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ITERATIVE METHOD OF DETERMINING STRESS INTENSITY COEFFICIENTS UNDER DYNAMIC LOADING OF THE CRACK SYSTEM

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An elastic isotropic body in a state of plane deformation, which contains a system of randomly placed cracks under the action of a dynamic (harmonic) loading, is considered. The authors set the problem of determining the stress field around the cracks under the conditions of their wave interaction. The solution method is based on the introduction of displacements in the body in the form of a superposition of discontinuous solutions of the equations of motion constructed for each crack. With this in mind, the initial problem is reduced to a system of singular integro-differential equations with respect to unknown displacement jumps on the crack surfaces. To solve this system, a new iterative method, which involves solving a set of independent integro-differential equations that differ only in their right-hand parts at each iteration, is proposed. For the zero approximation, solutions that correspond to individual cracks under the action of dynamic loading are chosen. Such a new approach allows to avoid the difficulties associated with the need to solve systems of integro-differential equations of large dimensions that arise when traditional methods are used. Based on the results of the iterations, formulas for calculating the stress intensity coefficients for each crack were obtained. In the partial case of four cracks, a good agreement between the results obtained during the direct solution of the system of eight integro-differential equations by the mechanical quadrature method and the results obtained by the iterative method was established. In general, numerical examples demonstrate the convergence and stability of the proposed method in the case of systems with a fairly large number of densely located cracks. The influence of the interaction between cracks on the stress intensity factor (SIF) value under dynamic loading conditions was studied. An important and new result for fracture mechanics is the detection of the absolute maximum of the normal stresses at certain frequencies of the oscillating normal loading. The number of interacting cracks and the configuration of the crack system itself affect the values of the frequencies at which SIF reaches a maximum and the maximum values. These maximum values significantly (by several times) exceed the SIF values of single cracks under a similar loading. At the same time, under conditions of static or low-frequency loading, it is possible to reduce the SIF values compared to the SIF for individual cracks. When cracks are sheared, the values of the tangential stresses have a tendency to decrease with increasing frequency, and their values do not significantly differ from the values of the tangential stress for an individual crack.

Keywords: *dynamic loading, cracks, stress intensity factors, iterative method.*

Introduction

In fracture mechanics, stress intensity coefficients are an important characteristic of the singularity of the stress field around cracks, which also determines the onset of fracture [1]. Their definition requires the solutions of the corresponding boundary value problems for the equations of the theory of elasticity, which are especially difficult in the presence of crack systems under conditions of dynamic loading of their surfaces. The relevance of solving dynamic problems of the theory of elasticity for bodies with cracks is explained by the effects caused by the reflection of waves generated by dynamic loading from the cracks surfaces. Reflected waves affect the distribution of stresses around the cracks, which, in turn, often leads to the fact that dynamic stress intensity factors (SIFs) can significantly exceed their static counterparts. Such an excess is observed even in the case of a single crack [2–5]. The situation is even more complicated if there is a system of interacting cracks in the body.

As of now, the availability of powerful computing equipment and special software products, such as AHSYS, ABAQUS, NASGROW, AFCROSS, contributes to the wide application of direct numerical methods for solving dynamic and static problems of crack mechanics [6–9]. The advantage of these methods is their universality, which allows consideration of bodies and cracks of arbitrary shape without simplifications. However, as can be seen from the cited works, the use of these methods requires a significant thickening of the grid or division into finite elements. This is especially true for the tops of cracks, where the stress singularity is observed. In the case of a system of rather densely placed cracks, such thickening should be carried out for each crack and cover a fairly large area. This, of course, creates significant difficulties when applying the methods of finite elements and finite differences, which are pointed out by the authors of papers [6–9]. Therefore, numerical results, as a rule, are given for one or two cracks.

An effective and widespread method of determining stress fields in bodies with cracks is the method of boundary integral equations and its discrete analogue, the method of boundary elements [3, 10–15]. However, the application of this method in the presence of a system of cracks in the body leads to the solution of systems of singular integral or integro-differential equations, the number of which is proportional to the number of cracks. In this regard, the first papers studied the interaction of dynamically loaded parallel cracks [16–20]. Difficulties associated with the numerical solution of systems of singular integral or integro-differential equations of large size can be avoided when considering periodic systems of cracks [21–23]. The determination of SIF in the case of two arbitrarily placed cracks under dynamic loading conditions was carried out in [24–26]. As for the solution of two-dimensional problems for bodies with arbitrary systems of dynamically loaded cracks, the papers [2, 11, 27–29] should be mentioned. In all these papers, the original problems are reduced to systems of integral or integro-differential equations. However, despite the fact that the solution was carried out using the general formulation, the numerical results are given in the case of several, usually only two, cracks. In view of this, it becomes relevant to develop a method for determining dynamic SIFs in the case of systems with a large number of cracks, which would not require the solution of systems of integral equations of large size. In [30], an iterative method was proposed to determine SIF under the action of shear SH-waves on an arbitrary system of cracks before solving the system of the corresponding integro-differential equations. When using this method, sets of equations for individual cracks are solved at each iteration. In this paper, the specified method is applied to the case of a system of cracks, the surfaces of which are under the influence of harmonic normal or shear loadings.

