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CONNECTED STABILITY ANALYSIS OF DELAY SYSTEMS VIA THE MATRIX-VALUED LYAPUNOV FUNCTION

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Abstract. By the method of combining the matrix-valued Lyapunov functional and comparison theorem, connected Lyapunov stability and practical stability of large scale delay system are studied deeply. A series of new sufficient conditions are proposed. These results are not only of theoretical but also of practical value.

Key words: delay system, connected Lyapunov stability, practical stability.

1. Designations.

Let $C = C([-\tau, 0], R^n)$, $J = [t_0, \infty)$, $t_0 \ge 0$. For any $\varphi \in C$ the norm $\|\varphi\| = \sup\limits_{-\tau \le s \le 0} |\varphi(s)|$ is used. For $x \in R^n$, $|x| = \max |x_r|$, $r = 1, 2, ..., n$. If $x \in C([t_0 - \tau, \infty), R^n)$, then $x_t \in C$ is determined as $x_t(s) = x(t+s)$, $-\tau \le s \le 0$. We designate $C_n^H = \{\varphi \in C : \|\varphi\| < H\}$, where $H > 0$ or $H = \infty$.

2. Description of the system and decomposition.

Consider the large scale system modeled by functional differential equation

$$\frac{dx}{dt} = f(t, x_t), x_{t_0} = \varphi_0 \in C_n^H, \tag{1}$$

where $f : J \times C_n^H \to R^n$. Provided that the vector-function f maps the bounded sets into the bounded sets, for each $t_0 \in J$ and $\varphi_0 \in C_n^H$ there exists a unique solution $x(t_0, x_{t_0})(t)$ determined on some interval $[t_0, t+\alpha], \alpha > 0$, and if $H_1 < H$ is such that $|x(t_0, \varphi_0(t)| \le H_1$, then $\alpha = \infty$.

The system (1) is decomposed into m interconnected subsystems

$$\frac{dx^i(t)}{dt} = f_i(t, x_t^i) + g_i(t, x_t^1, \cdots, x_t^m), \tag{2}$$

where $i \in I_m \underline{\Delta} \{1, 2, ..., m\}$, $f_i \in (J \times C_{n_i}^{H_i}, R^{n_i})$, $g_i \in C(J \times C_n^H, R^{n_i})$ and $\sum\limits_i n_i = n$. We assume that the functions $g_i(t, x_t)$ depend on $m \times m$-matrix of interconnections $E_t = [e_t^{ij}]$,

$$g_i(t, x_t) = g_i(t, e_t^{i1} x_t^1, e_t^{i2} x_t^2, \cdots, e_t^{im} x_t^m), \tag{3}$$

when $i \in I_m$, where the elements $e_t^{ij} \in C([-\tau,0],[0,1])$ depend in general case on the delay $e_t^{ij} x_t = e^{ij}(t+\theta)x^i(t+\theta), \theta \in [-\tau, 0]$.

We designate by $\overline{E_t}$ the fundamental matrix of interactions with the elements $\overline{e_t}^{ij} = 1$ if x_j is contained in $g_i(t, x_t)$; $\overline{e_t}^{ij} = 0$ if x_j is not contained in $g_i(t, x_t)$.

For $E_t = 0$ we get from system (2) the independent subsystems of functional differential equations of smaller dimensions

$$\frac{dx^i(t)}{dt} = f_i(t, x_t^i), \quad x_{t_0}^i = \varphi_0^i \in C, \tag{4}$$

where $x_i \in R^{n_i}$ and $f_i \in J \times C_{n_i}^{H_i} \to R^{n_i}$. Moreover, we assume $f_i(t,0) = 0$ and for subsystems (2) $f_i(t,0) + g_i(t,0) = 0$ for all $t \in J$ and $i \in I_m$, i.e. the state $x = x^1 = ... = x^m = 0$ is the unique equilibrium state of system (1) and subsystems (2).

For subsystem (2), whose functions $g_i(t, x_t), i = 1, 2, ..., m$, depend on the matrix of interactions E_t, the problem on practical stability of motion is reduced to the establishment of conditions under which the solution $x(t_0, x_{t_0})(t)$ of system (2) possesses certain qualitative properties for given estimates of initial and subsequent deviations on the infinite interval.

