshnikov, V. Yu. (2019). *Zmishana zadacha teorii pruzhnosti dlia sharu z tsylindrychnym vkliuchenniam* [Mixed problem of the theory of elasticity for a cylindrical inclusion layer]. *Naukovyi visnyk budivnytstva – Scientific Bulletin of Civil Engineering*, vol. 96, no. 2, pp. 247–252 (in Ukrainian). https://doi.org/10.29295/2311-7257-2019-96-2-247-252.
17. Miroshnikov, V. Yu. (2019). Investigation of the stress strain state of the layer with a longitudinal cylindrical thick-walled tube and the displacements given at the boundaries of the layer. *Journal of Mechanical Engineering*, vol. 22, no. 2, pp. 44–52. https://doi.org/10.15407/pmach2019.02.044.
18. Kantorovich, L. V. & Akilov, G. P. (1977). *Funktsionalnyy analiz* [Functional analysis]. Moscow: Nauka, 742 p. (in Russian).

Received 28 August 2019

Дослідження напруженого стану композиту у вигляді шару та півпростору з повздовжньою циліндричною порожниною, за заданих на граничних поверхнях напружень

В. Ю. Мірошніков

Харківський національний університет будівництва та архітектури,
61002, Україна, м. Харків, вул. Сумська, 40

Запропоновано аналітико-числовий підхід до розв'язання просторової задачі теорії пружності для півпростору, жорстко зчепленого з шаром. В півпросторі, паралельно його межам, розташована нескінченна кругова циліндрична порожнина. Півпростір та шар – однорідні ізотропні матеріали, відмінні один від одного. Необхідно дослідити напружено-деформований стан пружних тіл шару та півпростору. На поверхні порожнини та на верхній межі шару задані напруження. На плоскій поверхні контакту шару та півпростору виникають умови спряження.

Розв'язок просторової задачі теорії пружності отримано узагальненим методом Фур'є стосовно системи рівнянь Ламе в циліндричних координатах, пов'язаних із порожниною, та декартових координатах, пов'язаних із шаром та півпростором. Нескінченні системи лінійних алгебраїчних рівнянь, які отримані в результаті задовольняння граничних умов та умов спряження, розв'язано методом зрізання. В результаті отримані переміщення та напруження в різних точках пружного шару та пружного півпростору. Виконання граничних умов доведено до 10^{-4} за рахунок підібраного параметра зрізання для заданих геометричних характеристик. Проведено аналіз напружено-деформованого стану шару та півпростору за заданих фізичних та геометричних параметрів. Подані графіки напружень на межі шару та півпростору, на поверхні порожнини та верхній межі шару, а також на перешийку між порожниною та межею півпростору. Зазначені графіки напружень показують, що найбільші напруження концентруються на поверхні циліндричної порожнини та на поверхні півпростору. Запропонований метод може використовуватись для розрахунку деталей, підземних споруд та комунікацій, розрахункові схеми яких відповідають постановці задачі даної роботи. Наведений аналіз напруженого стану може бути використаний для підбору геометричних параметрів на стадії проектування, а графік напружень на межі шару та півпростору – для аналізу міцності з'єднання.

Ключові слова: *циліндрична порожнина в півпросторі, композит, рівняння Ламе, умови спряження, узагальнений метод Фур'є.*

