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THE INFLUENCE OF THE INHOMOGENEOUS RESIDUAL STRESSES ARISING FROM THE CONTACT OF THE CUT ON THE DISPERSION OF AXISYMMETRIC LONGITUDINAL WAVES IN THE TWO-LAYER HOLLOW CYLINDER

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Abstract. The paper deals with the study of the influence of the inhomogeneous residual stresses, caused by the «contact» of the cut, on the dispersion of the axisymmetric longitudinal waves in the two-layer hollow cylinder. According to the classification of Guz, this study is carried out using the second version of the small initial deformation of the linearized 3D theory of elastic waves in bodies with initial stresses. The discrete analytical method developed by the authors is used to solve the corresponding field equations. Numerical results illustrating the influence of residual stresses on the dispersion curves are presented and discussed. The magnitude of the residual stresses is estimated by the intersection (cutting) angle as a result of contacting, through which the residual stresses occur in the layers of the cylinder.

Key words: non-homogeneous residual stresses, wave dispersion, two-layer hollow cylinder, bandgap, axisymmetric longitudinal waves.

Introduction.

To indicate the place and significance of the present work among other related works, we try to distinguish the terms «initial stresses» and «residual stresses» from each other. According to references [1 – 5] and many others listed there, by «initial stresses» we mean the stresses generated by the action of the external load (let us call it «initial load») on the body before the «additional loads» act which caused small disturbances in it. Throughout the processing that takes place in the body as a result of the action of the additional loads, the action of the «initial load» on the body continues. However, «residual» stresses are understood as the stresses that occur in the body (composites) as a result of the various technological processes, and in the study of the problems associated with these cases, there are no «initial» and «additional» load cases. However, some parameters or characteristic values are introduced for the quantitative estimation of the mentioned residual stresses. Note that the various aspects of the occurrence of initial and residual stresses in the composites during the generation of these stresses and the influence of those on the mechanical behavior of the structural elements are studied in many investigations, such as in [3, 5 – 8] and others listed there.

One type of residual stress state in the cylinders appears as a result of the following procedures. It is assumed that the cylinder has a longitudinal cut and this cut is closed by applying some external forces or moments and is «contacted» through welding or in other ways. After contacting, the external forces and moments are removed and as a result of these procedures, the cylinder has axisymmetric inhomogeneous residual stresses, the values of which are determined through the expressions given in the monograph [9]. Note that such contacting processes are used in creating thick-walled layered cylinders which are used in various branches of modern industry, especially in oil platforms used for exploration and oil production from the sea and ocean shelves. Creating a scientific fundament for applying non-destructive defect-detection methods through ultrasonic waves in these cases requires a fundamental theoretical study of how these inhomogeneous residual stresses influence the wave dispersion propagated in such layered cylinders. The subject of the present paper is devoted namely to such problems and to the study of the influence of the foregoing type of axisymmetric inhomogeneous residual stresses on the dispersion of the axisymmetric longitudinal wave in the two-layered hollow cylinders within the scope of the so-called 3D linearized theory of elastic waves in bodies with initial stresses.

We now briefly review recent studies on the dispersion of waves propagating in layered hollow cylinders with inhomogeneous initial stresses. The overview and detailed consideration of the corresponding investigations for the case when the mentioned initial stresses are homogeneous can be found in the works [3, 5 – 8] and other works listed therein. We begin this review with the work [10], in which the longitudinal axisymmetric wave dispersion in a hollow cylinder is studied. The cylinder is assumed to be made of an incompressible, highly elastic, functionally graded material. It is assumed that the elasticity relations of the cylinder material are described by the Mooney – Rivlin potential and that in the initial state, the inner and outer surfaces of the cylinder are subjected to the action of hydrostatic pressures. It should be noted that in the work [10], to determine the initial stresses in the considered cylinder, the solution of the corresponding static problems studied in the works [11, 12] is used. Moreover, in the work [10] the equations of the linearized 3D theory are solved using the state space formalism together with the approximate laminate or multilayer technique.

