**A . O . K a m i n s k y**

# STUDIES OF SUBCRITICAL CRACK GROWTH IN THE VISCOELASTIC ANISOTROPIC BODIES USING THE CONTINUED FRACTION OPERATOR METHOD: SYNTHESIS AND SUMMARY

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**Abstract.** A technique of solving the problems of linear viscoelasticity is presented. Some basics are given on the application of continued fractions to solve some problems for viscoelastic anisotropic bodies with the slowly growing cracks. It is shown by examples that the operator continued fraction technique can be effectively used for solving the complex problems of fracture mechanics for the modern anisotropic composite materials.

**Key words**: linear viscoelastic medium, anisotropy, fracture, cracks, life term, integral operator, continued fraction.

## 1. Introduction.

Investigation of the slow quasi-static crack growth in the viscoelastic bodies has been started in the 1960-s (see surveys [16, 38, 54]).

During the initial period, several approximate methods were applied to solve boundary problems of linear viscoelasticity. In this context, it is worth noting the variable moduli method by [1], the quasi-elastic method by [52]. The main idea of these methods is the simplification of the problem by replacing Volterra integral operators (formalisms) in Boltzmann-Volterra theory [5, 63] with some functions of time (the creep or relaxation functions). Such methods are rather simple and can be applied to obtain solutions for some problems of fracture mechanics in the case of isotropic media, e.g., polymers (see bibliography in [37, 54] but they failed to be widely adopted because there are no strict criteria of their applicability [42].

Some stricter analytical and numerical methods were elaborated later on to study the deformation of the linear viscoelastic anisotropic bodies [10, 17, 20, 34, 57] as well as viscoelastic composites of various structures [2, 7, 24, 53]. These methods allow solving of some fracture mechanics problems of crack propagation in viscoelastic anisotropic bodies and composites (see bibliography in [6, 8, 17, 19, 44, 54, 67].

There are two main approaches in this field.

The first one is well-known from the literature Laplace-Carson transform approach (see books by [10] or [45]). The correspondence principle allows obtaining the solution of the original problem from the solution of some elastic problem in the transform domain. As it is known that the inverse transform is an ill-posed problem. The solution inevitably can be found with some simplifications and constrictions only [11].

The other approach is based on the theory of Boltzmann-Volterra [5, 63], properties of integral operators, and the Volterra principle which is an analog of the correspondence principle for viscoelastic operators [20]. It is possible to use some simple rules (resolvent operators algebra [13, 36, 48]) and methods to expand functions of integral operators into power series [48, 68]. It was found that such power series are not quite useful for a wide range of problems due to the slow convergence. To work around the convergence problem it was proposed to use continued fractions for approximations [17, 20]. It was proved that the continued fractions for several important problems of viscoelastic fracture converge very fast as compared to the corresponding power series. Since then, the continued fraction technique was successfully applied to solve several problems for holes and cuts in anisotropic plates and bulk bodies [46, 47], contact problems of viscoelasticity [21], and the problems of viscoelastic properties prediction for composite materials [28, 30, 55, 59].

In this work, a generalization of theoretical investigations in fracture mechanics of viscoelastic bodies is given for the results that were obtained using the operator continued fraction technique. This technique was elaborated in [17, 19, 20, 29].

Herein, we consider the applications of the continued fraction operator technique for theoretical studies of deformation and delayed fracture of anisotropic viscoelastic bodies caused by slow subcritical crack growth under quasi-static conditions (i.e. when inertial effects due to straining can be neglected). The highly damaged material at the tips of growing cracks is described using some models of cohesive zones [15, 19]. The analysis is simplified by using small strain theory for the linear viscoelastic continuum because many viscoelastic materials (polymers, glass-reinforced plastics, etc.) remain linear under high stress and some materials remain linear even up to failure, so the linear theory of viscoelasticity can be used to describe their deformation. It is also assumed, based on experimental data, that the viscoelastic strains beyond the growing crack are negligible compared the strains in the cohesive zone [24, 37, 54].

## 2. Methods of solving boundary-value problems of linear viscoelasticity for anisotropic bodies with growing cracks.

**2.1. Some Approaches and Methods in the Theory of Linear Viscoelasticity for bodies with growing cracks.** *Problem Formulation.* The linear theory of viscoelasticity (or the theory of hereditary elasticity) was intensively developed in the 20-th century [10, 48, 49] to study the rheological properties of new structural materials such as polymers and their composites, and concrete, reinforced concrete, rocks.

As it was noted above, of special importance is the Boltzmann – Volterra theory of linear viscoelasticity [63], which employs Volterra operators. These studies are based on the Volterra principle, the algebra of Volterra operators [48], and methods of determining functions of integral operators, i.e., reducing these functions to ordinary Volterra operators.

In [63], a function of an integral operator is considered as some power series (Taylor series) with an integral operator as the variable. He showed that such an operator series converges if the function is analytical. If viscoelastic bodies are homogeneous and isotropic, then the corresponding functions of integral operators are rational, and there are no fundamental difficulties to solve problems of linear viscoelasticity [48].

If, however, viscoelastic bodies are inhomogeneous or anisotropic, these functions are irrational and solving specific problems involve severe difficulties associated with the weak convergence of power series [48]. This is why many researchers failed to find effective solutions to boundary-value problems of linear viscoelasticity for anisotropic bodies.

In this connection, a new method to solve problems of linear viscoelasticity by expanding irrational functions of integral operators into continued fractions [17] has been developed.

In works [17, 20, 29], this method was proved for Volterra resolvent operators of the second kind. Owing to the high convergence of continued fractions, the method makes it possible to solve viscoelastic problems that failed to be solved before. It helped to solve a number of new problems related to fracture mechanics of and stress concentration in viscoelastic anisotropic bodies.

*Basic Equations and Methods.* The constitutive equations describing the linear viscoelastic behavior can be written as

$$\sigma_{ij} = E^{*}_{ijkl}\varepsilon_{kl}; \tag{1}$$

$\varepsilon_{ij}$ components of the deformation tensor $\sigma_{ij}$ components of the stress tensor

$$\varepsilon_{ij} = R^*_{ijkl}\sigma_{kl}, \tag{2}$$

where $E^*_{ijkl}$ and $R^*_{ijkl}$ are linear Volterra operators of the second kind;

$$E^*_{ijkl}\varepsilon_{kl} = E^0_{ijkl}\varepsilon_{kl}(t) + \int_0^t E_{ijkl}(t-\tau)\varepsilon_{kl}(\tau)\,\mathrm{d}\tau;$$

$$R^*_{ijkl}\sigma_{kl} = R^0_{ijkl}\sigma_{kl}(t) + \int_0^t R_{ijkl}(t-\tau)\sigma_{kl}(\tau)\,\mathrm{d}\tau;$$

$E_{ijkl}(t)$ is the relaxation tensor; $R_{ijkl}(t)$ is the creep tensor; $E^0_{ijkl}$ and $R^0_{ijkl}$ are the elastic constants of anisotropic body $E^0_{ijkl}$ instantaneous constants of anisotropic body.

Eqs (1) and (2) are usually used in the mechanics of polymers and their composites (such as glass-reinforced plastics, carbon fiber-reinforced plastics, etc.), rock mechanics, and studies of the creep of some metallic materials at high temperatures. If the tensors $E_{ijkl}(t)$ and $R_{ijkl}(t)$ are symmetric, the number of different functions is equal to the number of independent elastic constants, i.e., 21.