Problem statement

An isotropic elastic space, which is in a state of plane deformation and contains N through cracks, is considered. These cracks in the xOy plane (Fig. 1) are placed on segments $(-d_k, d_k)$, which do not intersect and have their centers at points $O_k(a_k, b_k)$, $k=1, 2, \dots, N$. Normal ones $N_k e^{-i\omega t}$ or shear $T_k e^{-i\omega t}$ self-balanced loadings are applied to the crack surfaces.

Then the multiplier $e^{-i\omega t}$, which determines the dependence on time, is discarded and only amplitude values are considered. Let's set that $u(x, y)$, $v(x, y)$ – displacements of the wave field created by crack loading. Then they satisfy the equations of motion, which under conditions of plane strain have the form

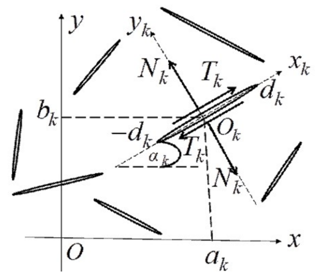


Fig. 1. The system of cracks under the action of dynamic loading

$$(\lambda + 2\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \mu\Delta u = -\rho\omega^2 u \;;$$

$$(\lambda + 2\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \mu\Delta v = -\rho\omega^2 v \,, \tag{1}$$

where Δ is the two-dimensional Laplace operator; ρ is the density of the elastic medium; λ, μ are Lamé coefficients.

To formulate the boundary conditions on the crack surfaces, a local coordinate system is associated with each crack $x_kO_ky_k$, $k = \overline{1, N}$ (Fig. 1) in a way that the angle between the axes Ox_k and Ox equals to α_k.

Let's denote $u^k(x_k, y_k)$, $v^k(x_k, y_k)$, $\sigma_y^k(x_k, y_k)$, $\sigma_x^k(x_k, y_k)$, $\tau_{yx}^k(x_k, y_k)$ displacements and stresses in the coordinate system associated with the k-th crack. Then, under the condition of the crack surfaces loading, the following equalities must be fulfilled

$$\sigma_y^k(x_k,0)=N_k;\ \tau_{yx}^k(x_k,0)=T_k;$$

$$x_k\in[-d_k,d_k];\ k=1, 2,\dots, N. \tag{2}$$

In addition, there are discontinuities on the surfaces of the displacement cracks, the unknown jumps of which are marked as

$$v^k(x_k,+0)-v^k(x_k,-0)=\chi_{3k}(x_k);\ u^k(x_k,+0)-u^k(x_k,-0)=\chi_{4k}(x_k);$$

$$x_k\in[-d_k,d_k];\ k=1, 2,\dots, N. \tag{3}$$

Under such conditions, it is necessary to determine displacements and stresses in a body with cracks and to obtain formulas for SIF calculating.

Reduction to a system of integro-differential equations and an iterative solution method

As in [27], the solution is based on the use of discontinuous solutions of equations (1) with jumps (3) constructed for each crack in the coordinate system associated with it $x_lO_ly_l$, l=1, 2,…, N

$$v^{dl}(x_l,y_l)=\int_{-d_l}^{d_l}\chi_{3l}(\eta)G_{33}(\eta-x_l,y_l)d\eta+\int_{-d_l}^{d_l}\chi_{4l}(\eta)G_{34}(\eta-x_l,y_l)d\eta;$$

$$u^{dl}(x_l,y_l)=\int_{-d_l}^{d_l}\chi_{3l}(\eta)G_{43}(\eta-x_l,y_l)d\eta+\int_{-d_l}^{d_l}\chi_{4l}(\eta)G_{44}(\eta-x_l,y_l)d\eta, \tag{4}$$

where $G_{33}=\dfrac{1}{\kappa_2^2}\dfrac{\partial}{\partial y_l}\left(\left(\kappa_2^2+2\dfrac{\partial^2}{\partial x_l^2}\right)r_1-2\dfrac{\partial^2 r_2}{\partial x_l^2}\right)$; $G_{34}=\dfrac{1}{\kappa_2^2}\dfrac{\partial}{\partial x_l}\left(2\left(\kappa_1^2+\dfrac{\partial^2}{\partial x_l^2}\right)r_1-\left(\kappa_2^2+2\dfrac{\partial^2}{\partial x_l^2}\right)r_2\right)$;