Until recently, there have been no studies on wave dispersion in a two-layer hollow cylinder with inhomogeneous initial stresses, and the study carried out in the paper [13] can be considered the first attempt in this field. Note that in this work, as in the work [10], it is assumed that the initial axisymmetric inhomogeneous stresses are caused by the action of the internal and external hydrostatic pressures, and the influence of these initial stresses on the dispersion of the axisymmetric longitudinal waves propagating in this cylinder is studied. However, unlike the work [10], the investigations in the work [13] are carried out within the framework of the second version of the initial small deformation version of the linearized 3D theory of elastic waves in bodies with initial stresses [3], and the equations of this theory are solved using the discrete-analytic method.

In the paper [14] the problem of axisymmetric wave propagation was investigated for the case when the hollow cylinder is surrounded by an infinite elastic medium and this medium is compressed in the initial state (i.e. before the wave propagation) by the uniformly distributed normal forces acting radially inward at infinity.

The influence of the initial inhomogeneous thermal stresses in the two-layer hollow cylinders on the dispersion of the axisymmetric torsional waves propagating in these cylinders is studied in the paper [15].

From the preceding brief review, it is evident that up to now, all the investigations have been concerned with the case where the inhomogeneous initial stresses are present in the two-layered cylinders, and these initial stresses are caused by the external hydrostatic pressure which acts during all procedure of the wave propagation. In this context, the present work is the first attempt in which the case is considered in which inhomogeneous residual stresses are assumed to be present in the two-layer hollow cylinder. The influence of these residual stresses on the dispersion of the axisymmetric longitudinal waves propagating in this cylinder is studied for the first time.

1. Formulation of the problem.

Let us assume that the inner and outer layers of the cylinder occupied the regions $R \le r \le R + h^{(2)}$ and $R + h^{(2)} \le r \le R + h^{(2)} + h^{(1)}$ respectively, in the cylindrical coordinate system $Or\theta z$ associated with the central axis of the cylinder and in the natural state each layer has a longitudinal cut marked by the central angles $\alpha^{(1)}$ and $\alpha^{(2)}$, respectively. Let us assume that $\alpha^{(1)} \ll 1$ and $\alpha^{(2)} \ll 1$. After removing the cuts by touching their ends, we obtain the hollow cylinders with residual stresses and according to the monograph [9], these residual stresses are calculated by the use of the following expressions.

In the outer layer of the cylinder.

$$\sigma_{rr}^{(1)0} = -4\frac{M^{(1)}}{N^{(1)}}\left[(R+h^{(2)})^2 (R+h^{(2)}+h^{(1)})^2 \frac{1}{r^2}\log\left(\frac{(R+h^{(2)}+h^{(1)})}{(R+h^{(2)})} \right) + \right.$$

$$\left. +(R+h^{(2)}+h^{(1)})^2 \log\left(\frac{r}{(R+h^{(2)}+h^{(1)})} \right) + (R+h^{(2)})\log\left(\frac{r}{(R+h^{(2)})} \right) \right]; \qquad (1)$$

$$\sigma_{\theta\theta}^{(1)0} = -4\frac{M^{(1)}}{N^{(1)}}\left[-(R+h^{(2)})^2 (R+h^{(2)}+h^{(1)})^2 \frac{1}{r^2}\log\left(\frac{(R+h^{(2)}+h^{(1)})}{r} \right) + (R+h^{(2)}+h^{(1)})^2 \times \right.$$

$$\left. \log\left(\frac{r}{(R+h^{(2)}+h^{(1)})} \right) + (R+h^{(2)})^2 \log\left(\frac{r}{(R+h^{(2)})} \right) + (R+h^{(2)}+h^{(1)})^2 - (R+h^{(2)})^2 \right];$$

$$\sigma_{zz}^{(1)0} = \nu^{(1)}(\sigma_{rr}^{(1)0} + \sigma_{\theta\theta}^{(1)0}) .$$

In the inner layer of the cylinder.