Eqs. (1) and (2) are derived from the Boltzmann principle [5] from which it follows that the constitutive equations of linear viscoelasticity can be derived from the constitutive equations of linear elasticity by replacing the elastic constants $E_{ijkl}$ and $R_{ijkl}$ with the linear integral operators $E^*_{ijkl}$ and $R^*_{ijkl}$. The experimental determination of the functions $E_{ijkl}(t)$ and $R_{ijkl}(t)$ involves severe difficulties and is mainly carried out for anisotropic materials with certain symmetry and, hence, fewer independent functions $E_{ijkl}(t)$ and $R_{ijkl}(t)$.

For viscoelastic orthotropic bodies with cylindrical anisotropy, the rheological equations (1) can be adapted as

$$\varepsilon_{rr} = \frac{1}{E^*_{rr}}\sigma_{rr} - \frac{\nu^*_{\varphi r}}{E^*_{rr}}\sigma_{\varphi\varphi} - \frac{\nu^*_{zr}}{E^*_{zz}}\sigma_{zz};\ \varepsilon_{\varphi\varphi} = -\frac{\nu^*_{r\varphi}}{E^*_{rr}}\sigma_{rr} + \frac{1}{E^*_{\varphi}}\sigma_{\varphi\varphi} - \frac{\nu^*_{z\varphi}}{E^*_{zz}}\sigma_{zz};$$

$$\varepsilon_{zz} = -\frac{\nu^*_{rz}}{E^*_{rr}}\sigma_{rr} - \frac{\nu^*_{\varphi z}}{E^*_{\varphi}}\sigma_{\varphi\varphi} + \frac{1}{E^*_{zz}}\sigma_{zz};\ \varepsilon_{\varphi z} = \frac{1}{G^*_{\varphi z}}\sigma_{\varphi z};\ \varepsilon_{rz} = \frac{1}{G^*_{rz}}\sigma_{rz};\ \varepsilon_{r\varphi} = \frac{1}{G^*_{r\varphi}}\sigma_{\varphi z}, \tag{3}$$

where the values marked with stars are the Volterra integral operators of second kind $(r, \varphi, z)$ cylindrical coordinates. It is assumed below that

$$E^*_{\varphi\varphi} = E^*_{rr};\ \nu^*_{z\varphi} = \nu^*_{\varphi z};\ G^*_{\varphi z} = G^*_{z\varphi}.$$

*Volterra Principle.* Volterra pointed out that in solving problems of linear viscoelasticity (hereditary elasticity), the operations to solve the differential equations and the time integration operations to calculate Volterra operators can be performed in any order. This leads to the Volterra principle: *to solve a viscoelastic problem, it is necessary to find the linear elastic solution and to replace the elastic constants in it with operators, expanding the resulting combinations of operators or functions of operators*.

It was established [15, 51] that the Volterra principle is applicable if the crack grows monotonically with time and the cohesive zone is constant. If the external load's domain in the boundary conditions does not change with time, this principle is applicable no matter how the crack length varies. For the cohesive zone model, not only the applicability conditions for the Volterra principle must be satisfied, but also the cohesive zone must grow monotonically.

*Rational Functions of Operators.* In the general case, the solution of the boundary value problem of linear viscoelasticity for homogeneous bodies is

$$f(x_j, t) = F(R^*_i)g(x_j, t), \tag{4}$$

where $g(x_j, t)$ is the external load; $F(R_i^*)$ is a function of operators $R_i^*$, which should be expanded somehow because it has been written formally, by replacing the elastic characteristics $R_i$ with the respective viscoelastic operators $R_i^*$.

If $F$ is a rational function of resolvent operators $R_i^*$, it can be expanded using the algebra of resolvent operators [48].

*Nonrational Functions of Operators.* If $F$ is not rational (irrational, transcendental, etc.) function of the resolvent operator $R^*$, then it can be expanded in a conventional way [48], using a power series:

$$F(\kappa R^*) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} (\varkappa R^*)^n, \tag{5}$$

$F^{(n)}(0)$ is the $n$-th derivative of the function $F$ with respect to $R^*$ at zero; $(R^*)^n$ is the $n$-th degree of the operator $R^*$, which is an operator whose kernel is iterated $n-1$ times.

In this case, the viscoelastic solution has the form

$$f(x_j, t) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} (\varkappa R^*)^n g(x_j, t). \tag{6}$$

In [48], it is shown that series eq. (6) converges slowly; therefore, for high accuracy of the solution, it is necessary to retain a great number of terms (up to several hundred) in the series. This leads to unwieldy calculations required to conduct with very high precision, which is not always possible or acceptable.

To overcome these shortcomings, it was proposed in [17] to use the method of operator-valued continued fractions, which is outlined below.

**2.2. A method of operator-valued continued fraction (for resolvent operators).** Rather than expanding a function of integral operators into some operator power series (an analogue of Taylor series) as it was proposed by Volterra [63], this function can be represented by a new mathematical object which is called an operator-valued continued fraction [17].

Thiele's formula [14] can be used to expand the function of integral operators $T(R_i^*)$ into a continued fraction of these operators

$$T(\varkappa_i R_i^*) = T^0 \left[ I + \mathop{\boldsymbol{K}}_{m=1}^{\infty} \frac{c_m \varkappa_i R_i^*}{I} \right] = T^0 \left[ I + \cfrac{c_1 \varkappa_i R_i^*}{I + \cfrac{c_2 \varkappa_i R_i^*}{I + \cfrac{c_3 \varkappa_i R_i^*}{I + \dots}}} \right]. \tag{7}$$

This allows us to introduce rapidly converging continued fractions of resolvent integral operators instead of weakly converging operator-valued series.

With this approach, the function $T(\varkappa_i R_i^*)$ can be represented as a linear combination of resolvent operators $R^*$. A few approximations are sufficient in the interpolation process for continued fractions to converge with an appropriate accuracy [14].

Denoting the finite fraction with $n$ chains as $F_n^* = \mathop{\boldsymbol{K}}_{r=1}^{n} \frac{c_r A^*}{1}$ (for $c_r > 0$) the following remarkable properties of operator continued fraction convergence can be obtained from the proof of the continued fraction convergence theorem [20]

$$1) \left\| F_{2m}^* \right\|_E \le \left\| F(A^*) \right\|_E \le \left\| F_{2n-1}^* \right\|_E ;$$

$$2) \left\| F(A^*) - F_n^* \right\|_E \le \left\| F_n^* - F_{n-1}^* \right\|_E ,$$

where $\|\cdot\|_E$ is a norm in space $E$. The precise value of the function is found to be between the finite fractions with the even and odd number of chains.

To solve eq. (7) one need to present operator $T^*$ as a standard convolution-type integral

$$T^* f(t) = T^0 \left[ f(t) + \int_0^t \Pi(t-\tau) f(\tau) \mathrm{d}\tau \right]. \tag{8}$$

Here, $T^0 = F(\varkappa_i R_i^0)$, $\Pi(t-\tau)$ is the kernel of operator $T^*$. As a rule of thumb, the continued fractions of operator $T^*$ converges rapidly to $T^*$ [14] and therefore it can be replaced with its approximant in eq. (7). Using the algebra of resolvent operators [48], it is possible to reduce eq. (7) to

$$T^* = T^0 \left[ I + \sum_{i=1}^{N} \mu_i R^*(\gamma_i) \right], \tag{9}$$

where $I$ is the identity $I$ identity operator, $T^0$, $\mu_i$, $\gamma_i$ are some constants.