$G_{43}=\dfrac{1}{\kappa_2^2}\dfrac{\partial}{\partial x_l}\left(\left(\kappa_2^2+2\dfrac{\partial^2}{\partial x_l^2}\right)r_1-2\left(\kappa_2^2+2\dfrac{\partial^2}{\partial x_l^2}\right)r_2\right)$; $G_{44}=\dfrac{1}{\kappa_2^2}\dfrac{\partial}{\partial y_l}\left(-2\dfrac{\partial^2 r_1}{\partial x_l^2}+\left(\kappa_2^2+2\dfrac{\partial^2}{\partial x_l^2}\right)r_2\right)$;

$r_j(\eta-x_l,y_l)=-\dfrac{i}{4}H_0^{(1)}\left(\kappa_j\sqrt{(\eta-x_l)^2+y_l^2}\right)$, j=1, 2; $\kappa_1^2=\dfrac{\omega^2\rho^2}{\lambda+2\mu}$; $\kappa_2^2=\dfrac{\omega^2\rho^2}{\mu}$;

$H_0^{(1)}(z)$ is the Hankel function.

The following stresses correspond to displacements (4) in the coordinate system $x_lO_ly_l$

$$\sigma_x^{dl}(x_l,y_l)=\mu\int_{-d_l}^{d_l}\chi'_{3l}(\eta)E_{03}(\eta-x_l,y_l)d\eta+\mu\int_{-d_l}^{d_l}\chi'_{4l}(\eta)E_{04}(\eta-x_l,y_l)d\eta+\mu(\kappa_2^2-2\kappa_1^2)\int_{-d_l}^{d_l}\chi_{3l}(\eta)r_1(\eta-x_l,y_l)d\eta;$$

$$\sigma_y^{dl}(x_l,y_l)=\mu\int_{-d_l}^{d_l}\chi'_{3l}(\eta)E_{13}(\eta-x_l,y_l)d\eta+\mu\int_{-d_l}^{d_l}\chi'_{4l}(\eta)E_{14}(\eta-x_l,y_l)d\eta+\mu\kappa_2^2\int_{-d_l}^{d_l}\chi_{3l}(\eta)r_1(\eta-x_l,y_l)d\eta;$$

$$\tau_{yx}^{dl}(x_l,y_l)=\mu\int_{-d_l}^{d_l}\chi'_{3l}(\eta)E_{23}(\eta-x_l,y_l)d\eta+\mu\int_{-d_l}^{d_l}\chi'_{4l}(\eta)E_{24}(\eta-x_l,y_l)d\eta+\mu\kappa_2^2\int_{-d_l}^{d_l}\chi_{4l}(\eta)r_2(\eta-x_l,y_l)d\eta. \tag{5}$$

When obtaining formulas (5) in order to reduce the order of singularity of integrable functions, integration by parts was carried out while taking into account $\chi_l(\pm d_l)=0$ and the following notation is adopted

$$E_{03}=E_{24}=-\frac{4}{\kappa_2^2}\left[\left(\kappa_1^2+\frac{\partial^2}{\partial\eta^2}\right)\frac{\partial r_1}{\partial\eta}-\left(\kappa_2^2+\frac{\partial^2}{\partial\eta^2}\right)\frac{\partial r_2}{\partial\eta}\right];$$

$$E_{04}=-\frac{2}{\kappa_2^2}\left[\left(2\frac{\partial^2}{\partial\eta^2}+2\kappa_1^2-\kappa_2^2\right)\frac{\partial r_1}{\partial y_l}-\left(2\frac{\partial^2}{\partial\eta^2}+\kappa_2^2\right)\frac{\partial r_2}{\partial y_l}\right];$$

$$E_{13}=\frac{4}{\kappa_2^2}\left[\left(\kappa_2^2+\frac{\partial^2}{\partial\eta^2}\right)\frac{\partial r_1}{\partial\eta}-\left(\kappa_2^2+\frac{\partial^2}{\partial\eta^2}\right)\frac{\partial r_2}{\partial\eta}\right];$$

$$E_{14}=E_{23}=\frac{2}{\kappa_2^2}\left[\left(\kappa_2^2+2\frac{\partial^2}{\partial\eta^2}\right)\frac{\partial r_1}{\partial y_l}-\left(\kappa_2^2+2\frac{\partial^2}{\partial\eta^2}\right)\frac{\partial r_2}{\partial y_l}\right].$$