$$\sigma_{rr}^{(2)0} = -4\frac{M^{(2)}}{N^{(2)}}\left[R^2 (R+h^{(2)})^2 \frac{1}{r^2}\log\left(\frac{(R+h^{(2)})}{R} \right) + \right.$$

$$\left. +(R+h^{(2)})^2 \log\left(\frac{r}{(R+h^{(2)})} \right) + R^2 \log\left(\frac{r}{R} \right) \right] R^2 \log\left(\frac{r}{R} \right);$$

$$\sigma_{\theta\theta}^{(2)0} = -4\frac{M^{(2)}}{N^{(2)}}\left[-R^2 (R+h^{(2)})^2 \frac{1}{r^2}\log\left(\frac{(R+h^{(2)})}{r} \right) + \right.$$

$$\left. +(R+h^{(2)})^2 \log\left(\frac{r}{(R+h^{(2)})} \right) + R^2 \log\left(\frac{r}{R} \right) + (R+h^{(2)})^2 - R^2 \right]; \qquad (2)$$

$$\sigma_{zz}^{(2)0} = \nu^{(2)}(\sigma_{rr}^{(2)0} + \sigma_{\theta\theta}^{(2)0}) .$$

In expressions (1) and (2) the following notation is used:

$$M^{(1)} = -\frac{\alpha^{(1)}\mu^{(1)}}{4\pi(1-\nu^{(1)})}\left(\left((R+h^{(2)}+h^{(1)})^2 - (R+h^{(2)})^2 \right)^2 - 4(R+h^{(2)}+h^{(1)})^2 \times \right.$$

$$\left. \times (R+h^{(2)})^2 \left(\log\frac{(R+h^{(2)}+h^{(1)})}{(R+h^{(2)})} \right)^2 \right) \frac{1}{2\left((R+h^{(2)}+h^{(1)})^2 - (R+h^{(2)})^2 \right)};$$

$$N^{(1)} = \left((R+h^{(2)}+h^{(1)})^2 - (R+h^{(2)})^2 \right)^2 -$$

$$-4(R+h^{(2)}+h^{(1)})^2(R+h^{(2)})^2\left(\log\frac{(R+h^{(2)}+h^{(1)})}{(R+h^{(2)})}\right)^2;$$

$$M^{(2)}=-\frac{\alpha^{(2)}\mu^{(2)}}{4\pi(1-\nu^{(2)})}\left(\left((R+h^{(2)})^2-R^2\right)^2-4(R+h^{(2)})^2\times\right.$$

$$\left.\times R^2\left(\log\frac{(R+h^{(2)})}{R}\right)^2\right)\frac{1}{2\left((R+h^{(2)})^2-R^2\right)};$$

$$N^{(2)}=\left((R+h^{(2)})^2-R^2\right)^2-4(R+h^{(2)})^2R^2\left(\log\frac{(R+h^{(2)})}{R}\right)^2. \tag{3}$$

In relations (1), (2) and (3), and below, the upper index (2) (the upper index (1)) is used to indicate the corresponding quantities belonging to the inner (outer) layer of the cylinder shown and upper index 0 means that the stresses are the residual ones. Moreover, in (1), (2) and (3), the following notation is used: $\nu^{(n)}$ is Poisson's ratio, $\mu^{(n)}$ and $\lambda^{(n)}$ are the Lame constants of the n-th $(n=1,2)$ material.

Thus, we seek to investigate how the residual stresses acting in the two-layer hollow cylinder compounded from the aforementioned hollow cylinders and determined by expressions (1), (2), and (3) affect the dispersion of axially symmetric longitudinal waves propagating in the two-layer cylinder. Assuming that the materials of the layers are moderately rigid, we perform this investigation in the framework of the second version of the small initial deformation of the 3D linearized theory of elastic waves in bodies with initial stresses [3]. For this purpose, we use the aforementioned cylindrical $Or\theta z$ coordinate systems associated with the central axis of the cylinder under consideration and determine the position of the points of this cylinder with the Lagrange coordinates in this coordinate system.

Let us write, according to the reference [3], the complete system of equations and relations of the second version of the small initial deformation theory of the 3D linearized theory of elastic waves in bodies with initial stresses.