As it follows from eq. (9), the kernel of $T^*$ can be written in the convolution form as

$$\Pi(t-\tau) = \sum_{i=1}^{N} \mu_i R(t-\tau, \gamma_i). \tag{10}$$

Moreover, the expression for $T(\varkappa_i R_i^*)$ remains in the class of resolvent operators $R_i^*$. Finally, the definition by the continued-fraction expansion of a non-rational function of a resolvent operator is a more general definition than the expansion into power series eq. (6) because it is applicable to both holomorphic and meromorphic functions. For holomorphic functions, both definitions are equivalent.

**2.3. The continued fraction expansion.** A theorem of convergence of Volterra integral operators continued fraction was formulated and proven in [17]. To represent the function $F(E_{ij}^*)$ in standard form, expand square root function into continued fraction. Such an expansion has the form [14] of eq. (7).

Formal substitution of operator $\sum_{i=1}^{n} \xi_i R^*(\eta_i)$ for $x$ in eq. (7) yields

$$\sqrt{I + \sum_{i=1}^{n} \xi_i R^*(\chi_i)} = I + \mathop{\boldsymbol{K}}_{r=1}^{\infty} \frac{c_r \sum_{i=1}^{n} \xi_i R^*(\chi_i)}{I} = I + \cfrac{c_1 \sum_{i=1}^{n} \xi_i R^*(\chi_i)}{I + \cfrac{c_2 \sum_{i=1}^{n} \xi_i R^*(\chi_i)}{I + \cfrac{c_3 \sum_{i=1}^{n} \xi_i R^*(\chi_i)}{I + \ldots}}}. \tag{11}$$

Following the resolvent operators algebra rules [48] the $M$-th finite fraction approximant for eq. (11) can be reduced to the form

$$I + \mathop{\boldsymbol{K}}_{r=1}^{M} \frac{c_r \sum_{i=1}^{n} \xi_i R^*(\chi_i)}{I} = I + \sum_{j=1}^{m} \gamma_j R^*(\delta_j). \tag{12}$$

Whence,

$$\sqrt{I + \sum_{i=1}^{n} \xi_i R^*(\chi_i)} \overset{M}{\approx} I + \sum_{j=1}^{m} \gamma_j R^*(\delta_j). \tag{13}$$

**2.4. Determination of a Superposition of Irrational Functions of Resolvent Operators [20].** If it is necessary to determine a superposition $F$ of two irrational functions $F_1$ and $F_2$ of the resolvent operator

$$F = F_1\left\{F_2\left[\kappa R^*(\lambda)\right]\right\} \tag{14}$$

then the internal function $F_2$, which is represented by a linear combination of resolvent operators

$$F_2\left[\kappa R^*(\lambda)\right] = c_0 + \underset{r_2}{\mathbf{K}}\frac{c_{r_2}\kappa R^*(\lambda)}{I} \approx c_0 + \sum_{i_2} a_{i_2} R^*(\lambda_{i_2}) \tag{15}$$

can be determined first. After that, the external function $F_1$ of the linear combination obtained is determined,

$$F \approx F_1\left[\sum_{i_2} a_{i_2} R^*(\lambda_{i_2})\right] = c_0' + \underset{r_1}{\mathbf{K}}\frac{c_{r_1}\sum_{i_2} a_{i_2} R^*(\lambda_{i_2})}{I} \approx c_0' + \sum_{i_1} a_{i_1} R^*(\lambda_{i_1}') \,. \tag{16}$$

If it is necessary to determine a superposition $F$ of several functions $F_i$ ($i = \overline{1,m}$) of the resolvent operator

$$F = F_1\left\{F_2\left\{\dots F_m\left[\kappa R^*(\lambda)\right]\right\}\right\} \tag{17}$$

then the function $F_m$ is determined first, after that the function $F_{m-1}$, and so on, until the superposition $F$ is determined. The determination of a superposition of two or more irrational functions of a linear combination of resolvent operators may be similar: the function $F_m\left[\sum_{s=1}^{s_0}\kappa_s R^*(\lambda_s)\right]$ is determined first, after that $F_{m-1}$ of the obtained linear combination of resolvent operators.

Below, the effectiveness of the proposed method is shown for one of the theries describing the subcritical crack growth in viscoelastic anisotropic bodies and composites [15, 17, 19].

**3. Mechanics of slow crack growth in viscoelastic bodies.**

**3.1. Modelling crack in viscoelastic media.** Numerous experimental studies [24, 38, 65] indicate that there is a cohesive zone at the crack front that moves together with the crack. Cohesive zones appear due to the high level of stress at the crack front. The material in a cohesive zone is damaged (for example, "crazes" in polymers) (fig. 1) [54].

These cohesive zone models can be called two-phase models because the material goes through two phases of fracture, unlike sin-glephase models such as the Griffith – Irwin model, where the solid material abruptly (with-out transient phase) becomes damaged. Among two-phase models, it is worth to mention the Leonov – Pana-syuk model [41], the Barenblatt model [3], the Dugdale model [12].

Generalizing the above models to viscoelastic materials leads to a new kinetic model that differs from the static model which describes the critical state of elastic bodies.

According to modified Leonov – Panasyuk – Barenblatt model (see [15, 16, 24]) that can describe the fracture behaviour of some viscoelastic polymers and composites, the non-small cohesive zone at the crack tip is replaced by a notch on the crack continuation with selfbalanced compressive stresses $\sigma(t)$ distributed along the zone and applied to its edges (fig. 2). Here, $a(t)$ is the half

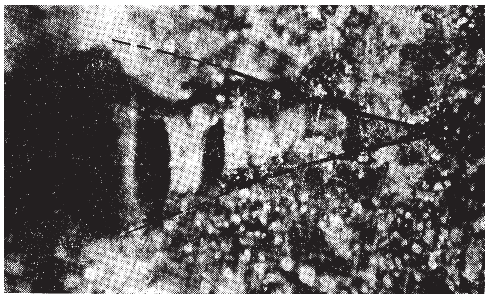

*Fig. 1*

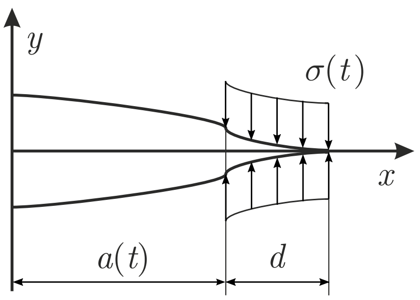


*Fig. 2*

crack, $d$ is the constant cohesive length $a(t)$ halflength of the crack at time $t$.

This model has an experimental background for orthotropic polymeric composites (fiber-glass plastics) [18, 19].

In this paper, the critical COD [39, 50, 51, 64] is used as a criterion indicating when a mode I crack starts growing. It is also can be assumed that the used criterion is valid for a growing crack at every instant. Thus,

$$\delta(x,t)\big|_{x=a(t)} = \delta_{Ic}, \tag{18}$$

where $\delta = 2v$ is the crack-tip opening displacement $\delta$ crack-tip opening displacement ( $v$ is vertical displacement of the crack edges), $\delta_{Ic}$ is its critical value $\delta_{Ic}$ critical value of the crack tip opening displacement.

