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TO THE SOLUTION OF GEOMETRIC INVERSE HEAT CONDUCTION PROBLEMS

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On the basis of A. N. Tikhonov's regularization theory, a method is developed for solving inverse heat conduction problems of identifying a smooth outer boundary of a two-dimensional region with a known boundary condition. For this, the smooth boundary to be identified is approximated by Schoenberg's cubic splines, as a result of which its identification is reduced to determining the unknown approximation coefficients. With known boundary and initial conditions, the body temperature will depend only on these coefficients. With the temperature expressed using the Taylor formula for two series terms and substituted into the Tikhonov functional, the problem of determining the increments of the coefficients can be reduced to solving a system of linear equations with respect to these increments. Having chosen a certain regularization parameter and a certain function describing the shape of the outer boundary as an initial approximation, one can implement an iterative process. In this process, the vector of unknown coefficients for the current iteration will be equal to the sum of the vector of coefficients in the previous iteration and the vector of the increments of these coefficients, obtained as a result of solving a system of linear equations. Having obtained a vector of coefficients as a result of a converging iterative process, it is possible to determine the root-mean-square discrepancy between the temperature obtained and the temperature measured as a result of the experiment. It remains to select the regularization parameter in such a way that this discrepancy is within the measurement error. The method itself and the ways of its implementation are the novelty of the material presented in this paper in comparison with other authors' approaches to the solution of geometric inverse heat conduction problems. When checking the effectiveness of using the method proposed, a number of two-dimensional test problems for bodies with a known location of the outer boundary were solved. An analysis of the influence of random measurement errors on the error in identifying the outer boundary shape is carried out.

Keywords: *geometric inverse heat conduction problem, A. N. Tikhonov's regularization method, stabilizing functional, regularization parameter, identification, approximation, Schoenberg's cubic splines.*

Introduction

Today, inverse problems, that is, the problems in which the causal characteristics of physical processes are determined by the results of measurements or other investigative manifestations, are widely used in the study of physical processes of various nature, including thermophysical ones [1]. Solution of geometric inverse heat conduction problems (IHCP) of identifying the outer body boundary with a known boundary condition is of particular importance at the stage of constructing mathematical models in the presence of experimental information about the thermal process under investigation. In this paper, a geometric IHCP is considered as the problem of identifying the outer body boundary both with the temperature-dependent thermal conductivity coefficient and heat capacity, and with a known boundary condition at this boundary. Works [1–5] give classifications of IHCPs and consider methods of their solution. Monograph [1] also gives a classification of geometric IHCPs. This paper, according to [1], considers the problem that belongs to the class of geometric IHCPs of determining the shape and location of region boundaries. Article [6] proposes a unified methodological approach to the formulation and solution of geometric IHCPs, one of the stages of which is the parametrization of the required geometric information, ie. the solution of a geometric IHCP is reduced to the determination of a finite set of geometric parameters.

In this paper, the search for the desired boundary in the two-dimensional case is reduced to finding the equation of a smooth boundary in the form $x=f(y)$. Having approximated the required function $f(y)$ by a linear combination of Schoenberg's cubic splines with unknown coefficients, one can reduce the solution of a geometric IHCP to the search for these coefficients. Work [7] proposes a method for solving an internal IHCP (identification of the temperature-dependent thermal conductivity coefficient). In this work, to obtain a stable solution, the author uses M. M. Lavrentyev's α-regularization method [8], which is less flexible than A. N. Tikhonov's regularization method [4], since when using it, it is more difficult to take into account the a-priori information about the required function.

In [9], we used both the iterative process proposed in [7] and A. N. Tikhonov's regularization method [4], as well as an approach to selecting the regularization parameter for the simultaneous identification of several required thermophysical characteristics. Using this approach, this paper proposes a method for finding part of a smooth boundary by solving nonlinear geometric IHCPs.

Problem Formulation

In this paper, a two-dimensional nonlinear geometric IHCP is considered, which can be formalized as follows:

$$A[f(y)] = T^{ex},$$

where $f(y)$ is the right side of the required equation of the outer boundary $x=f(y)$; $T^{ex} = T(x, y, \tau)$ is temperature, in many cases known from the experiment (initial data); A is the operator that connects the desired function $f(y)$ with the initial data.