The equations of motion:

$$\frac{\partial t_{rr}^{(m)}}{\partial r}+\frac{\partial t_{zr}^{(m)}}{\partial z}+\frac{1}{r}(t_{rr}^{(m)}-t_{\theta\theta}^{(m)})=\rho^{(m)}\frac{\partial^2 u_r^{(m)}}{\partial t^2};$$

$$\frac{\partial t_{rz}^{(m)}}{\partial r}+\frac{1}{r}t_{rz}^{(m)}+\frac{\partial t_{zz}^{(m)}}{\partial z}=\rho^{(m)}\frac{\partial^2 u_z^{(m)}}{\partial t^2},\quad m=1,2, \tag{4}$$

where

$$t_{rr}^{(m)}=\sigma_{rr}^{(m)}+\sigma_{rr}^{(m)0}(r)\frac{\partial u_r^{(m)}}{\partial r};\quad t_{rz}^{(m)}=\sigma_{rz}^{(m)}+\sigma_{rr}^{(m)0}(r)\frac{\partial u_z^{(m)}}{\partial r};$$

$$t_{\theta\theta}^{(m)}=\sigma_{\theta\theta}^{(m)}+\sigma_{\theta\theta}^{(m)0}(r)\frac{u_r^{(m)}}{r};\quad t_{zr}^{(m)}=\sigma_{zr}^{(m)}+\sigma_{zz}^{(m)0}(r)\frac{\partial u_r^{(m)}}{\partial z}; \tag{5}$$

$$t_{zz}^{(m)}=\sigma_{zz}^{(m)}+\sigma_{zz}^{(m)0}(r)\frac{\partial u_z^{(m)}}{\partial z}.$$

The elasticity relations and the relations between the strains and displacements:

$$\sigma_{(ij)}^{(m)}=\lambda^{(m)}(\varepsilon_{rr}^{(m)}+\varepsilon_{\theta\theta}^{(m)}+\varepsilon_{zz}^{(m)})+2\mu^{(n)}\varepsilon_{(ij)}^{(m)};\quad (jj)=rr;\,\theta\theta;\,zz;\quad \sigma_{rz}^{(m)}=2\mu^{(m)}\varepsilon_{rz}^{(m)}; \tag{6}$$

$$\varepsilon_{rr}^{(m)} = \frac{\partial u_r^{(m)}}{\partial r}; \quad \varepsilon_{\theta\theta}^{(m)} = \frac{u_r^{(m)}}{r}; \quad \varepsilon_{zz}^{(m)} = \frac{\partial u_z^{(m)}}{\partial z}; \quad \varepsilon_{rz}^{(m)} = \frac{1}{2}\left(\frac{\partial u_r^{(m)}}{\partial z} + \frac{\partial u_z^{(m)}}{\partial r}\right). \tag{7}$$

Equations (4) – (7) are the complete system of field equations in the framework of which the present investigations are carried out. Note that the conventional notation is used in these equations.

We add to these equations the following boundary and contact conditions

$$t_{rr}^{(1)}\Big|_{r=R+h^{(1)}+h^{(2)}} = 0; \quad t_{rz}^{(1)}\Big|_{r=R+h^{(1)}+h^{(2)}} = 0; \quad t_{rr}^{(1)}\Big|_{r=R+h^{(1)}} = t_{rr}^{(2)}\Big|_{r=R+h^{(1)}};$$

$$t_{rz}^{(1)}\Big|_{r=R+h^{(1)}} = t_{rz}^{(2)}\Big|_{r=R+h^{(1)}}; \quad u_r^{(1)}\Big|_{r=R+h^{(1)}} = u_r^{(2)}\Big|_{r=R+h^{(1)}}; \tag{8}$$

$$u_z^{(1)}\Big|_{r=R+h^{(1)}} = u_z^{(2)}\Big|_{r=R+h^{(1)}}; \quad t_{rr}^{(2)}\Big|_{r=R} = 0, \; t_{rz}^{(2)}\Big|_{r=R} = 0.$$

This completes the formulation of the problem.

2. Method of solution.

For solution to the formulated problem we attempt to employ the discrete-analytical method developed and employed in the papers [13 – 15], according to which, the intervals $[R, R+h^{(2)}]$ and $[R+h^{(2)}, R+h^{(2)}+h^{(1)}]$ are divided into N_2 and N_1 numbers of subintervals or sublayers, respectively. The thickness of the sublayers of the region $[R, R+h^{(2)}]$ is equal to $h^{(2)}/N_2$ and in the n_2-th sub-layer, the relation $(R+(n_2-1)h^{(2)}/N_2) \le r \le (R+n_2 h^{(2)}/N_2)$ takes place, where $1 \le n_2 \le N_2$. We can also conclude that the thickness of the sublayers of the region $[R+h^{(2)}, R+h^{(2)}+h^{(1)}]$ is equal to $h^{(1)}/N_1$ and in the n_1-th sub-layer, the relation

$$(R+h^{(2)}+(n_1-1)h^{(1)}/N_1) \le r \le (R+h^{(2)}+n_1 h^{(1)}/N_1)$$

takes place, where $1 \le n_1 \le N_1$. Thus, after the foregoing division, the inhomogeneous initial stresses, determined by the expressions (1), (2) and (3) in each of the foregoing sublayers, are taken as constants, the values of which are determined by the following relations:

In the n_2-th sub-layer

$$\sigma_{rr}^{(2)0}(r) \approx \sigma_{rr}^{(2)0}(r_{n_2}); \quad \sigma_{\theta\theta}^{(2)0}(r) \approx \sigma_{\theta\theta}^{(2)0}(r_{n_2}); \quad \sigma_{zz}^{(2)0}(r) \approx \sigma_{zz}^{(2)0}(r_{n_2});$$

$$r_{n_2} = R+(n_2-1)h^{(2)}/N_2 + h^{(2)}/(2N_2). \tag{9}$$

In the n_1-th sub-layer

$$\sigma_{rr}^{(1)0}(r) \approx \sigma_{rr}^{(1)0}(r_{n_1}); \quad \sigma_{\theta\theta}^{(1)0}(r) \approx \sigma_{\theta\theta}^{(1)0}(r_{n_1}); \quad \sigma_{zz}^{(1)0}(r) \approx \sigma_{zz}^{(1)0}(r_{n_1});$$

$$r_{n_1} = R+h^{(2)}+(n_1-1)h^{(1)}/N_1 + h^{(1)}/(2N_1). \tag{10}$$

Below, we use the upper indices $(2)\,n_2$ and $(1)\,n_1$ for indicating the values belonging to the inner and outer layers of the cylinder, respectively.

The above division of the intervals $[R, R+h^{(2)}]$ and $[R+h^{(2)}, R+h^{(2)}+h^{(1)}]$ into subintervals requires the formulation of the contact conditions between these subintervals and the paraphrasing of the corresponding boundary conditions on the sub-interfaces with the new notation for the upper indices. We assume that the mentioned contact conditions are perfect, and in order to reduce the size of the paper, the mathematical expressions for them are not written here. Although, these expressions are explicitly presented in the papers [13, 14]. By direct verification, it is found that the number of contact and boundary conditions mentioned is equal to $4(N_1+N_2)$ and, as in the investigations in the work [13], the concrete values of the numbers N_1 and N_2 are determined from the convergence requirement of the numerical results.

Thus, if we take the assumptions formulated in (9) and (10) and substitute the expressions in (5) into the equations (4), we obtain the following equations of motion which are satisfied within each sublayer.

$$\frac{\partial\sigma_{rr}^{(m)n_m}}{\partial r}+\sigma_{rr}^{(m)0}(r_{n_m})\frac{\partial^2 u_r^{(m)n_m}}{\partial r^2}+\frac{\partial\sigma_{zr}^{(m)n_m}}{\partial z}+\sigma_{zz}^{(m)0}(r_{n_m})\frac{\partial^2 u_r^{(m)n_m}}{\partial z^2}+\frac{1}{r}(\sigma_{rr}^{(m)n_m}-\sigma_{\theta\theta}^{(m)n_m})+$$

$$+\sigma_{rr}^{(m)0}(r_{n_m})\frac{1}{r}\frac{\partial u_r^{(m)n_m}}{\partial r}-\sigma_{\theta\theta}^{(m)0}(r_{n_m})\frac{u_r^{(m)n_m}}{r^2}=\rho^{(m)}\frac{\partial^2 u_r^{(m)n_m}}{\partial t^2};$$

$$\frac{\partial\sigma_{rz}^{(m)n_m}}{\partial r}+\sigma_{rr}^{(m)0}(r_{n_m})\frac{\partial^2 u_z^{(m)n_m}}{\partial r^2}+\frac{1}{r}\sigma_{rz}^{(m)n_m}+\sigma_{rr}^{(m)0}(r_{n_m})\frac{1}{r}\frac{\partial u_z^{(m)n_m}}{\partial r}+\frac{\partial\sigma_{zz}^{(m)n_m}}{\partial z}+ \tag{11}$$

$$+\sigma_{zz}^{(m)0}(r_{n_m})\frac{\partial^2 u_z^{(m)n_m}}{\partial z^2}=\rho^{(m)}\frac{\partial^2 u_z^{(m)n_m}}{\partial t^2},\quad m=1,\,2\,.$$