3. Matrix-valued functional.

For system (2) we construct the matrix-valued functional

$$U(t,\varphi) = [v_{ij}(t,\varphi)], \; i, j \in I_m \tag{5}$$

with the elements satisfying the following conditions.

H_1. The elements $v_{ii}(t, \varphi^i) \in C(J \times C_{n_i}^{H_i}, R_+^{m_i})$, $1 \le m_i \le n_i$, $v_{ii}(t,0) = 0$, are locally Lipschitz in φ^i;

H_2.The elements $v_{ij}(t, \varphi^i, \varphi^j) \in C(J \times C_{n_i}^{H_I} \times C_{n_i}^{H_I}, R^{m_i \times m_j})$, are locally Lipschitz in φ^i and φ^j for all $(i \ne j) \in I_m$.

By means of the real vector $\eta \in R_+^m, \eta > 0$, we construct the functional

$$V(t, \varphi, \eta) = \eta^T U(t, \varphi)\eta, \tag{6}$$

which is continuous and definite on the set $J \times C_n^H$ by conditions $H_1 - H_2$. The upper derivative of functional (6) along solutions of system (2) is determined by the formula

$$D^+V(t, \varphi, \eta) = \eta^T D^+U(t, \varphi)\eta, \tag{7}$$

where $D^+U(t,\phi) = \lim\limits_{\delta \to 0^+} \sup \frac{1}{\delta}\{U(t+\delta, x_{t+\delta}(t, \varphi)) - U(t, \varphi)\}$. Note that $D^+U(t,\varphi)$ is computed element-wise.

4. Definitions of connected stability of system (2).

Taking into account the results of paper [3] we shall cite the definitions of stability notion incorporated in this paper.

Definition 1. The equilibrium state $x = 0$ of system (1) is called

a) connectedly stable if for every $\varepsilon > 0$ and $t_0 \ge 0$ there exists $\delta = \delta(\varepsilon, t_0)$, such that $\|x(t_0, \varphi)(t)\| < \varepsilon$ whenever $[\varphi \in C_n^\delta, t \ge t_0]$ for all $E_t \subset \overline{E_t}$;

b) uniformly connectedly stable if in definition (a) the value δ does not depend on t_0;

c) asymptotically connectedly stable if it is connectedly stable and for any $t_0 \geq 0$ there exists $\Delta > 0$ such that $\|x(t_0, \varphi(t)\| \to 0,$ as $t \to \infty,$ whenever $\varphi \in C_n^{\Delta}$, for all $E_t \subset \overline{E_t}$;

d) uniformly asymptotically connectedly stable if it is uniformly connectedly stable and there exists some $\eta > 0$ and for every $\gamma > 0$ there exists $\tau > 0$ such that $\|x(t_0, \varphi(t)\| < \gamma,$ whenever $[\varphi_0 \in C_n^{\delta}, t \geq t_0]$ for all $E_t \subset \overline{E_t}$.

5. Conditions of connected stability of system (2).

Using matrix-valued functional (5) and its derivative (7) and applying the theorems of comparison principle for functional-differential equations (see [1]) we shall set out a series of sufficient conditions for connected stability of the equilibrium state $x = 0$ of system (1).

Theorem 1. Let system of functional-differential equations (1) be such that

1) there exists the matrix-valued functional $U(t,\varphi) \in C(J \times C_n^H, R^{m\times m})$, $U(t,0) = 0$ for all $t \in J$ and $U(t, \varphi)$ is locally Lipschitz in φ for every $t \in J$;

2) there exist $m \times m$ constant matrices $A_1(\eta)$ and $B_1(\eta)$, real vector $\eta \in R_+^m$, $\eta > 0$ and comparison functions $u_{1i}(|\varphi^i(0)|),\ u_{2i}(\|\varphi^i\|),\ i \in I_m$, of Hahn class K so that $u_1^T(|\varphi(0)|)\ A_1(\eta)\ u_1(|\varphi(0)|) \leq \sum_{i,j=1}^{m} \eta_i\eta_j u_{ij}(t, \varphi) \leq u_2^T(\|\varphi\|)B_1(\eta)u_2(\|\varphi\|)$ for all $t \in J$ and $\varphi \in C_n^H$;