Due to the violation of causal relationships, such a problem, like any other IHCP, is an ill-posed one, according to Hadamard, which is the reason for the instability of the solution obtained. To solve such a problem, it is either reduced to conditionally well-posed or left ill-posed, but with the application of one of the regularization methods [2–5]. Here, we use A. N. Tikhonov's regularization method [4].

Consider a thermal process in a two-dimensional body (Fig. 1).

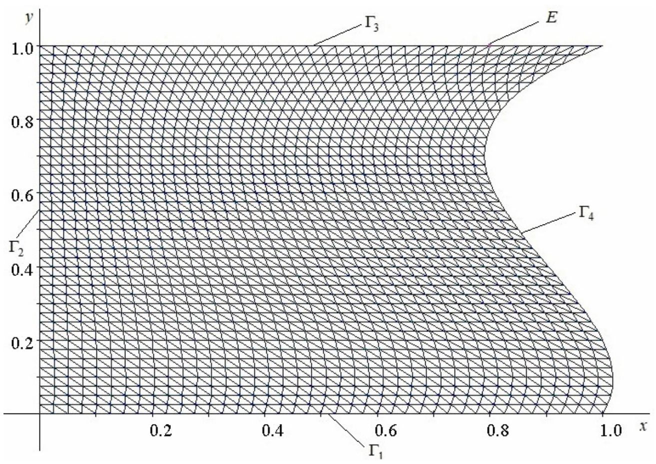

Fig. 1. Geometry of a body investigated in a test problem with a finite element mesh

This process is described by the following equations [2, 10]:

$$C(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda(T)\frac{\partial T}{\partial y}\right), \quad (x,y) \in D, \tag{1}$$

$$-\lambda(T)\frac{\partial T}{\partial \nu}\bigg|_{M \in \Gamma_1 \cup \Gamma_2} = 0, \tag{2}$$

$$-\lambda(T)\frac{\partial T}{\partial \nu}\bigg|_{M \in \Gamma_3} = \alpha_1\left(T - T_{S_1}\right), \tag{3}$$

$$-\lambda(T)\frac{\partial T}{\partial \nu}\bigg|_{M \in \Gamma_4} = \alpha_2\left(T - T_{S_2}\right), \tag{4}$$

$$T(x, y, \tau)\big|_{\tau=0} = T_0, \tag{5}$$

$$\text{at } T(x_k, y_k, \tau_l) = T_{lk}^{ex}, \quad l = \overline{1, n_\tau}, \quad k = \overline{1, m}, \tag{6}$$

where $T = T(x, y, \tau)$ is the body temperature; D is the space area occupied by the body; $\Gamma_i, i = \overline{1,4}$ are parts of the boundary of the area D; (x, y) is a point in the area D; $\lambda(T)$ and $C(T)$ are the temperature-dependent thermal conductivity coefficient and heat capacity; α_1 and α_2 are the heat transfer coefficients on the body surfaces Γ_3 and Γ_4, respectively; T_{S_1} and T_{S_2} are the specified temperatures of the external environment in contact with the body surfaces Γ_3 and Γ_4, respectively; ν is the outer normal to the body boundary; Γ_4 is the body boundary to be identified, described by the equation $x=f(y)$; T_0 is the initial body temperature; n_τ is the number of measurements in time; m is the number of measurement points in the body; (x_k, y_k) are points of the area D in which the temperature T_{lk}^{ex} is measured. The measurement error is a random variable distributed according to the normal law with zero mathematical expectation and σ^2 dispersion.

According to the thermophysical experiment, the equation of the outer boundary $x=f(y)$ is determined taking into account the available a-priori information about this function.

Below we consider a methodological approach to solving the problem.