The following displacements and stresses correspond to discontinuous solutions (4), (5) in the system *Oxy*:

$$u^{gl}=u^{dl}\cos\alpha_l-v^{dl}\sin\alpha_l;\ v^{gl}=u^{dl}\sin\alpha_l-v^{dl}\cos\alpha_l;$$

$$\sigma_x^{gl}=\sigma_x^{dl}\cos^2\alpha_l+\sigma_y^{dl}\sin^2\alpha_l-\tau_{xy}^{dl}\sin 2\alpha_l;$$

$$\sigma_y^{gl}=\sigma_x^{dl}\sin^2\alpha_l+\sigma_y^{dl}\cos^2\alpha_l-\tau_{xy}^{dl}\sin 2\alpha_l;$$

$$2\tau_{xy}^{gl}=\sigma_x^{dl}\sin 2\alpha_l+\sigma_y^{dl}\sin 2\alpha_l+2\tau_{xy}^{dl}\cos 2\alpha_l.$$

The general displacements field is given in the form

$$u(x,y)=\sum_{l=1}^{N}u^{gl}(x,y);\ v(x,y)=\sum_{l=1}^{N}v^{gl}(x,y). \tag{6}$$

Stresses σ_x, σ_y, τ_{yx} are also determined by similar formulas. Thus, according to formulas (6), displacements and stresses in the body will be determined, provided that the unknown jumps (3) are determined. For this, conditions (2) should be used. The stresses included in them are given through stresses (4) corresponding to discontinuous solutions according to the following formulas:

$$\sigma_y^k(x_k,y_k)=\sum_{l=1}^{N}\sigma_y^{kl}(x_k,y_k);\ \tau_{yx}^k(x_k,y_k)=\sum_{l=1}^{N}\tau_{yx}^{kl}(x_k,y_k), \tag{7}$$

where

$$\sigma_y^{kl}=\sigma_x^{dl}\sin^2\alpha_{kl}+\sigma_y^{dl}\cos^2\alpha_{kl}-\tau_{yx}^{dl}\sin 2\alpha_{kl};$$

$$2\tau_{xy}^{kl}=-\sigma_x^{dl}\sin 2\alpha_{kl}+\sigma_y^{dl}\sin 2\alpha_{kl}+2\tau_{yx}^{dl}\cos 2\alpha_{kl};\ \alpha_{kl}=\alpha_k-\alpha_l.$$

Substitution of (7) in (2) leads to a system of $2N$ singular integro-differential equations with respect to unknown displacements jumps. As a result of the extraction of the singular component and a series of transformations, this system has the form

$$\frac{1}{2\pi}\int_{-1}^{1}\varphi'_{3k}(\tau)\left[-\frac{2(1-\xi^2)}{\tau-\varsigma}+R_{13}^k(\tau-\varsigma)\right]d\tau+\frac{1}{2\pi}\int_{-1}^{1}\varphi_{3k}(\tau)\left[\kappa_0^2\gamma_k^2\ln|\tau-\varsigma|+V_{13}^k(\tau-\varsigma)\right]d\tau+$$
$$+\sum_{\substack{l=1\\l\neq k}}^{N}\left[\frac{1}{2\pi}\int_{-1}^{1}\varphi'_{3l}(\tau)F_{kl}^{13}(\tau,\varsigma)d\tau+\frac{1}{2\pi}\int_{-1}^{1}\varphi'_{4l}(\tau)F_{kl}^{14}(\tau,\varsigma)d\tau\right]+$$
$$+\sum_{\substack{l=1\\l\neq k}}^{N}\left[\frac{1}{2\pi}\int_{-1}^{1}\varphi_{3l}(\tau)U_{kl}^{13}(\tau,\varsigma)d\tau+\frac{1}{2\pi}\int_{-1}^{1}\varphi_{4l}(\tau)U_{kl}^{14}(\tau,\varsigma)d\tau\right]=N_{0k};$$

$$\frac{1}{2\pi}\int_{-1}^{1}\varphi'_{4k}(\tau)\left[\frac{2(1-\xi^2)}{\tau-\varsigma}+R_{24}^k(\tau-\varsigma)\right]d\tau+\frac{1}{2\pi}\int_{-1}^{1}\varphi_{4k}(\tau)\left[\kappa_0^2\gamma_k^2\ln|\tau-\varsigma|+V_{24}^k(\tau-\varsigma)\right]d\tau+$$
$$+\sum_{\substack{l=1\\l\neq k}}^{N}\left[\frac{1}{2\pi}\int_{-1}^{1}\varphi'_{3l}(\tau)F_{kl}^{23}(\tau,\varsigma)d\tau+\frac{1}{2\pi}\int_{-1}^{1}\varphi'_{4l}(\tau)F_{kl}^{24}(\tau,\varsigma)d\tau\right]+$$
$$+\sum_{\substack{l=1\\l\neq k}}^{N}\left[\frac{1}{2\pi}\int_{-1}^{1}\varphi_{3l}(\tau)U_{kl}^{23}(\tau,\varsigma)d\tau+\frac{1}{2\pi}\int_{-1}^{1}\varphi_{4l}(\tau)U_{kl}^{24}(\tau,\varsigma)d\tau\right]=T_{0k}. \tag{8}$$