At the same time, within each sublayer, the relations in (6) and (7) remain valid without change, and in this way we obtain the complete system of equations consisting of (11), (6), and (7). As in the paper [13], we use the classical Lame decomposition (see, e.g., the monograph [2]) to solve these equations, which can be presented for the axisymmetric problems as follows.

$$u_r^{(m)n_m}=\frac{\partial\Phi^{(m)n_m}}{\partial r}+\frac{\partial^2\Psi^{(m)n_m}}{\partial r\partial z};\quad u_z^{(m)n_m}=\frac{\partial\Phi^{(m)n_m}}{\partial z}-\frac{\partial^2\Psi^{(m)n_m}}{\partial r^2}-\frac{\partial\Psi^{(m)n_m}}{r\partial r}\,. \tag{12}$$

By the usual procedure, we obtain the following equations for the potentials $\Phi^{(m)n_m}$ and $\Psi^{(m)n_m}$ from equations (11), (6) and (7).

$$\left(1+\frac{\sigma_{rr}^{(m)0}(r_{n_m})}{\lambda^{(m)}+2\mu^{(m)}}\right)\frac{\partial^2\Phi^{(m)n_m}}{\partial r^2}+\left(1+\frac{\sigma_{\theta\theta}^{(m)0}(r_{n_m})}{\lambda^{(m)}+2\mu^{(m)}}\right)\frac{\partial\Phi^{(m)n_m}}{r\partial r}+$$

$$+\left(1+\frac{\sigma_{zz}^{(m)0}(r_{n_m})}{\lambda^{(m)}+2\mu^{(m)}}\right)\frac{\partial^2\Phi^{(m)n_m}}{\partial z^2}=\frac{1}{(c_1^{(m)})^2}\frac{\partial^2\Phi^{(m)n_m}}{\partial t^2};$$

$$\left(1+\frac{\sigma_{rr}^{(m)0}(r_{n_m})}{\mu^{(m)}}\right)\frac{\partial^2\Psi^{(m)n_m}}{\partial r^2}+\left(1+\frac{\sigma_{\theta\theta}^{(m)0}(r_{n_m})}{\mu^{(m)}}\right)\frac{\partial\Psi^{(m)n_m}}{r\partial r}+ \tag{13}$$

$$+\left(1+\frac{\sigma_{zz}^{(m)0}(r_{n_m})}{\mu^{(m)}}\right)\frac{\partial^2\Psi^{(m)n_m}}{\partial z^2}=\frac{1}{(c_2^{(m)})^2}\frac{\partial^2\Psi^{(m)n_m}}{\partial t^2},$$

where $c_1^{(m)}=\sqrt{(\lambda^{(m)}+2\mu^{(m)})/\rho^{(m)}}$ and $c_2^{(m)}=\sqrt{\mu^{(m)}/\rho^{(m)}}$ are the speed of dilatation and distortion wave propagation velocities, respectively in the m-th material.

As we consider the harmonic wave propagation along the Oz axis, we can present the sought functions $\Phi^{(m)n_m}$, $u_r^{(m)n_m}$, $\sigma_{rr}^{(m)n_m}$, $\sigma_{\theta\theta}^{(m)n_m}$ and $\sigma_{zz}^{(m)n_m}$ with the multiplying $\cos(kz-\omega t)$ as well the sought functions $\Psi^{(m)n_m}$, $u_z^{(m)n_m}$ and $\sigma_{rz}^{(m)n_m}$ with the multiplying $\sin(kz-\omega t)$ and, by denoting the amplitudes of these quantities with the same symbols, we obtain the equation:

$$\frac{d^2\Phi^{(m)n_m}}{d(r_2^{(m)})^2}+\frac{\alpha_1^{(m)}(r_{n_m})}{r_2^{(m)}}\frac{d\Phi^{(m)n_m}}{dr_2^{(m)}}+\Phi^{(m)n_m}=0\,;$$