Regularizing Algorithm for Solving the Geometric IHCP

To solve the nonlinear geometric IHCP (1–6), we use A. N. Tikhonov's regularization method, which reduces to minimizing the functional

$$J = \int_0^{\tau_0}\int_D \left[T(x,y,\tau) - T^{ex}(x,y,\tau)\right]^2 dxdyd\tau + \beta\cdot\Omega[f(y)] + \Delta[f(y)], \tag{7}$$

where $T=T(x, y, \tau)$ is the temperature obtained in the process of solving the geometric IHCP; $T^{ex}(x, y, \tau)$ is the experimentally obtained temperature; τ_0 is the completion moment of thermal process analysis; β is the regularization parameter; $\Omega[f(y)]$ is the stabilization functional; $\Delta[f(y)]$ is the quadratic functional characterizing the discrepancy between the required function $f(y)$ and the a-priori information given about it.

We represent the required function $f(y)$ in the form

$$f(y) = \sum_{k=1}^{n} \varphi_k B_3^k(y), \tag{8}$$

where $(\varphi_1, \varphi_2, ..., \varphi_n) = \vec{\Phi}$ is the vector of required parameters, and $B_3^k(y)$ are Schoenberg's cubic splines. Then the identification of the required function reduces to the determination of the unknown vector $\vec{\Phi}$.

We minimize functional (7) by the iterative method [7]. Since the temperature $T(x, y, \tau)$ depends on the vector $\vec{\Phi}$, it can be represented in the $(p+1)$th iteration using the Taylor series as follows:

$$T^{p+1}\left(x,y,\tau,f^{p+1}(y)\right) \approx T^{p}\left(x,y,\tau,f^{p}(y)\right) + \sum_{k=1}^{n}\frac{\partial T^p}{\partial\varphi_k}\Delta\varphi_k^{p+1}, \tag{9}$$

where $(\Delta\varphi_1^{p+1}, \Delta\varphi_2^{p+1}, ..., \Delta\varphi_n^{p+1}) = \overrightarrow{\Delta\Phi^{p+1}}$ is the vector of increments $\overrightarrow{\Delta\Phi^{p+1}} = \overrightarrow{\Phi^{p+1}} - \overrightarrow{\Phi^{p}}$.

We represent the stabilizing functional $\Omega[f(y)]$ in the $(p+1)$th iteration in the form

$$\Omega\left[f^{p+1}(y)\right] = \int_0^1 \left(w_0\left(f^{p+1}\right)^2 + w_1\left(\frac{\partial f^{p+1}}{\partial y}\right)^2 + w_2\left(\frac{\partial^2 f^{p+1}}{\partial y^2}\right)^2 \right) dy, \tag{10}$$

where w_0, w_1, w_2 are the weight factors that are selected using the a-priori information about the desired function $f(y)$.

In this problem, we used the second order regularization [5], when $w_0=0$, $w_1=0$ and $w_2=1$.

The quadratic functional $\Delta[f(y)]$ characterizing the discrepancy between the values of the desired function and its same values that are a-priori specified for some values of the variable y, can be constructed as follows.

Let $y_1, y_2, \ldots, y_s$ be some arguments of the required function, and $f_1, f_2, \ldots, f_s$, respectively, the a-priori specified values of this function. Then

$$\Delta[f(y)] = \sum_{j=1}^{s} P_j\left(f(y_j) - f_j\right)^2, \tag{11}$$

where P_j are the weight factors that are selected based on how accurately the a-priori values $f_1, f_2, \ldots, f_s$ are specified.

If we substitute expressions (8), (10), (11) into functional (7) and replace $T(x, y, \tau)$ with an approximate temperature value (9) at temperature measurement points, then, using the necessary condition for the minimum of functional (7), we can obtain a system of linear equations with respect to $\Delta\varphi_i^{p+1}$, $i=\overline{1,n}$, in the (*p*+1)th iteration

$$\left(M_0+\beta\cdot M_1+M_2\right)\overrightarrow{\Delta\varphi^{p+1}}=\vec{V_0}-\beta\cdot\vec{V_1}+\vec{V_2},$$