Other criteria for the limiting state in viscoelastic bodies have been proposed in works by [38, 43, 54, 66] (for a survey one is referred to [37]).

**3.2. Mode I crack opening displacement in viscoelastic anisotropic body.** According to the Volterra principle, the boundary of a crack in a homogeneous viscoelastic body can be described by the equation [15]

$$\delta(x,t) = T^{*}\delta_0(x,t), \tag{19}$$

where $\delta_0(x,t)$ is a function of mechanical and geometrical parameters; $T^{*}$ is a function of integral operators

$$T^{*} = f(R^{*}_{ijkl}), \tag{20}$$

where $f$ is a specific function; $T^{*}$ and $R^{*}_{ijkl}$ are linear Volterra operators of the second kind.

The standard form of this integral operator is

$$T^{*}s(t) = T_0\left[s(t) + \int_0^t \Pi(t-\tau)s(\tau)\mathrm{d}\tau\right]. \tag{21}$$

Hence, it is possible to express viscoelastic mode-I crack opening displacement in a viscoelastic body as

$$\delta(x,t) = T_0\left[\delta_0(x,t) + \int_0^t \Pi(t-\tau)\delta_0(t,\tau)\mathrm{d}\tau\right], \tag{22}$$

where $T_0\delta_0(x,t)$ is the elastic mode I crack opening displacement.

Substituting the expansion from eq. (10) into eq. (19), yields

$$\delta(x,t) = T_0\left[\delta_0(x,t) + \int_0^t \sum_{i=1}^{N}\mu_i R\left(t-\tau,\gamma_i\right)\delta_0(t,\tau)\mathrm{d}\tau\right]. \tag{23}$$

Structure of $\delta_0(x,t)$ depends on the crack path, external loads and the body geometry. For the orthotropic viscoelastic plate with a rectilinear through-thick crack of length $2a$ located along one of the plate orthotropy axes (denoted $x$) and the self-balanced quasi-static loading on the crack faces $q_0(x,t)$ ($|x|<b$, $b \le a$), $\delta(x,t)$ is as follows [17]

$$\delta(x,t) = -T^{*}\int_{-L}^{L} q(\xi,t)\Gamma(L,x,\xi)\mathrm{d}\xi, \tag{24}$$

where

$$\Gamma(L, x, \xi) = \ln \frac{L^2 - x\xi - \sqrt{(L^2 - x^2)(L^2 - \xi^2)}}{L^2 - x\xi + \sqrt{(L^2 - x^2)(L^2 - \xi^2)}};$$

$$q(x, t) = \begin{cases} q_0(x,t), & |x| < b, \\ 0, & b < |x| \le a(t), \quad L(t) = a(t) + d, \\ -\sigma(t), & a(t) < |x| < L(t), \end{cases}$$

$d$ prefracture zone length $T^*$ is an integral operator with translation kernel which can be written as a function of other operators as

$$T^* = \frac{1}{\pi\sqrt{E_{11}^* E_{22}^*}} \sqrt{2\left(\sqrt{\frac{E_{11}^*}{E_{22}^*}} - \nu_{21}^*\right) + \frac{E_{11}^*}{G_{12}^*}}. \tag{25}$$

It should be noted that $T^*$ depends on the kind of anisotropy.

According to the chosen crack model, the stresses at the crack tip should be finite. This condition can only be met if

$$\lim_{x \to L(t)} \int_{L(t)}^{L(t)} \frac{q_0(\xi, t)\sqrt{L^2(t) - \xi^2}}{x - \xi} \mathrm{d}\xi = 0. \tag{26}$$

This equation is an additional dependence between the external load, the crack length and the length of its process zone.

For macrocracks ( $d \ll a$ ), it is possible to use SIF and eq. (24) can be rewritten as

$$\delta(x, t) = T^* \frac{2}{\pi} \sigma(t) \left\{ 2\sqrt{d[d - x + a(t)]} + [x - a(t)] \ln \frac{\sqrt{d} - \sqrt{d - x + a(t)}}{\sqrt{d} + \sqrt{d - x + a(t)}} \right\}, \tag{27}$$

where $\sigma^2(t) = (\pi K_I^2)/(8d)$ and $K_I$ stress intensity factor for mode-I crack is the value of SIF for the given problem parameters; $\sigma(t)$ uniform stress level in the process zone at the moment $t$.

**3.3. Subcritical quasi-static crack growth in viscoelastic bodies.** In the general case, a mode I crack in viscoelastic body under subcritical loads goes through the incubation (preparatory) stage when the crack gradually opens but does not grow; initiation (transition) stage when the crack grows through the initial process zone; and the main period of quasi-static growth that lasts until the crack starts to grow dynamically [17].

The governing equations for all those stages can be obtained by substituting eq. (23) or eq. (27) into the criterion eq. (18).

In what follows, only the main stage of crack growth is studied because as it follows from works [16, 54], this stage duration is very close to the life term of a viscoelastic body with crack.

During the main stage of slow quasi-static crack growth under the constant or slowly changing external loads, the crack gradually tears continuous material that once forming a cohesive zone. The crack begins to open in the cohesive zone after the tip of the cohesive zone reaches a certain point in the viscoelastic material. This moment of time is denoted by $t'$. For non-small cohesive zone, the equation of slow crack growth at this stage is derived assuming that criterion eq. (18) is satisfied at every instant of crack growth. In this case, this yields [15, 16]

$$\delta_{Ic} = T^0 \left\{ \delta_0\left[a(t)\right] + \int_{t'}^{t} \Pi\left(t - \tau\right) \delta_0\left[a(t), a(\tau)\right] \mathrm{d}\tau \right\}, \tag{28}$$

where $\delta\left[x, a(t)\right] = T_0 \delta_0 \left[x, a(t)\right]$; $\delta[a(t), a(\tau)]$ is the elastic opening displacement for the crack of length $a(\tau)$ for $x = a(t)$; $t'$ can be found from the equation

$$a(t) - a(t') = d.$$

Eq. (28) is a nonlinear integral equation of complex structure.

In general terms, eq. (28) cannot be solved analytically. However, they can be solved numerically using discretization by the space coordinate. It is shown in [27] that the most effective implementation of the numerical scheme is to determine the time when the crack tip arrives at some given mesh node. Assuming that the speed of crack tip on each step is constant, the time $t_K$ when the crack tip arrives at the *K*-th node can be determined from a nonlinear equation.

For macrocracks ($d \ll a(t)$) [15, 16]

$$\frac{K_{Ic}}{K_I} = 1 + \frac{d}{\dot{a}} \int_0^1 \Pi\left(\frac{d \cdot s}{\dot{a}}\right) F(s) \mathrm{d}s, \tag{29}$$

where $K_I$ is SIF, $K_{Ic}$ is its critical value for $a = a_*$ $K_{Ic}$ critical value of stress intensity factor for mode-I crack, when the crack starts to grow dynamically; $\dot{a}$ is the crack growth velocity; $F(s) = \sqrt{1-s} + s/2\left[\ln(1-\sqrt{1-s})/(1+\sqrt{1-s})\right]$.

Eq. (29) is a general dependence between the SIF and crack velocity for every growing crack because it does not depend on geometry parameters and kind of mode I loading. The existence of such a general dependence is experimentally proven in work [24] for a wide range of polymers and polymer-based composites. The integro-differential eq. (29) allows analytical solutions for crack growth kinetics and durability of a viscoelastic body for some problems.