In system (8), the following notations are adopted: $\varphi_{sk}(\tau) = d_k^{-1}\chi_{sk}(d_k\tau)$; s=3, 4; k=1, 2, …, N; $\gamma_k = d^{-1}d_k$; d=max(d_1, d_2, …, d_N); $N_{0k} = \mu^{-1}N_k$; $T_{0k} = \mu^{-1}T_k$; k=1, 2, …, N; $\xi^2 = \frac{\kappa_2^2}{\kappa_1^2} = \frac{1-2\nu}{2(1-\nu)}$; ν is the Poisson's ratio of an elastic material. Functions R_{js}^k, V_{js}^k, F_{kl}^{js}, U_{kl}^{js}; j=1, 2; s=3, 4; l, k=1, 2, …, N are continuous at $-1 \le \varsigma, \tau \le 1$.

Equalities, which are a condition for closing cracks and ensure the unity of the solution of the system (8), should be added to system (8)

$$\int_{-1}^{1} \varphi'_{sk}(\tau)d\tau = 0\text{; } s=3,\,4;\ k=1,\,2,\,\ldots,\,N. \tag{9}$$

The system of integro-differential equations (8) requires a numerical solution. In order to prevent the solution of a large-dimensional system, it is suggested to apply an iterative method similar to that done in [30]. It consists in the fact that at the i-th iteration step, $i \ge 1$, N pairs of independent integro-differential equations are solved

$$\frac{1}{2\pi}\int_{-1}^{1}\left(\varphi_{3k}^i(\tau)\right)'\left[-\frac{2(1-\xi^2)}{\tau-\varsigma} + R_{13}^k(\tau-\varsigma)\right]d\tau + \frac{1}{2\pi}\int_{-1}^{1}\varphi_{3k}^i(\tau)\left[\kappa_0^2\gamma_k^2\ln|\tau-\varsigma| + V_{13}^k(\tau-\varsigma)\right]d\tau =$$

$$= N_{0k} - \sum_{\substack{l=1\\ l\ne k}}^{N}\sum_{s=3}^{4}\left[\frac{1}{2\pi}\int_{-1}^{1}\left(\varphi_{sl}^{i-1}(\tau)\right)' F_{kl}^{1s}(\tau,\varsigma)d\tau + \frac{1}{2\pi}\int_{-1}^{1}\varphi_{sl}^{i-1}(\tau)U_{kl}^{1s}(\tau,\varsigma)d\tau\right];$$

$$\frac{1}{2\pi}\int_{-1}^{1}\left(\varphi_{4k}^i(\tau)\right)'\left[\frac{2(1-\xi^2)}{\tau-\varsigma} + R_{24}^k(\tau-\varsigma)\right]d\tau + \frac{1}{2\pi}\int_{-1}^{1}\varphi_{4k}^i(\tau)\left[\kappa_0^2\gamma_k^2\ln|\tau-\varsigma| + V_{24}^k(\tau-\varsigma)\right]d\tau =$$

$$= T_{0k} - \sum_{\substack{l=1\\ l\ne k}}^{N}\sum_{s=3}^{4}\left[\frac{1}{2\pi}\int_{-1}^{1}\left(\varphi_{sl}^{i-1}(\tau)\right)' F_{kl}^{2s}(\tau,\varsigma)d\tau + \frac{1}{2\pi}\int_{-1}^{1}\varphi_{sk}^{i-1}(\tau)U_{kl}^{2s}(\tau,\varsigma)d\tau\right];$$

$$\int_{-1}^{1}\left(\varphi_{sk}^i(\tau)\right)' d\tau = 0\text{; } s=3,\,4;\ k=1,\,2,\,\ldots,\,N;\ i=1,\,2,\,3. \tag{10}$$

Solutions of equations (10) with right-hand parts where there are no sums, i.e. equal to N_{0k}, T_{0k}, are chosen for zero approximation $\varphi_{3k}^0(\tau)$, $\varphi_{4k}^0(\tau)$. Therefore, solutions corresponding to individual cracks under the influence of dynamic loading are chosen as the zero approximation.

The numerical solution of equations (10) is carried out in the same way as in [27, 30]. Derivatives of unknown functions taking into account the root singularity are given in the form

$$\left(\varphi_{sk}^i(\tau)\right)' = \frac{\psi_{sk}^i(\tau)}{\sqrt{1-\tau^2}}.$$

Further unknown functions $\psi_{sk}^i(\tau)$ are approximated by interpolation polynomials

$$\psi_{sk}^i(\tau) = \sum_{m=1}^{n}\psi_{skm}^i\frac{T_n(\tau)}{(\tau-\tau_m)T_n'(\tau)}\text{; } \psi_{skm}^i = \psi_{sk}^i(\tau_m),$$

where $T_n(\tau)$ – is the Chebyshev polynomial of the second kind; τ_m, m=1, 2, …, n are its roots.