where $M_0=\left\{m_{0ij}\right\}_{\mathrm{i,j-1}}^{n}$, $M_1=\left\{m_{1ij}\right\}_{\mathrm{i,j-1}}^{n}$, $M_2=\left\{m_{2ij}\right\}_{\mathrm{i,j-1}}^{n}$ are the symmetric matrices of dimension *n*, and $\vec{V_0}=\left\{v_{0i}\right\}_{\mathrm{i-1}}^{n}$, $\vec{V_1}=\left\{v_{1i}\right\}_{\mathrm{i-1}}^{n}$, $\vec{V_2}=\left\{v_{2i}\right\}_{\mathrm{i-1}}^{n}$ are the vectors of dimension *n*. The corresponding components of the matrices and vectors are as follows:

$$m_{0ij}=\int\limits_0^{\tau_0}\int\limits_D\left[\frac{\partial T^p}{\partial\varphi_i}\cdot\frac{\partial T^p}{\partial\varphi_j}\right]dxdyd\tau,$$

$$m_{1ij}=\int\limits_0^1\left(w_0\cdot B_i^3(y)\cdot B_j^3(y)+w_1\cdot\frac{\partial B_i^3(y)}{\partial y}\cdot\frac{\partial B_j^3(y)}{\partial y}+w_2\cdot\frac{\partial^2 B_i^3(y)}{\partial y^2}\cdot\frac{\partial^2 B_j^3(y)}{\partial y^2}\right)dy,$$

$$m_{2ij}=\sum_{l=1}^{s}z_l\cdot B_i^3(y_l)\cdot B_j^3(y_l),$$

$$v_{0i}=\int\limits_0^{\tau_0}\int\limits_D\left(T^{ex}(x,y,\tau)-T^p(x,y,\tau,f^p(y))\right)\cdot\frac{\partial T^p}{\partial\varphi_i}dxdyd\tau,$$

$$v_{1i}=\int\limits_0^1\left(w_0\cdot B_i^3(y)\cdot f^p(y)+w_1\cdot\frac{\partial B_i^3(y)}{\partial y}\cdot\frac{\partial f^p(y)}{\partial y}+w_2\cdot\frac{\partial^2 B_i^3(y)}{\partial y^2}\cdot\frac{\partial^2 f^p(y)}{\partial y^2}\right)dy,$$

$$v_{2i}=\sum_{l=1}^{s}P_l\cdot\left(f_l-f^p(y_l)\right)\cdot B_i^3(y_l).$$

This system includes the regularization parameter β, which is determined, as in [9, 11, 12], based on the condition

$$\left(1-\sqrt{\frac{2}{N}}\right)\sigma\le\delta\le\left(1+\sqrt{\frac{2}{N}}\right)\sigma, \qquad (12)$$

which was proposed in [5]. Here, *N* is the total number of thermometric measurements; σ is the root mean square error of measurement; δ is the root-mean-square deviation at temperature measurement points, obtained from the temperature measured.

It is considered that the regularization parameter is chosen correctly if, for the solution obtained according to the iterative scheme proposed above, the two-sided inequality (12) is satisfied.

Numerical Experiment

Let us consider the process of cooling a body (Fig. 1) by convective heat flux. To carry out a numerical experiment, as *f*(*y*) at the boundary Γ_4 we take:

– the dependency

$$f(y)=1+0.4y-2.4y^2+2y^3, \qquad (13)$$

which is fairly accurately approximated by Schoenberg's cubic splines with a small number of required parameters;

– the nonlinear thermal conductivity coefficient

$$\lambda(T)=1+3.2T-7T^2+4T^3$$

– the nonlinear heat capacity

$$C(T)=1-1.2T+7.2T^2-5.7T^3.$$

Let us impose constraints on the right side of the equation of the boundary to be identified. Let the required function $f(y)$ satisfy the two-sided inequality

$$F_1 \le f(y) \le F_2,$$

where F_1 and F_2 are selected based on the technological process. Let us assume that the temperature measurement points are arranged evenly along the coordinates x and y so that they are strictly in the region under study when we search for the boundary. On the obtained numerical solution at the temperature measurement points, we impose a random error distributed according to the normal law at σ=0.02.

To solve the geometric IHCP under consideration, as the a-priori information, we use the values of $f(y)$ at the ends of the interval $[0,1]$: $f(0)=1$ and $f(1)=1$.