In what follows, solutions to several problems of long-term fracture of anisotropic plates and bodies are given by the operator continued fractions method.

**4. Subcritical growth of a mode I crack in an anisotropic viscoelastic plate.**

To analyze some advantages of the proposed approach consider a mode-I macrocrack growth along an orthotropy axis in the orthotropic plate under the static subcritical loads. For simplicity, it is assumed that the material of the plate exhibits the shear creep only. As it is shown by [48], such properties are common for plastic reinforced by orthogonal glass fibers. This assumption is not restrictive for the method.

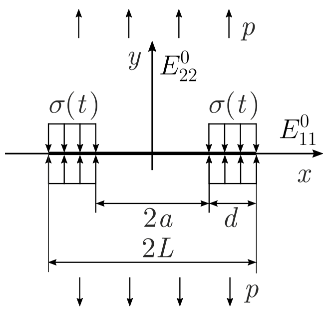


*Fig. 3*

The problem notation is shown in fig. 3.

The simplified version of the constitutive eqs. (1) then can be written as [40]

$$\varepsilon_{11} = \sigma_{11} / E_{11}^0 - \nu_{21}^0 / E_{22}^0 \sigma_{22};$$

$$\varepsilon_{22} = -\nu_{21}^0 / E_{22}^0 \sigma_{11} + \sigma_{22} / E_{22}^0;$$

$$\varepsilon_{12} = 2 / G_{12}^* \sigma_{12},$$

where $E_{11}^0$ and $E_{22}^0$ are the instantaneous elastic moduli for axes $x$ and $y$, respectively, $\nu_{21}^0$ is Poisson's ratio in $x$ direction for the tension along $y$, and

$$\frac{1}{G_{12}^*} = \frac{1}{G_{12}^0}\left[1 + \lambda_G R^*(\beta_G)\right]. \tag{30}$$

Substituting eq. (30) into eq. (25) yields

$$T^* = T_0 \sqrt{1 + \lambda_G' R^*(\beta_G)}, \tag{31}$$

where

$$T_0 = \sqrt{\frac{1}{E_{11}^0 E_{22}^0}} \sqrt{2\left(\sqrt{\frac{E_{11}^0}{E_{22}^0}} - \nu_{21}^0\right) + \frac{E_{11}^0}{G_{12}^0}}; \quad \lambda'_G = \lambda_G \frac{\frac{E_{11}^0}{G_{12}^0}}{2\left(\sqrt{\frac{E_{11}^0}{E_{22}^0}} - \nu_{21}^0\right) + \frac{E_{11}^0}{G_{12}^0}}.$$

As it follows from [14],

$$\sqrt{1+x} = 1 + \underset{m=1}{\overset{\infty}{\mathbf{K}}} \frac{c_m x}{1} = 1 + \cfrac{c_1 x}{1 + \cfrac{c_2 x}{1 + \cfrac{c_3 x}{1 + \ldots}}}, \tag{32}$$

where $c_1 = 0,5$ and $c_m = 0,25$ for $r = 2,3,\ldots$ $\underset{m=1}{\overset{\infty}{\mathbf{K}}}$ continued fraction notation

Hence,

$$\sqrt{I + \lambda'_G R^*(\beta_G)} = I + 2\underset{i=1}{\overset{\infty}{\mathbf{K}}} \frac{0,25\lambda'_G R^*(\beta_G)}{I} \approx 1 + 2\underset{i=1}{\overset{m}{\mathbf{K}}} \frac{0,25\lambda'_G R^*(\beta_G)}{I}, \tag{33}$$

where

$$\underset{i=1}{\overset{m}{\mathbf{K}}} \frac{a_i}{b_i} = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ldots}}.$$

For $m = 1$,

$$\sqrt{I + \lambda'_G R^*(\beta_G)} \approx I + 2\frac{0,25\lambda'_G R^*(\beta_G)}{I} = I + 0,5\lambda'_G R^*(\beta_G), \tag{34}$$

for $m = 2$,

Hence, Eq. (33) can be written as

$$\sqrt{I + \lambda'_G R^*(\beta_G)} = I + \sum_{i=1}^{m_1} C_i R^*(\beta_i), m_1 = \left[\frac{m-1}{2}\right] + 1, \tag{35}$$

where $[(m-1)/2]$ is an integral part of $(m-1)/2$. So the kernel of this integral operator can be expressed

$$\Pi(t-\tau) = \sum_{i=1}^{m_1} C_i R(t-\tau, \beta_i), \tag{36}$$

where $R(t-\tau, \beta_i)$ are the kernels of resolvent operators $R^*(\beta_i)$. $\Pi(t-\tau)$ kernel of Volterra integral operator

It can be shown [48, 49] that within the framework of the fractional calculus and some other non-restrictive limitations

$$R(t-\tau, \beta_i) = E_\alpha(t-\tau, \beta_i) = \sum_{n=0}^{\infty} \frac{\beta_i^n (t-\tau)^{n(1-\alpha)-\alpha}}{\Gamma[(n+1)(1-\alpha)]}, \tag{37}$$

where $\beta_i$, $\alpha$ are rheological parameters, $0 \le \alpha < 1$, $\beta_i < 0$.

Substituting eqs. (36), (37) into eq. (29) gives an equation of subcritical crack growth

$$\frac{K_{Ic}}{K_I} = 1 + \frac{d}{\dot{a}} \int_0^1 \sum_{i=1}^{m_1} C_i \sum_{n=0}^{\infty} \frac{\beta_i^n \left(\frac{d}{\dot{a}} s\right)^{n(1-\alpha)-\alpha}}{\Gamma\left[(n+1)(1-\alpha)\right]} F(s) \mathrm{d}s =$$

$$= 1 + \sum_{n=0}^{\infty} \frac{\sum_{i=1}^{m_1} C_i \beta_i^n}{\Gamma\left[(n+1)(1-\alpha)\right]} \left(\frac{d}{\dot{a}}\right)^{(n+1)(1-\alpha)} \int_0^1 s^{n(1-\alpha)-\alpha} F(s)\mathrm{d}s = 1 + \sum_{n=0}^{\infty} A_n(\alpha, C_i, \beta_i) \left(\frac{d}{\dot{a}}\right)^{(n+1)(1-\alpha)}, \tag{38}$$

where

$$A_n(\alpha, C_i, \beta_i) = \frac{\sqrt{\pi}}{2} \frac{\sum_{i=1}^{m_1} C_i \beta_i^n}{\left[(n+1)(1-\alpha)+1\right] \Gamma\left[(n+1)(1-\alpha)+1,5\right]}.$$

The duration of the main period of crack evolution can be found as

$$\Delta t_{II} = \int_{a_0}^{a_*} \frac{1}{\dot{a}} \mathrm{d}a, \tag{39}$$

where $a_*$ is the critical half-length of the crack $a_*$ critical half-length of the crack (the value of $a$ when the dynamical growth begins). Taking into account two terms in eq. (38), one can find that

$$\frac{K_{Ic}}{K_I} = 1 + A_1 \left(\frac{d}{\dot{a}}\right)^{1-\alpha} - A_2 \left(\frac{d}{\dot{a}}\right)^{2(1-\alpha)}, \tag{40}$$

where

$$A_1 = \frac{\sqrt{\pi}}{2} \frac{\sum_{i=1}^{m_1} C_i}{(2-\alpha)\Gamma(2,5-\alpha)}; \quad A_2 = -\frac{\sqrt{\pi}}{2} \frac{\sum_{i=1}^{m_1} C_i \gamma_i}{(3-2\alpha)\Gamma(3,5-2\alpha)} \quad (\gamma_i < 0), \tag{41}$$

whence

$$\frac{d}{\dot{a}} = a_1 \left[1 - \sqrt{1 - a_2 \left(\frac{K_{Ic}}{K_I} - 1\right)}\right]^{1/(1-\alpha)}, \tag{42}$$

where

$$a_1 = \left(\frac{A_1}{2A_2}\right)^{1/(1-\alpha)}; \; a_2 = \frac{4A_2}{A_1^2}.$$