$$\frac{d^2\Psi^{(m)n_m}}{d(r_1^{(m)})^2}+\frac{\alpha^{(m)}(r_{n_m})}{r_1^{(m)}}\frac{d\Psi^{(m)n_m}}{dr_1^{(m)}}+\Psi^{(m)n_m}=0 \qquad (14)$$

for the amplitudes $\Phi^{(m)n_m}$ and $\Psi^{(m)n_m}$, where

$$\alpha^{(m)}(r_{n_m})=\frac{1+\sigma_{\theta\theta}^{(m)0}(r_{n_m})/\mu^{(m)}}{1+\sigma_{rr}^{(m)0}(r_{n_m})/\mu^{(m)}};\quad \beta^{(m)}(r_{n_m})=\frac{1+\sigma_{zz}^{(m)0}(r_{n_m})/\mu^{(m)}}{1+\sigma_{rr}^{(m)0}(r_{n_m})/\mu^{(m)}}\,;$$

$$r_1^{(i)n_i}=kr\sqrt{\frac{c^2}{(c_2^{(i)})^2(1+\sigma_{rr}^{(i)0}(r_{n_i})/\mu^{(i)})}-(\beta^{(i)}(r_{n_i}))^2}\,,\ c=\omega/\kappa\,; \qquad (15)$$

$$\alpha_1^{(m)}(r_{n_m})=\frac{1+\sigma_{\theta\theta}^{(m)0}(r_{n_m})/(\lambda^{(m)}+2\mu^{(m)})}{1+\sigma_{rr}^{(m)0}(r_{n_m})/(\lambda^{(m)}+2\mu^{(m)})};\quad \beta_1^{(m)}(r_{n_m})=\frac{1+\sigma_{zz}^{(m)0}(r_{n_m})/(\lambda^{(m)}+2\mu^{(m)})}{1+\sigma_{rr}^{(m)0}(r_{n_m})/(\lambda^{(m)}+2\mu^{(m)})};$$

$$r_2^{(m)n_m}=kr\sqrt{\frac{c^2}{(c_1^{(m)})^2(1+\sigma_{rr}^{(m)0}(r_{n_m})/(\lambda^{(m)}+2\mu^{(m)}))}-(\beta_1^{(m)}(r_{n_m}))^2}\,.$$

According to [16], the solution to the equations in (15) are found as follows.

$$\Phi^{(m)n_m}=A_1^{(m)n_m}(r_2^{(m)})^{\gamma_1^{(m)}(r_{n_m})}E_{\gamma_1^{(m)}(r_{n_m})}(r_2^{(m)n_m})+A_2^{(m)n_m}(r_2^{(m)})^{\gamma_1^{(m)}(r_{n_m})}F_{\gamma_1^{(m)}(r_{n_m})}(r_2^{(m)n_m})\,; \quad (16)$$

$$\Psi^{(m)n_m}=B_1^{(m)n_m}(r_1^{(m)})^{\gamma^{(m)}(r_{n_m})}E_{\gamma^{(m)}(r_{n_m})}(r_1^{(m)n_m})+B_2^{(m)n_m}(r_1^{(m)})^{\gamma^{(m)}(r_{n_m})}F_{\gamma^{(m)}(r_{n_m})}(r_1^{(m)n_m})\,, \quad (17)$$

where

$$\gamma_1^{(m)}(r_{n_m})=\left(1-\alpha_1^{(m)}(r_{n_m})\right)/2\,;\quad \gamma^{(m)}(r_{n_m})=(1-\alpha^{(m)}(r_{n_m}))/2\,;$$

$$E_{\gamma_1^{(m)}(r_{n_m})}(r_2^{(m)n_m})=\begin{cases}J_{\gamma_1^{(m)}(r_{n_m})}(r_2^{(m)n_m}), \text{ if } (r_2^{(m)n_m})^2/r^2>0;\\ I_{\gamma_1^{(m)}(r_{n_m})}(r_2^{(m)n_m}), \text{ if } (r_2^{(m)n_m})^2/r^2<0;\end{cases}$$

$$F_{\gamma_1^{(m)}(r_{n_m})}(r_2^{(m)n_m})=\begin{cases}Y_{\gamma_1^{(m)}(r_{n_m})}(r_2^{(m)n_m}), \text{ if } (r_2^{(m)n_m})^2/r^2>0;\\ K_{\gamma_1^{(m)}(r_{n_m})}(r_2^{(m)n_m}), \text{ if } (r_2^{(m)n_m})^2/r^2<0;\end{cases},$$