Fig. 2 shows the functions $f(y)$ of the boundary to be identified for the following dimensionless data: $n_\tau=400$, m=36, $\Delta\tau=0.0025$, $\alpha_1=5.0$, $\alpha_2=5.0$, $T_{s_1}=0$, $T_{s_2}=0$, $T_0=1.0$, $n=13$, $w_0=0$, $w_1=0$, $w_2=1$, $F_1=0.7$, $F_2=1.2$, $s=2$, $y_1=0$, $y_2=1$, $f_1=1$, $f_2=1$, $z_1=10^5$, $z_2=10^5$.

The dependencies $T(\tau)$ at the point E (Fig. 1) located on the outer boundary Γ_3 are shown in (Fig. 3).

The selection of the regularization parameter β began with β=80.0. The iterative selection process β after four iterations ended at $\beta=0.8\cdot10^{-4}$ when the mean square error $\delta\approx0.005186$ was reached. All the boundary value problems of determining the temperature field in the body under investigation were solved using the finite element method and an implicit difference scheme.

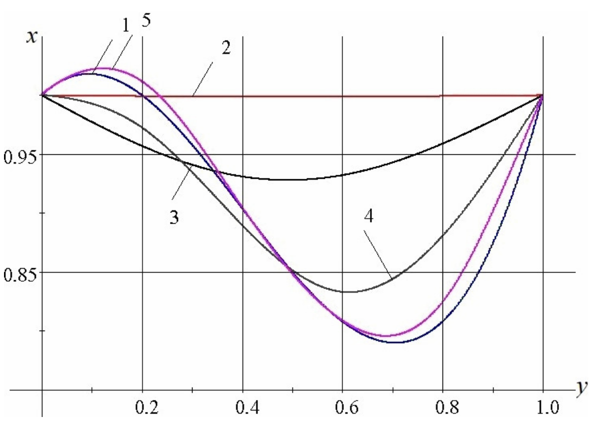

Fig. 2. Dependencies of the right side of the equation of the boundary to be identified x=f(y):
1 – in the form (13); 2 – boundary in the 1st iteration at β=80; 3 – boundary in the 2nd iteration at β=0.8; 4 – boundary in the 3rd iteration at $\beta=0.8\cdot10^{-2}$; 5 – boundary in the 4th iteration at $\beta=0.8\cdot10^{-4}$

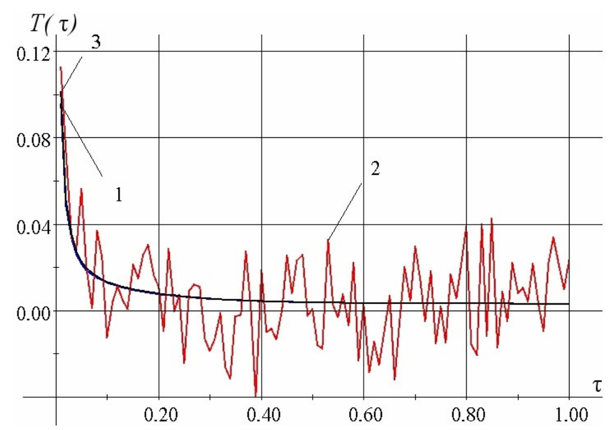

Fig. 3. Dependencies T(τ) at the point E (Fig. 1) with the coordinates x=0,7, y=1:
1 – obtained using the equation for the boundary Γ_4 in the form (13); 2 – the "noisy" solution of the direct problem; 3 – obtained by solving the geometric IHCP

Conclusions

The presented solution of the nonlinear two-dimensional geometric IHCP of identifying part of the outer boundary indicates that the proposed method can be successfully used in the presence of the a-priori information about the desired function even with sufficiently large errors in temperature measurements (Fig. 3).

Although this paper is of a theoretical nature, the approaches and methods presented in it are used (see [1]) and can be applied in the design of radio electronic equipment, when it is necessary to determine the areas of location of heat sources and sinks. They can also be useful in the study of such technological processes as the induction heating of parts or activation annealing of a semiconductor wafer. They can also be used to carry out nondestructive diagnostics (for example, to determine the size of cracks and other defects in the body under study).