3) there exists the comparison function $W \in C(J \times R_+, R)$ such that

$$D^+V(t, \varphi, \eta) \leq W(t, V(t, \varphi, \eta)) \tag{8}$$

for all $(t, \varphi) \in J \times C_n^H$ and all matrices of interaction $E_t \subset \overline{E_t}$. Then the certain type of stability of zero solution to the comparison equation

$$\frac{du}{dt} = W(t, u),\ u(t_0) = u_0 \geq 0 \tag{9}$$

and the restrictions on the matrices $A_1(\eta), B_1(\eta)$ imply the corresponding type of connected stability of the equilibrium state of system (1) with decomposition (2).

Proof. Provided that the matrices $A_1(\eta)$ and $B_1(\eta)$ are positive definite, functional (6) is positive definite and decreasing. Further, we apply Theorem 4.4.3 from [1] and determine certain type of connected stability of system (1).

Corollary 1. Let

1) conditions (1) and (2) of Theorem 1 be satisfied;

2) the matrix $A_1(\eta)$ be positive definite, the matrix $B_1(\eta) \equiv 0$ and the comparison function $W(t, V(t, \varphi, \eta)) \equiv 0$.

Then the equilibrium state $x = 0$ of system (1) with decomposition (2) is connectedly stable.

Corollary 2. Let

1) conditions (1) and (2) of Theorem 1 be satisfied;

2) the matrices $A_1(\eta)$ and $B_1(\eta)$ be positive definite and the comparison function $W(t, V(t, \varphi, \eta)) \equiv 0$.

Then the equilibrium state $x = 0$ of system (1) with decomposition (2) is uniformly connectedly stable.

Corollary 3. Let

1) conditions (1) and (2) of Theorem 1 be satisfied;

2) the matrices $A_1(\eta)$ and $B_1(\eta)$ be positive definite;

3) the zero solution of comparison equation (9) be uniformly asymptotically stable.

Then the equilibrium state $x=0$ of system (1) with decomposition (2) is uniformly asymptotically connectedly stable.

Theorem 2. Let system of functional differential equations (1) be such that
1) conditions (1) and (2) of Theorem 1 are satisfied;
2) there exist a constant $m\times m$ matrix $C_1(\eta),\ \eta\in R_+^m,\ \eta>0$ and functions $u_{3i}(\|x_t^i\|)$, u_{3i} is of class K for all $i\in I_m$, such that $D^+V(t,\ \varphi,\ \eta)\le u_3^T(\|x_t\|)C_1(\eta)u_3(\|x_t\|)$ for any $(t,\ \varphi)\in J\times C_n^H$ and any matrices of interactions $E_t\subset\overline{E_t}$, where $u_3^T(\|x_t\|)= =(u_{31}(\|x_t^l\|),...,\ u_{3m}(\|x_t^m\|))$;
3) the matrices $A_1(\eta)$ and $B_1(\eta)$ are positive definite and the matrix $C_1(\eta)$ is negative definite.
Then the equilibrium state $x=0$ of system (1) with decomposition (2) is uniformly asymptotically connectedly stable.
Theorem 3. Let in system of equations (1) the vector function $f(t,\phi)$ be bounded in ϕ and
1) conditions (1) and (2) of Theorem 1 are satisfied;
2) there exist a constant $m\times m$ matrix $C_2(\eta),\ \eta\in R_+^m,\ \eta>0$ and functions $u_{4i}(|x_t^i|)$ of class K for all $i\in I_m$ such that $D^+V(t,\ \varphi,\ \eta)\le u_4^T(|x_t|)C_2(\eta)u_4(|x_t|)$ for all $(t,\varphi)\in J\times C_n^H$ and any matrices of interconnections $E_t\subset\overline{E_t}$;
3) the matrices $A_1(\eta)$ and $B_1(\eta)$ are positive definite and the matrix $C_2(\eta)$ is negative definite.
Then the equilibrium state $x=0$ of system (1) with decomposition (2) is uniformly asymptotically connectedly stable.