To define ψ_{skm}^i, as in [27, 30], a system of linear algebraic equations was obtained from (10) by the method of mechanical quadrature. After its solution, the approximate values of SIF based on the results of the i-th iteration are found by the formulas

$$K_{pl}^{\pm i} = \mu\sqrt{d_l}\,k_{pl}^{\pm i};\ l=1,2,\ldots,N;\ p=1,2;$$

$$k_{pl}^{+i} = -(1-\xi^2)n^{-1}\sum_{m=1}^{n}(-1)^m \psi_{2+plm}^{i}\,\mathrm{ctg}(0{,}5\beta_m);$$

$$k_{pl}^{-i} = -(1-\xi^2)n^{-1}(-1)^{p+n}\sum_{m=1}^{n}(-1)^m \psi_{2+plm}^{i}\,\mathrm{tg}(0{,}5\beta_m);$$

$$\beta_m = \frac{\pi(2m-1)}{2n};\ m=1,2,\ldots,n. \qquad (11)$$

In formulas (11), the superscript plus corresponds to SIF in the vicinity of the top of the l-th crack ($+d_l$, 0), the minus index corresponds to SIF in the vicinity of the top ($-d_l$, 0).

Analysis of the results of numerical studies

The numerical implementation of the proposed iterative method was carried out in order to study its practical convergence, the stability of the iterative process, the ability to study the influence of the wave interaction of cracks on the SIF value. In order to confirm the reliability of the results obtained by the iterative method, as in [17], a system of four cracks of the same length (Fig. 2), which configuration is determined by parameters θ, h, β, was considered. The results obtained by directly solving system (8) were compared, as in [12, 16–18, 25, 26], by the mechanical quadrature method and the iterative method.

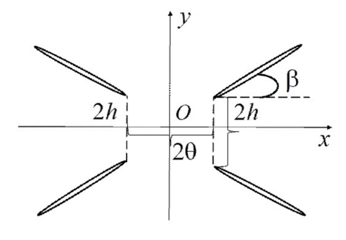


Fig. 2. A system of four cracks of the same length

All cracks are under the action of the same normal $N_{0k}=1$ or shear $T_{0k}=1$ loadings, $k=1, 2, 3, 4$. Under such conditions, SIFs near the tops of all cracks coincide. Calculations were carried out at $\theta=h=0.75d$, $\beta=30°$, and the results are shown in Fig. 3. They show change of $|k_1^+|$ – the absolute value of the SIF of normal stresses depending on the dimensionless wave number $\kappa_0=\kappa_2 d$.

When applying the iterative method, the number of iterations varied from 2 to 16. The dashed curve corresponds to the SIF values for a separate crack. The curve numbered 0 shows the SIF values found as a result of the direct solution of the system of integro-differential equations (8). Other curves are plotted at the specified number of iterations. At $\kappa_0\to 0$, SIF values go to their static counterpart. It can be seen that the results obtained by different methods practically coincide after the fourth iteration, except for the first most significant maximum, where the coincidence is observed after 16 iterations. This confirms the reliability of the results obtained by the iterative method and the convergence of the iterative process. Extreme values of SIF significantly exceed both their static counterpart and SIF values for a separate crack. The convergence of the iterative method deteriorates when $\beta\to 0$, that is, when closely spaced parallel cracks enter the system. In general, $\beta=0$ convergence is observed at $h>0.5d$. Results of comparison of various methods and studies of convergence at $\beta=0$, $\theta=h=0.75d$ are shown in Fig. 4.