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References

1. Matsevytyy, Yu. M. & Kostikov, A. O. (2014). *Geometricheskiye obratnyye zadachi teploobmena* [Geometric inverse problems of heat transfer]. Kiyev: Naukova dumka. 223 p. (in Russian).

2. Matsevityy, Yu. M. (2002). *Obratnyye zadachi teploprovodnosti. T. 1. Metodologiya.* [Inverse problems of thermal conductivity: In 2 vols. Vol. 1. Methodology. Kiyev: Naukova dumka, 408 p. (in Russian).
3. Alifanov, O. M., Artyukhin, Ye. A., & Rumyantsev, S. V. (1988). *Ekstremalnyye metody resheniya nekorrektnykh zadach* [Extreme methods for solving ill-posed problems]. Moscow: Nauka, 288 p. (in Russian).
4. Tikhonov, A. N. & Arsenin, V. Ya. (1979). *Metody resheniya nekorrektnykh zadach* [Methods for solving ill-posed problems]. Moscow: Nauka, 288 p. (in Russian).
5. Beck, J. V., Blackwell B., & St. Clair, C, R. (Jr.) (1985). Inverse heat conduction. Ill-posed problems. New York etc.: J. Wiley & Sons, 308 p. https://doi.org/10.1002/zamm.19870670331.
6. Kostikov, A. O. (2004) *Yedinyy metodologicheskiy podkhod k postanovke i resheniyu geometricheskikh obratnykh zadach teploprovodnosti* [A unified methodological approach to the formulation and solution of geometric inverse problems of heat conduction]. *Problemy Mashinostroyeniya – Journal of Mechanical Engineering*, vol. 7, no. 4, pp. 52-60 (in Russian).
7. Krukovskiy, P. G. (1998). *Obratnyye zadachi teplomassoperenosa (obshchiy inzhenernyy podkhod)* [Inverse problems of heat and mass transfer (general engineering approach)]. Kiyev: Institute of Technical Thermophysics, National Academy of Sciences of Ukraine, 224 p. (in Russian).
8. Lavrentyev, M. M. (1962). *O nekotorykh neofitsialnykh zadachakh matematicheskoy fiziki* [On some ill-posed problems of mathematical physics]. Novosibirsk: Publishing house of the Siberian Branch of the USSR Academy of Sciences, 68 p. (in Russian).
9. Matsevitiyy, Yu. M. & Ganchin, V. V. (2020). Multiparametric identification of several thermophysical characteristics by solving the internal inverse heat conduction problem. *Journal of Mechanical Engineering*, vol. 23, no. 2, pp. 14-20. https://doi.org/10.15407/pmach2020.02.014.
10. Tikhonov, A. N. & Samarskiy, A. A. (1966). *Uravneniya matematicheskoy fiziki* [Equations of mathematical physics]. Moscow: Nauka, 596 p. (in Russian).
11. Matsevityy, Yu. M., Slesarenko, A. P., & Ganchin V. V. (1999). *Regionalno-analiticheskoye modelirovaniye i identifikatsiya teplovykh potokov s ispolzovaniyem metoda regulyarizatsii A. N. Tikhonova* [Regional analytical modeling and identification of heat fluxes using the A. N. Tikhonov regularization method]. *Problemy mashinostroyeniya – Journal of Mechanical Engineering*, vol. 2, no. 1–2, pp. 34–42 (in Russian).
12. Matsevityy, Yu. M., Safonov, N. A., & Ganchin V. V. (2016). *K resheniyu nelineynykh obratnykh granichnykh zadach teploprovodnosti* [On the solution of nonlinear inverse boundary problems of heat conduction]. *Problemy mashinostroyeniya – Journal of Mechanical Engineering*, vol. 19, no. 1, pp. 28–36 (in Russian). https://doi.org/10.15407/pmach2016.01.028.