Substituting eq. (42) into eq. (39) yields

$$\Delta t_{II} = \frac{a_1}{d} \int_{a_0}^{a_*} \left[1 - \sqrt{1 - a_2 \left(\frac{K_{Ic}}{K_I} - 1\right)}\right]^{1/(1-\alpha)} \mathrm{d}a =$$

$$= \frac{a_0}{d} a_1 \int_1^{\frac{a_*}{a_0}} \left[1 - \sqrt{1 - a_2 \left(\frac{K_{Ic}}{K_I} - 1\right)}\right]^{1/(1-\alpha)} \mathrm{d}\frac{a}{a_0}. \tag{43}$$

For the crack in an infinite plate,

$$K_I = P\sqrt{\pi a}. \tag{44}$$

From eq. (44), one can find that

$$a_* / a_0 = \left(K_{Ic} / K_{Ic}\right)^2, \tag{45}$$

where $K_I^0$ is an initial value of $K_I$. $K_I^0$ initial value of stress intensity factor for mode-I crack.

Substituting eq. (45) into eq. (43) and replacing $a/a_0 = \xi$ gives an equation to determine the duration of the main period of crack growth

$$\Delta t_{II} = \frac{aa_1}{d} \int_1^{\beta} \left[ 1 - \sqrt{1 - a_2 \left( \frac{B}{\xi} - 1 \right)} \right]^{1/(1-\alpha)} \mathrm{d}\xi; \quad B = \left( \frac{K_{Ic}}{K_I^0} \right). \tag{46}$$

A life term $T$ for the plate with a macrocrack is determined by the duration of the main period of the crack growth. Thus

$$T \approx \frac{a_0}{d} a_1 \int_1^{B} \left[ 1 - \sqrt{1 - a_2 \left( \frac{B}{\xi} - 1 \right)} \right]^{1/(1-\alpha)} \mathrm{d}\xi. \tag{47}$$

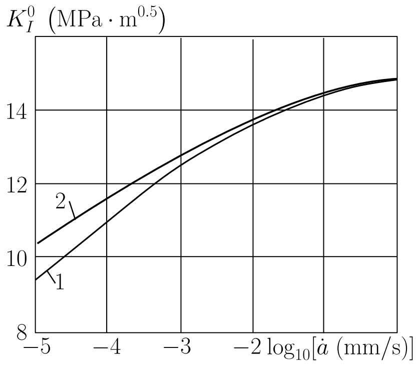

*Fig. 4*

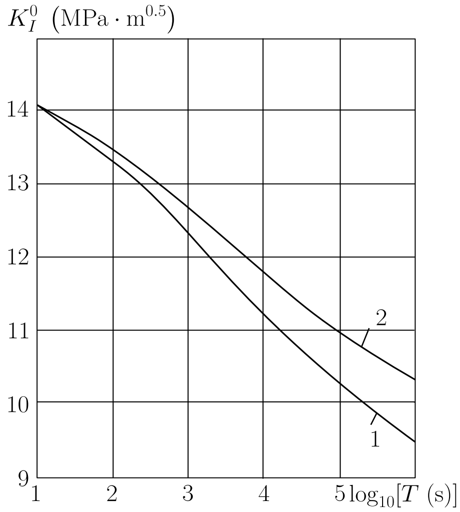

*Fig. 5*

For a composite material of tree layers longitudinal reinforcement by alumoborosilicate glass fabric TC-10 and polyethylene matrix with $E_{11}^0 = 19{,}7$ GPa; $E_{22}^0 = 11{,}7$ GPa; $\nu_{21}^0 = 0{,}14$; $G_{12}^0 = 0{,}637$ GPa; $\alpha = 0{,}717$; $\lambda_G = 0{,}1398\ \mathrm{s}^{\alpha-1}$; $\beta_G = -0{,}0407\ \mathrm{s}^{\alpha-1}$; $K_{Ic} = 15{,}4\mathrm{MPa} \cdot m^{0{,}5}$, $d = 0{,}59$ mm, a dependence $K_I(\dot{a})$ vs. $\dot{a}$ is given in fig. 4. Fig. 5 shows the plot of $K_I^0(T)$ $T$ life term of the cracked body for $2a_0 = 10$ mm ("1" and "2" are the first and second approximations obtained by the operator continued fractions method). It should be noted that the second and third approximations are indistinguishable on these figures (see Table) as a result of the quick convergence. Moreover, as it follows from [14], the exact value of $K_I(\dot{a})$ lies between any even and odd approximations. This is why the second approximation for the studied material gives almost precise coincidence with the exact solution.

| | | $\dot{a}$, mm/s |
|---|---|---|
| 2-nd approximation | 3-rd approximation | |
| 14,815 | 14,815 | 1 |
| 14,355 | 14,353 | 0,1 |
| 13,629 | 13,622 | 0,01 |
| 12,625 | 12,598 | 0,001 |
| 11,464 | 11,376 | 0,0001 |

**5. A crack in a viscoelastic body of cylindrical anisotropy.**

Consider fracture of viscoelastic body of axisymmetric (cylindrical) anisotropy with a penny-shaped crack of diameter $2a$ under uniform tension of intensity $p$ applied at infinity (fig. 6).

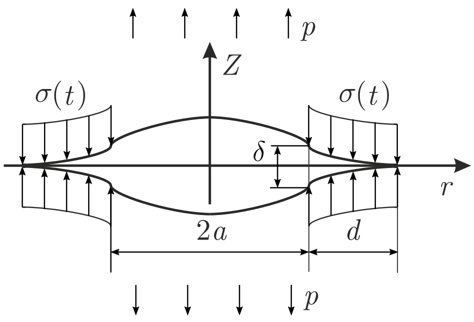

*Fig. 6*

The problem is solved in the axisymmetric statement, i.e., the material is cylindrically anisotropic [40] and the crack stays flat and circular through all the period of its growth. This holds true provided that the anisotropy axis is normal to the crack plane and goes through its center.

It is assumed that the material properties of the body can be described using eq. (3) and fracture properties $K_{Ic}$, $\delta_{Ic}$ do not depend on the location of the crack front and keep the same values during the life term of the bulk material.