6. Matrix-valued function on space product.
For system (4) we construct the matrix-valued function

$$U(t,x,x_t)=[v_{ij}(t,x,x_t)],\ i,j=1,2,...,\ m\ , \tag{10}$$

with the elements satisfying the following conditions.
H_3. The elements $v_{ii}\in C(J\times C_{n_i}^{H_i}\times C,R_+),\ v_{ii}(t,\ 0,\ 0)=0$ are locally Lipschitz in x_i;
H_4. The elements $v_{ij}\in C(J\times C_{n_i}^{H_i}\times C_{n_j}^{H_j}\times C\times C,R),\quad v_{ij}(t,\ 0,\ 0,\ 0)=0$ are locally Lipschitz in $x_i,\ x_j$ for all $(i\ne j)\in I_m$.

By means of the real vector $\eta\in R_+^m,\eta>0$, we construct the function

$$V(t,\ x,\ x_t,\ \eta)=\eta^T U(t,\ x,\ x_t)\eta, \tag{11}$$

which is definite on the space product $R^n\times C$ and locally Lipschitz in x, providing conditions of assumptions H_3 and H_4 are satisfied. Further we define

$$D^+V(t,\ x,\ x_t,\ \eta)=\eta^T D^+U(t,\ x,\ x_t)\eta, \tag{12}$$

where

$$D^+U(t,\ x,\ x_t)=\lim\left\{\sup\left[U(t+\theta,\ x+\theta f(t,\ x_t),\ x_{t+h}(\cdot))-U(t,\ x,\ x_t)\right]\theta^{-1}:\ \theta\to 0^+\right\}. \tag{13}$$

Note that when formula (12) is properly applied, $D^+U(t,\ x,\ x_t)$ is computed element-wise.

7. Conditions of connected practical stability of system (2).
In view of the results from [1, 4] we shall formulate the following definitions.
Definition 2. System (2) is called

a) connectedly practically stable, if given estimates of (λ, A), $0 < \lambda < A$, the condition $\varphi_0 \in C_n^{\lambda}$ implies $|x(t_0, \varphi_0)(t)| < A$ for all $t \geq t_0$ and all $E_t \subset \overline{E_t}$;

b) connectedly asymptotically practically stable, if conditions of definition (a) are satisfied and $\lim_{t\to\infty} |x(t_0, \varphi_0)(t)| = 0$.

The other definitions of connected practical stability can be formulated in terms of Definition 2.

Theorem 4. Let system of functional differentional equations (1) be such that

1) there exists a matrix-valued function $U \in C(J \times C_n^H \times C, R^{m\times m})$, $U(t,0,0) = 0$ for all $t \in J$ and $U(t, x, x_t)$ is locally Lipschitz in x for $(t, x, x_t) \in J_+ \times S(A) \times C(A)$;

2) there exist a real vector $\eta \in R^+$, $\eta > 0$, constant $m \times m$ matrices $A(\eta)$ and $B(\eta)$ and a comparison function $u_{1i}(|x|), u_{2i}(|x_t(\cdot)|), i = 1, 2, ..., m$, $u_{1i}, u_{2i} \in K$, such that $u_1^T(|x|)A(\eta)u_1(|x|) \leq$ $\leq \sum_{i,j=1}^{m} \eta_i \eta_j v_{ij}(t,x,x_t) \leq u_2^T(|x_t(\cdot)|)B(\eta)u_2(|x_t(\cdot)|)$ for all $(t,x,x_t) \in J \times S(A) \times C(A)$;

3) there exists a comparison function $W \in C(J \times R_+, R)$ such that $D^+V(t,x,x_t,\eta) \leq$ $\leq W(t, V(t, x, x_t, \eta))$ for all $(t, x, x_t) \in J \times S(A) \times C(A)$ and all matrices of interactions $E_t \subset \overline{E_t}$;

4) the matrices A and B are positive definite and $\lambda_M(B)\, a(\lambda) < \lambda_m(A)\, b(\lambda)$ where $\lambda_m(A)$ is the minimal and $\lambda_M(B)$ is the maximal eigenvalues of the matrices A and B respectively and a, b are of class K.