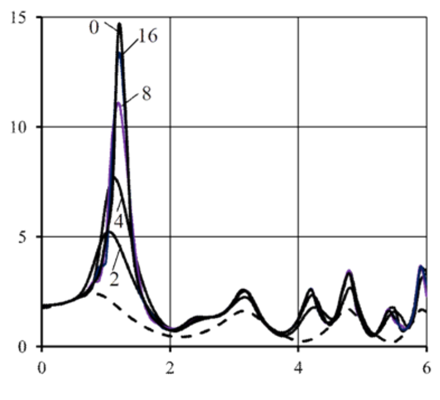


Fig. 3. Convergence of the iterative method

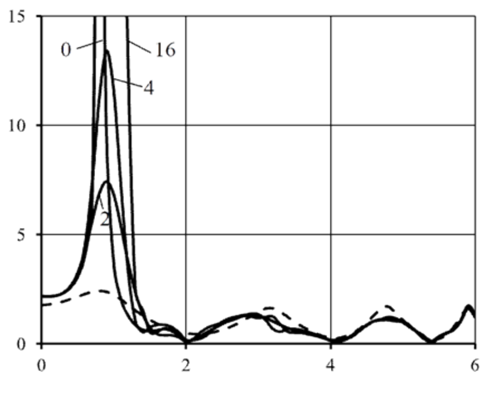
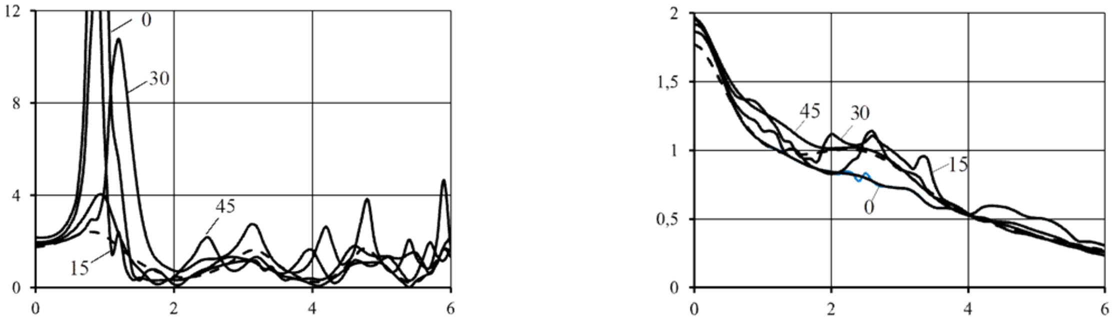


Fig. 4. Convergence for a system of four parallel cracks

They prove that calculations by different methods, except frequency $\kappa_0 \approx 0.9$, at which resonance occurs, are in good agreement. The influence of the angle β on the frequency dependence of SIF is shown in Fig. 5. Curves in Fig. 5, a show the frequency dependence of SIF of normal stresses $|k_1|$ with normal crack loading, and in Fig. 5, b – frequency dependence of SIF of tangential stresses $|k_2|$ under shear loading at the specified angle values β.



a b

Fig. 5. Effect on the SIF value of the angle between the cracks:
a – under normal loading; b – under shear loading

In the case of a normal loading, at all values of the angle β, the presence of a global maximum of SIF is observed, the frequency of reaching which depends on the angle β. The most extreme value of SIF is obtained at β=0, but with high-frequency oscillations $\kappa_0>1.8$, the largest values are reached by SIF at β=30°. When cracks are sheared, the values of the tangential stresses have a tendency to decrease with increasing frequency, and their values do not significantly differ from the values of the tangential stresses for a single crack. For a more detailed study of the influence of the interaction of the dynamic loading of cracks on the SIF value, the following system of 9 cracks was considered (Fig. 6).

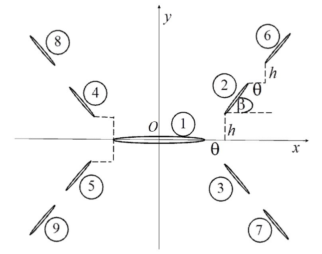


Fig. 6. A system of nine cracks

The system configuration is determined by parameters θ, h, β, to which the following data was provided during calculations $\theta = h = d_1/3$; $d_k = d_1/3$, $k=2, \ldots, 9$, and the cracks were under the influence of normal loading. The results are shown in Fig. 7 in the form of graphs of the dependence of SIF of the normal stresses of the first crack $k_1^+|$ for the specified angle β values.

As can be seen, the configuration of the crack system significantly affects the SIF behavior in the frequency domain. In particular, there is a global maximum of the SIF value, the largest of which is reached at β=0, when all cracks are parallel. Also, in case of static loading and at low frequencies ($\kappa_0<1.2$) the presence of other cracks near the top of the crack will lead to a decrease in the SIF values.

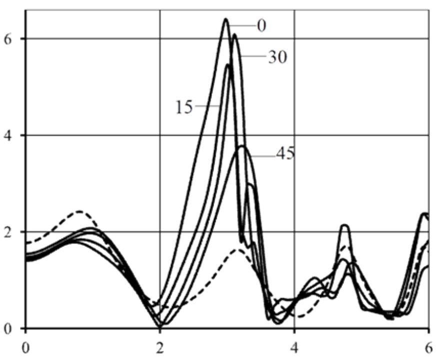


Fig. 7. The influence of the configuration of the crack system on the SIF value

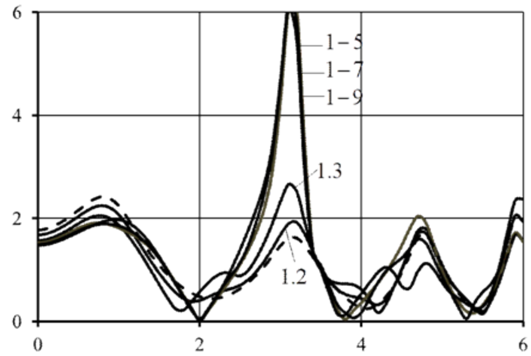


Fig. 8. The influence of the number of cracks on the SIF value

Fig. 8 shows the results of the influence of the number of cracks interacting with one crack (in this case, with the first one).