Let a crack to be located in the isotropy plane of the transversely isotropic body at a large distance from its edges. It is assumed that the crack remains circular in its initial plane during its propagation. The body is loaded by distributed forces of intensity $p$ acting along the normal to the crack plane. In this case, the viscoelastic opening of the crack is described [23] by

$$\delta(r,t)=\tilde{T}^{*}\delta_0\left[p,r,a(t)\right];\ \tilde{T}^{*}=T^{*}/T^{0}, \tag{48}$$

where $\delta_0\left(p,r,a\right)=T^{0}a_0p\zeta\left(r/a_0\,,a/a_0\right)$ is the elastic opening displacement of a crack of radius $a(t)$ at a point $r$, $a_0=a(0)$; $\zeta(\xi,\eta)$ is a function expressed as follows

$$\zeta(\xi,\eta)=\frac{4}{\pi}\left(\frac{\eta}{\rho}+1\right)\left(2\frac{\eta}{\rho}+1\right)^{-1/2}\int\limits_{\arcsin\frac{\eta}{\eta+\rho}}^{\arcsin\frac{\eta}{\xi}}\sqrt{\eta^2-\xi^2\sin^2\alpha}\,\mathrm{d}\alpha, \tag{49}$$

for $0<\eta\le\xi\le\eta+\rho$; $\rho=d/a_0$ is the dimensionless width of the ring-shaped cohesive zone; $\tilde{T}^{*}$ is a function of integral operators $E_{ij}^{*}$ characterizing the viscoelastic properties of the material

$$\tilde{T}^{*}=F(E_{ij}^{*})=\sqrt{\frac{1-\nu_{rz}^{*}\nu_{zr}^{*}}{E_{rr}^{*}E_{zz}^{*}}}\sqrt{2\left[\sqrt{\frac{(1-\nu_{r\varphi}^{*2})(1-\nu_{rz}^{*}\nu_{zr}^{*})}{E_{rr}^{*}E_{zz}^{*}}}-(1+\nu_{r\varphi}^{*})\nu_{rz}^{*}\right]+\frac{E_{rr}^{*}}{G_{rz}^{*}}}, \tag{50}$$

where $E_{rr}^{*}$, $E_{zz}^{*}$, $G_{rz}^{*}$, $\nu_{r\varphi}^{*}$, $\nu_{rz}^{*}$, $\nu_{zr}^{*}$ are viscoelasticity operators.

Consider the case when viscoelastic properties of anisotropic material can be described by the resolvent Rabotnov operators [48] (it is usually enough to use just one operator for one property but as it is shown in [29] the method works for arbitrary number of terms in each operator)

$$\frac{1}{E_{rr}^{*}}=\frac{1}{E_{rr}^{0}}\left[1+\lambda_rR^{*}(\beta_r)\right];\ \frac{1}{E_{zz}^{*}}=\frac{1}{E_{zz}^{0}}\left[1+\lambda_zR^{*}(\beta_z)\right];$$

$$\frac{1}{G_{rz}^{*}}=\frac{1}{G_{rz}^{0}}\left[1+\lambda_GR^{*}(\beta_G)\right];\ \nu_{r\varphi}^{*}=\nu_{r\varphi}^{0}\left[1+\lambda_{r\varphi}R^{*}(\beta_{r\varphi})\right]; \tag{51}$$

$$\nu_{rz}^{*}=\nu_{rz}^{0}\left[1+\lambda_{rz}R^{*}(\beta_{rz})\right];\ \nu_{zr}^{*}=\nu_{zr}^{0}\left[1+\lambda_{zr}R^{*}(\beta_{zr})\right].$$

The rheological parameters $1/E_{rr}^0$, $1/E_{zz}^0$, $1/G_{rz}^0$, $\nu_{r\varphi}^0$, $\nu_{rz}^0$, $\nu_{zr}^0$, $\lambda_r$, $\beta_r$, $\lambda_z$, $\beta_z$, $\lambda_G$, $\beta_G$, $\lambda_{r\varphi}$, $\beta_{r\varphi}$, $\lambda_{rz}$, $\beta_{rz}$, $\lambda_{zr}$, $\beta_{zr}$ can be determined from creep tests [24].

It is possible to use the expansions from the earlier sections. To simplify things, it should be noted that if $\nu_{ij}^0\nu_{kl}^0 \ll 1$ it is possible to use an approximation $\nu_{ij}^\infty$ long-term values of viscoelastic moduli

$$\frac{\nu_{ij}^0+\nu_{ij}^\infty}{2}\frac{\nu_{kl}^0+\nu_{kl}^\infty}{2} \cong \nu_{ij}^*\nu_{kl}^*$$

where

$$\nu_{ij}^\infty = (\nu_{ij}^* \cdot 1)_{t\to\infty}.$$

Using expansion eq. (32) one can obtain that

$$F(E_{ij}^*) \approx F(E_{ij}^0)\left[1+\sum_{j=1}^{7}\omega_j R^*(\rho_j)\right], \qquad (52)$$

where

$$\rho_1 = \beta_r - 0,25\lambda_r;\ \rho_2 = \beta_z - 0,25\lambda_z;\ \rho_j = \beta'_{j-2}\,(j = 3\,...\,7);$$

$$\omega_1 = 0,5\lambda_r\left(1+\frac{0,5\lambda_z}{\rho_1-\rho_2}\right)\left(1+0,5\sum_{j=1}^{5}\frac{\lambda_j}{\rho_1-\beta'_j}\right);$$

$$\omega_2 = 0,5\lambda_r\left(1+\frac{0,5\lambda_r}{\rho_2-\rho_1}\right)\left(1+0,5\sum_{j=1}^{5}\frac{\lambda'_j}{\rho_2-\beta'_j}\right);$$

$$\omega_j = 0,5\lambda'_{j-2}\left(1+\frac{0,5\lambda_r}{\beta'_{j-2}-\rho_1}\right)\left(1+\frac{0,5\lambda_z}{\beta'_{j-2}-\rho_2}\right) \quad (j = 3\,...\,7).$$

$\beta'_j\,(j = 1\,...\,5)$ are the roots of

$$1-0,25\sum_{i=1}^{5}\frac{\lambda_i}{\beta_i-\beta'} = 0;$$

$\lambda'_j\,(j = 1\,...\,5)$ are the solution of

$$1-\sum_{j=1}^{5}\frac{\lambda_j}{\beta_i-\beta'_j} = 0 \quad (i = 1\,...\,5);$$

$$\beta_1 = \beta_2 - 0,25\lambda_z;\ \beta_2 = \beta_r;\ \beta_3 = \beta_r - \lambda_r;\ \beta_4 = \beta_r - 0,75\lambda_r;\ \beta_5 = \beta_G;$$

$$\lambda_1 = \lambda_z\left(1+\frac{0,5\lambda_r}{\beta_4-\beta_1}\right)\frac{\varkappa_1}{\varkappa};\ \lambda_2 = -2\lambda_{rz}\frac{\nu_{rz}^0}{\varkappa};\ \lambda_3 = -\lambda_r\left(1+\frac{\lambda_G}{\beta_3-\beta_5}\right)\frac{\varkappa_2}{\varkappa};$$

$$\lambda_4 = -\lambda_r\left(1+\frac{0,5\lambda_z}{\beta_4-\beta_1}\right)\frac{\varkappa_1}{\varkappa};\ \lambda_5 = \lambda_G\left(1+\frac{\lambda_r}{\beta_3-\beta_5}\right)\frac{\varkappa_2}{\varkappa};$$

$$\nu = \nu_{r\varphi}\nu_{rz} + \nu_{rz}^0;\ \varkappa_1 = \sqrt{(1-\nu_{r\varphi}^2)(1-\nu_{rz}\nu zr)\frac{E_{rr}^0}{E_{zz}^0}};\ \varkappa_2 = \frac{E_{rr}^0}{G_{rz}^0};\ \varkappa = 2(\varkappa_1-\nu)+\varkappa_2.$$