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До розв'язання геометричних обернених задач теплопровідності

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На основі теорії регуляризації А. М. Тихонова розроблена методика розв'язання обернених задач теплопровідності з ідентифікації гладкої зовнішньої межі двовимірної області за відомих на ній граничних умов. Для цього гладка межа апроксимується кубічними сплайнами Шьонберга, внаслідок чого її ідентифікація зводиться до визначення невідомих коефіцієнтів в цій апроксимації. За відомих граничних і початкових умов температура в тілі буде залежати тільки від цих коефіцієнтів. Виразивши її за формулою Тейлора для двох членів ряду і підставивши в функціонал Тихонова, задачу визначення збільшень коефіцієнтів можна звести до розв'язання системи лінійних рівнянь щодо цих збільшень. Вибравши певний параметр регуляризації і деяку функцію, яка описує форму зовнішньої межі, як початкове наближення, можна реалізувати ітераційний процес. У цьому процесі вектор невідомих коефіцієнтів для поточної ітерації буде дорівнювати сумі вектора коефіцієнтів з попередньої ітерації і вектора приростів цих коефіцієнтів, отриманих в результаті розв'язання системи лінійних рівнянь. Отримавши вектор коефіцієнтів в результаті збіжного ітераційного процесу, можна визначити середньоквадратичний відхил між одержуваною температурою і температурою, що вимірюється в результаті проведеного експерименту. Залишається підібрати параметр регуляризації таким чином, щоб цей відхил був у межах середньоквадратичної похибки помилки вимірювань. У самій методиці та шляхах її реалізації полягає новизна викладеного у статті матеріалу в порівнянні з підходами інших авторів до розв'язання обернених геометричних задач теплопровідності. Під час перевірки ефективності використання запропонованої методики розв'язано низку двовимірних тестових задач для тіл з відомим розта-

шуванням зовнішньої межі. Проведено аналіз впливу випадкових похибок вимірювань на похибку ідентифікації форми зовнішньої межі.

Ключові слова: *геометрична обернена задача теплопровідності, метод регуляризації А. М. Тихонова, стабілізуючий функціонал, параметр регуляризації, ідентифікація, апроксимація, кубічні сплайни Шьонберга.*

Література

1. Мацевитый Ю. М., Костиков А. О. Геометрические обратные задачи теплообмена. Киев: Наук. думка, 2014. 223 с.
2. Мацевитый Ю. М. Обратные задачи теплопроводности: в 2-х т. Т. 1: Методология. Киев: Наук. думка, 2002. 408 с.
3. Алифанов О. М., Артюхин Е. А., Румянцев С. В. Экстремальные методы решения некорректных задач. М.: Наука, 1988. 288 с.
4. Тихонов А. Н., Арсенин В. Я. Методы решения некорректных задач. М.: Наука, 1979. 288 с.
5. Бек Дж., Блакуэлл Б., Сент-Клэр Ч. (мл.) Некорректные обратные задачи теплопроводности. М.: Мир, 1989. 312 с.
6. Костиков А. О. Единый методологический подход к постановке и решению геометрических обратных задач теплопроводности. *Пробл. машиностроения*. 2004. Т. 7. № 4. С. 52–60.
7. Круковский П. Г. Обратные задачи тепломассопереноса (общий инженерный подход). Киев: Ин-т техн. теплофизики НАН Украины, 1998. 224 с.
8. Лаврентьев М. М. О некоторых некорректных задачах математической физики. Новосибирск: Изд-во Сиб. отд-ния АН СССР, 1962. 68 с.
9. Matsevytyi, Yu. M. & Hanchyn, V. V. Multiparametric identification of several thermophysical characteristics by solving the internal inverse heat conduction problem. *J. Mech. Eng*. 2020. Vol. 23. No. 2. P. 14–20. https://doi.org/10.15407/pmach2020.02.014.
10. Тихонов А. Н., Самарский А. А. Уравнения математической физики. М.: Наука, 1966. 596 с.
11. Мацевитый Ю. М., Слесаренко А. П., Ганчин В. В. Регионально-аналитическое моделирование и идентификация тепловых потоков с использованием метода регуляризации А. Н. Тихонова. *Пробл. машиностроения*. 1999. Т. 2. № 1–2. С. 34–42.
12. Мацевитый Ю. М., Сафонов Н. А., Ганчин В. В. К решению нелинейных обратных граничных задач теплопроводности. *Пробл. машиностроения*. 2016. Т. 19. № 1. С. 28–36. https://doi.org/10.15407/pmach2016.01.028.