The cases of two cracks (first and second ones), three (from the first to the third one), five (from the first to the fifth one), seven and nine were considered (Fig. 6). If we compare the SIF values at the point of the global maximum, then when the number of cracks increases to five, their growth is observed first, and then the maximum values of SIF practically do not change. In all cases, in the region of low frequencies (κ_0<1,2), the presence of other cracks next to each other leads to a decrease in the SIF values.

General conclusions

An iterative method for SIF determining in the conditions of interaction of a system of dynamically loaded cracks, which avoids the need for numerical solution of systems of large integro-differential equations, is proposed. At each step of the iteration, a set of independent equations corresponding to the cases of individual cracks is solved. The numerical study showed that the results obtained by the iterative method are in good agreement with the results of the direct solution of the system of integro-differential equations by other methods. The stability and convergence of the method for systems of rather densely placed cracks of a complex configuration are demonstrated. The existence of frequencies at which the absolute maxima of SIF values are observed is important for assessing the performance and predicting the destruction of machine parts under the action of dynamic, particularly vibrational, loadings. These maximum values significantly (by several times) exceed the SIF value of individual cracks under a similar loading, i.e. if there are cracks in the part of the system at these frequencies, the critical value of SIF may be exceeded and destruction of the part may occur. At the same time, under conditions of static or low-frequency loading, it is possible to reduce the SIF values compared to SIF for individual cracks. The proposed method can be extended to systems of other defects, in particular, thin hard inclusions.

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Ітераційний метод визначення коефіцієнтів інтенсивності напружень при динамічному навантаженні системи тріщин

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Розглянуто пружне ізотропне тіло у стані плоскої деформації, яке містить систему довільно розміщених тріщин під дією динамічного (гармонічного) навантаження. Автори поставили задачу – визначити поле напружень в околі тріщин в умовах їх хвильової взаємодії. Метод розв'язання ґрунтується на поданні переміщень у тілі у вигляді суперпозиції розривних розв'язків рівнянь руху, побудованих для кожної тріщини. З огляду на це вихідна задача приводиться до системи сингулярних інтегро-диференціальних рівнянь відносно невідомих стрибків переміщень на поверхнях тріщин. Для розв'язання цієї системи запропоновано новий ітераційний метод, який передбачає розв'язання на кожній ітерації сукупності незалежних інтегро-диференціальних рівнянь, що відрізняються тільки правими частинами. За нульове наближення обираються розв'язки, які відповідають окремим поодиноким тріщинам під дією динамічного навантаження. Такий новий підхід дозволяє уникнути труднощів, пов'язаних з необхідністю розв'язання систем інтегро-диференціальних рівнянь великої розмірності, що виникають при застосуванні традиційних методів. За результатами ітерацій отримані формули для розрахунку коефіцієнтів інтенсивності напружень для кожної тріщини. У частинному випадку чотирьох тріщин встановлено добре узгодження результатів, отриманих при безпосередньому розв'язанні системи восьми інтегро-диференціальних рівнянь методом механічних квадратур, і результатів, отриманих ітераційним методом. У цілому числові приклади демонструють збіжність і стійкість запропонованого методу у випадку систем досить великої кількості щільно розташованих тріщин. Досліджено вплив взаємодії між тріщинами на значення коефіцієнта інтенсивності напружень (КІН) в умовах динамічного навантаження. Важливим для механіки руйнування і новим результатом є виявлення абсолютного максимуму КІН нормальних напружень при деяких частотах осцилюючого нормального навантаження. На значення частот, за яких КІН сягають максимуму, і на максимальні значення впливають кількість взаємодіючих тріщин і конфігурація самої системи тріщин. Ці максимальні значення суттєво (у кілька разів) перевищують значення КІН поодиноких тріщин при аналогічному навантаженні. У той саме час в умовах статичного або низькочастотного навантаження можливе зменшення значень КІН порівняно з КІН для окремих тріщин. При зсувному навантаженні тріщин значення КІН дотичних напружень мають тенденцію до спадання при зростанні частоти, а їх значення несуттєво відрізняються від КІН для окремої тріщини.

***Ключові слова**: динамічне навантаження, тріщини, коефіцієнти інтенсивності напружень, метод ітерацій.*

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