Then the certain type of practical stability of zero solution to the equation

$$\frac{du}{dt} = W(t, u),\ u(t_0) = u_0 \geq 0 \tag{14}$$

implies the certain type of connected practical stability of system (2).

Proof. Note first that under conditions (1) and (2) of Theorem 4 for the function $V(t,x,x_t)$ determined by (11) the estimate

$$\lambda_m(A)\, b(|x|) < V(t, x, x_t) < \lambda_M(B)\, a(|x_t(\cdot)|) \tag{15}$$

is true. This follows from the fact that for function $u_{1i}, u_{2i} \in K$, $i = 1, 2, ..., m$, there exist functions $a(|x_t(\cdot)|)$ and $b(|x|)$ of class K such that $b(|x|) \leq u_1^T(|x|)u_1(|x|)$ and $a(|x_t(\cdot)|) \geq u_2^T(|x_t(\cdot)|)u_2(|x_t(\cdot)|)$. Further we have from condition (3) of Theorem 4 for the function $m(t) = V(t, x(t_0, x_{t_0})(t), x(t_0, x_{t_0}))$ $D^+m(t) \leq W(t, m(t))$ which together with the condition $V(t, x_0, x_{t_0}) \leq u_0$ yield the estimate

$$V(t, x(t_0, x_{t_0})(t), x_t(t_0, x_{t_0})) \leq r(t, t_0, u_0),\ t \geq t_0 \tag{16}$$

according to the comparison principle (see[1] Theorem 4.1.1). Let the zero solution of equation (14) be practically stable. Given $(\lambda_M(B)\, a(\lambda), \lambda_m(A)\, b(A))$, we have

$$u(t, t_0, u_0) < \lambda_m(A)\, b(A), \tag{17}$$

provided that

$$u_0 < \lambda_M(B)\, a(\lambda). \tag{18}$$

Let

$$|x_0| < \lambda \text{ and } |x_{t_0}(\cdot)| < \lambda . \tag{19}$$

We shall demonstrate that $|x(t_0, x_{t_0})(t)| < A$ for all $t \geq t_0$.

Assume that this is not true and that there exists $t_1 > t_0$ such that for the solution $x(t_0, x_{t_0})(t)$ with initial condition (19) the correlations $|x(t_0, x_{t_0})(t_1)| = A$ and $|x(t_0, x_{t_0})(t)| \leq A$ hold for $t_0 \leq t \leq t_1$.

Estimate (15) yields

$$V(t_1, x(t_0, x_{t_0})(t_1), x_{t_1}(t_0, x_{t_0})) \geq \lambda_m(A)\, b(A) \tag{20}$$

Let $u_0 = V(t_0, x(t_0, x_{t_0})(t_0), x_{t_0}(t_0, x_{t_0}))$. Then for all $t_0 \leq t \leq t_1$, estimate (16) is valid, where $r(t, t_0, u_0)$ is the maximal solution of equation (14). Since $u_0 < \lambda_M(B)\, u_2^T \times \times(|x_{t_0}(\cdot)|)\, u_2(|x_{t_0}(\cdot)|) < \lambda_M(B) a(\lambda),$ we find by the comparison principle and inequalities (15).

$$\begin{aligned} &\lambda_m(A) b(A) \leq \lambda_m(A) u_1^T(|x_0|) u_1(|x_0|) \leq \\ &\leq V(t_1, x\,(t_0, x_{t_0})(t_1), x_{t_1}(t_0, x_{t_0})) \leq r(t_1, t_0, u_0) < \lambda_m(A) b(A) . \end{aligned} \tag{21}$$

The obtained contradiction shows that $t_1 \notin J$ and therefore system (2) is connectedly practically stable.

Р Е З Ю М Е. Методом об‘єднання матрично-значних функціоналів Ляпунова і теореми порівняння досліджено зв‘язну стійкість за Ляпуновим і практичну стійкість великих систем з запізненням. Запропоновано ряд нових достатніх умов. Результати мають не лише теоретичний сенс, але також практичне значення.

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