Then, the opening displacement can be written as

$$\delta(r,t) = F(E_{ij}^0)\left[ q(r,t) + \int_0^t \sum_{j=1}^{7} \omega_j R(t-\tau,\rho_j) q(r,\tau) rmd\tau \right], \tag{53}$$

where $R(t-\tau,\rho_j)$ are the kernels of $R^*(\rho_j)$, $j=1..7$,

$$q(r,t) = \frac{2}{\pi}\sigma(t)\left\{ 2\sqrt{d[d-r+a(t)]} + (r-a(t))\ln\frac{\sqrt{d}-\sqrt{d-r+a(t)}}{\sqrt{d}+\sqrt{d-r+a(t)}} \right\}.$$

For the macrocrack, it yields

$$\frac{K_{Ic}}{K_I} = 1 + \frac{d}{\dot{a}}\int_0^1 \sum_{j=1}^{7} \omega_j R\left(\frac{d}{\dot{a}}s,\rho_j\right) F(s)\mathrm{d}s. \tag{54}$$

After some trivial integration by $s$, eq. (54) it can be simplified to

$$\frac{K_{Ic}}{K_I} = 1 + \sum_{n=0}^{\infty} c_n(\alpha,\omega_j,\rho_j)\left(\frac{d}{\dot{a}}\right)^{(n+1)(1-\alpha)}, \tag{55}$$

where

$$c_n(\alpha,\omega_j,\rho_j) = \frac{0{,}5\sqrt{\pi}\sum_{j=1}^{7}\omega_j \rho_j^n}{\left[(n+1)(1-\alpha)+1\right]\Gamma\left[(n+1)(1-\alpha)+1{,}5\right]}.$$

Taking into account only two first terms in the series in eq. (55), it is possible to find the duration of the main period of subcritical growth of the crack [17]. As it is shown using the more precise solutions [31], this duration is very close to the life term of the bulk body, $T$. Thus

$$T \approx \frac{a_0}{d}\left(\frac{c_0}{2c_1}\right)^{1/(1-\alpha)} \int_0^{\left[(K_{Ic})/(K_I^0)\right]^2} \left[1 - \sqrt{1 - \frac{4c_1}{c_0^2}\left(\frac{K_{Ic}}{K_I^0}\frac{1}{\sqrt{z}} - 1\right)}\right]^{1/(1-\alpha)} \mathrm{d}z; \tag{56}$$

For the numerical example, experimentally determined properties for a polymeric visco-elastic composite [24] were used

$$E_{rr}^0 = 15{,}3\,\text{GPa};\ \lambda_r = 0{,}0202\,\text{s}^{(\alpha-1)};\ \beta_r = -0{,}0835\,\text{s}^{(\alpha-1)};$$
$$E_{zz}^0 = 10{,}8\,\text{GPa};\ \lambda_r = 0{,}0501\,\text{s}^{(\alpha-1)};\ \beta_r = -0{,}1011\,\text{s}^{(\alpha-1)};$$
$$G_{rz}^0 = 5{,}28\,\text{GPa};\ \lambda_r = 0{,}0985\,\text{s}^{(\alpha-1)};\ \beta_r = -0{,}0281\,\text{s}^{(\alpha-1)};$$

$$\nu_{rz} = 0{,}13;\ \nu_{zr} = 0{,}092;\ \nu_{r\varphi} = 0{,}11;\ \alpha = 0{,}75;\ K_{Ic} = 8{,}97\,\text{GPa}\cdot\text{m}^{0.5};\ d = 0{,}65\,\text{mm};$$

Fig. 7 shows the kinetic curve of the subcritical crack growth and fig. 8 can be used to determine the life term by the given initial value of SIF.

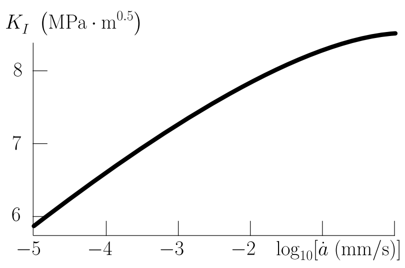


*Fig. 7*

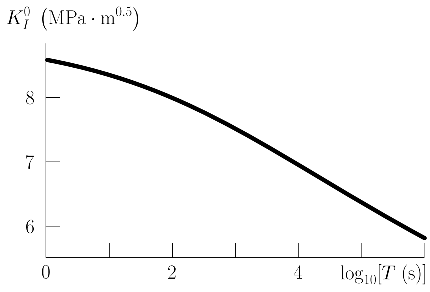


*Fig. 8*

It is worth to note that the operator continued fraction technique allows to solve the following classes of the fracture problems for the linear viscoelastic materials, in particular for composite materials

• the problems for through-thick cracks (mode-I/II/III and mixed mode) for small and non-small cohesive zones under the constant and slowly changing loads [4, 18, 19, 23, 24, 30, 32, 33, 35, 60];

• the problems for internal mode-I penny-shaped cracks for small and non-small cohesive zones in anisotropic bulk bodies under the constant and slowly changing tensile loads [24, 27, 31, 34,];

• the problems for wedging of cracks [9, 22].

**Results and conclusions.**

1. The operator continued fraction technique allows to get rid of one of the shortcom-ings of the analytical method of Boltzmann-Volterra theory, i.e. ill-convergence of the power series of Volterra integral operators. This shortcoming prevents using this analytical method to obtain effective solutions for the problems of subcritical crack growth in anisotropic viscoelastic materials and other problems of anisotropic viscoelastic body deformation.

2. It should be noted that this technique allows obtaining semi-analytical formulae with a manageable precision (it is shown that the second approximation can give an appropriate result in many cases) for practical engineering applications in the complex multi-parameter problems.

3. It is not the only advantage of the operator continued fractions technique when compared with Volterra's proposal to determine function of integral operators as a power series, that the continued fractions converge faster but that the expansion of the function into the continued fraction is more general as it can be applied to the non-analytical (meromorphic) functions as well as to the analytical (holomorphic) ones. In particular, this is important for multiply-connected domains when there are many cracks voids or inclusions in the body. This can potentially be used to study micromechanics of composite deformation and fracture.

4. The operator continued fraction methods allows investigation in case of the more complex crack models (see [25, 26, 56, 58, 61, 62]).

5. The technique that is presented in this survey can be used to solve various problems of composite long-term fracture (provided that the conditions of composite homogenization hold true) for various reinforcement patterns and various viscoelastic properties. The solution obtained can be used for the theoretical prediction of life time and crack growth resistance of composite structural elements.

Наукові дослідження, результати яких опубліковано в даній статті, виконано за рахунок бюджетної програми «Підтримка пріоритетних напрямків наукових досліджень» (КПКВК 6541230).

РЕЗЮМЕ. Висвітлено методику розв'язування задач лінійної в'язко-пружності. Наведено основи застосування ланцюгових дробів до розв'язування деяких задач щодо повільного поширення тріщин у в'язкопружних анізотропних тілах. На прикладах показано, що метод операторних ланцюгових дробів можна ефективно застосовувати до розв'язання складних задач механіки руйнування для сучасних анізотропних композитних матеріалів.

КЛЮЧОВІ СЛОВА: лінійно в'язкопружне середовище, анізотропія, руйнування, тріщина, довговічність, інтегральний оператор, ланцюговий дріб.

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