Література

1. Vaysfel'd N., Popov G., Reut V. The axisymmetric contact interaction of an infinite elastic plate with an absolutely rigid inclusion. *Acta Mech.* 2015. Vol. 226. P. 797–810. https://doi.org/10.1007/s00707-014-1229-7.
2. Попов Г. Я., Вайсфельд Н. Д. Осесимметричная задача теории упругости для бесконечной плиты с цилиндрическим включением при учете ее удельного веса. *Прикл. механика*. 2014. Т. 50. № 6. С. 27–38.
3. Grinchenko V. T., Ulitko A. F. An exact solution of the problem of stress distribution close to a circular hole in an elastic layer. *Soviet Appl. Mechs*. 1968. No. 10. P. 31–37. https://doi.org/10.1007/BF00886618.
4. Гринченко В. Т., Улитко А. Ф. Пространственные задачи теории упругости и пластичности. Равновесие упругих тел канонической формы. Киев: Наук. думка, 1985. 280 с.
5. Гузь А. Н., Кубенко В. Д., Черевко М. А. Дифракция упругих волн. Киев: Наук. думка, 1978. 307 с.
6. Гринченко В. Т., Мелешко В. В. Гармонические колебания и волны в упругих телах. Киев: Наук. думка, 1981. 284 с.
7. Волчков В. В., Вуколов Д. С., Сторожев В. И. Дифракция волн сдвига на внутренних туннельных цилиндрических неоднородностях в виде полости и включения в упругом слое со свободными гранями. *Механика твердого тела*. 2016. Вып. 46. С. 119–133.
8. Николаев А. Г., Проценко В. С. Обобщенный метод Фурье в пространственных задачах теории упругости. Харьков: Нац. аэрокосм. ун-т им. Н. Е. Жуковского «ХАИ», 2011. 344 с.
9. Николаев А. Г., Орлов Е. М. Решение первой осесимметричной термоупругой краевой задачи для трансверсально-изотропного полупространства со сфероидальной полостью. *Проблеми обчислювальної механіки і міцності конструкцій*. 2012. Вип.20. С. 253–259.
10. Miroshnikov V. Yu. First basic elasticity theory problem in a half-space with several parallel round cylindrical cavities. *J. Mech. Eng*. 2018. Vol. 21. No. 2. P. 12–18. https://doi.org/10.15407/pmach2018.02.012.
11. Protsenko V., Miroshnikov V. Investigating a problem from the theory of elasticity for a half-space with cylindrical cavities for which boundary conditions of contact type are assigned. *Eastern-European J. Enterprise Techn. Appl. Mech.* 2018. Vol. 4. No. 7. P. 43–50. https://doi.org/10.15587/1729-4061.2018.139567.
12. Николаев А. Г., Щербакова А. Ю., Юхно А. И. Действие сосредоточенной силы на трансверсально-изотропное полупространство с параболоидальным включением. *Вопр. проектирования и пр-ва конструкций летат. аппаратов.* 2006. Вып. 2. С. 47–51.
13. Miroshnikov V. Yu. Evaluation of the stress-strain state of half-space with cylindrical cavities. *Visnyk Dniprovskoho universytetu. Ser. Mekhanika – Bulletin of the Dnipro University*. 2018. Vol. 26. No. 5. P. 109–118.
14. Николаев А. Г., Танчик Е. А. Распределение напряжений в ячейке однонаправленного композиционного материала, образованного четырьмя цилиндрическими волокнами. *Вісн. Одес. нац. ун-ту. Математика. Механіка*. 2013. Т. 18. Вип. 4(20). С. 101–111.
15. Мірошніков В. Ю. Дослідження другої основної задачі теорії пружності для шару з циліндричною порожниною. *Опір матеріалів і теорія споруд*. 2019. № 102. С. 77–90.
16. Мірошніков В. Ю. Змішана задача теорії пружності для шару з циліндричним включенням. *Наук. вісн. буд-ва*. 2019. Т. 96. № 2. С. 247–252. https://doi.org/10.29295/2311-7257-2019-96-2-247-252.
17. Miroshnikov V. Investigation of the Stress Strain State of the Layer with a Longitudinal Cylindrical Thick-Walled Tube and the Displacements Given at the Boundaries of the Layer. *J. Mec. Eng*. 2019. Vol. 22. No. 2. P. 44–52. https://doi.org/10.15407/pmach2019.02.044.
18. Канторович Л. В., Акилов Г. П. Функциональный анализ. М.: Наука, 1977. 742с.