$$E_{\gamma^{(m)}(r_{n_m})}(r_1^{(m)n_m}) = \begin{cases} J_{\gamma^{(m)}(r_{n_m})}(r_1^{(m)n_m}), \text{ if } (r_1^{(m)n_m})^2 / r^2 > 0; \\ I_{\gamma^{(m)}(r_{n_m})}(r_1^{(m)n_m}), \text{ if } (r_1^{(m)n_m})^2 / r^2 < 0; \end{cases}$$

$$F_{\gamma^{(m)}(r_{n_m})}(r_1^{(m)n_m}) = \begin{cases} Y_{\gamma^{(m)}(r_{n_m})}(r_1^{(m)n_m}), \text{ if } (r_1^{(m)n_m})^2 / r^2 > 0; \\ K_{\gamma^{(m)}(r_{n_m})}(r_1^{(m)n_m}), \text{ if } (r_1^{(m)n_m})^2 / r^2 < 0. \end{cases} \quad (18)$$

In (18), $J_\delta(x)$ and $I_\delta(x)$ ($Y_\delta(x)$ and $K_\delta(x)$) are the Bessel and modified Bessel functions of the first (second) kind. Moreover, in (16) and (17) $A_1^{(m)n_m}$, $A_2^{(m)n_m}$, $B_1^{(m)n_m}$ and $B_2^{(m)n_m}$ are unknown constants.

Thus, using (16) (18), we obtain the expressions from (5) and (6) for the displacements and stresses for each sublayer, and substituting these expressions into the above-mentioned contact and boundary conditions, we obtain the homogeneous system of linear algebraic equations in terms of the unknown constants $A_1^{(m)n_m}$, $A_2^{(m)n_m}$, $B_1^{(m)n_m}$ and $B_2^{(m)n_m}$ ($m = 1,2$, $n_1 = 1, 2, ..., N_1$, $n_2 = 1, 2, ..., N_2$). Equating the determinant of the coefficient matrix to zero (denote it by (a_{qp})), we obtain the dispersion equation. Formally, this equation can be written as follows.

$$\det(a_{qp}(c, kR, \alpha^{(1)}, \alpha^{(2)}, \mu^{(1)}/\mu^{(2)}, h^{(1)}/R, h^{(2)}/R)) = 0; \quad q, p = 1, 2, ..., 4(N_1 + N_2). \quad (19)$$

In order to reduce the size of the paper, we omit here the explicit expressions for the components a_{qp}, which follow easily from the corresponding expressions presented and discussed above.

This concludes the consideration of the discrete-analytic solution method used in the present investigation.

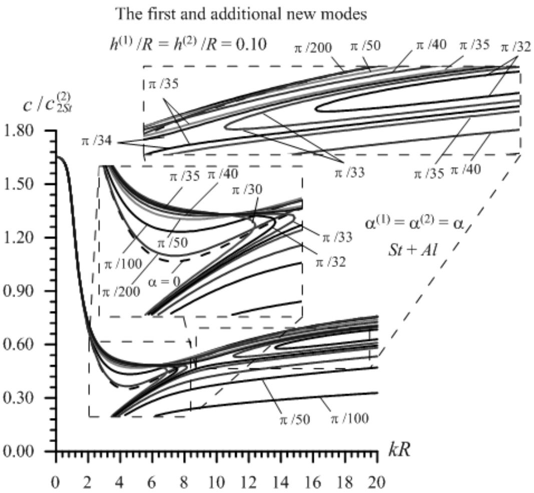
3. Numerical results and discussions.

Now we consider the numerical results related to the influence of the inhomogeneous residual stresses produced by contacting the section described in the first and second sections on the dispersion curves of the longitudinal axisymmetric wave propagating in the two-layer hollow cylinder. Note that these results were obtained from the numerical solution of the dispersion equation (19) using the well-known «bi-section» method. All numerical results discussed below were obtained for the case where $N_1 = N_2 = 27$ for the first lowest mode and some other new lowest modes. The reason for choosing such a number of sublayers will be discussed based on the convergence of the numerical results.

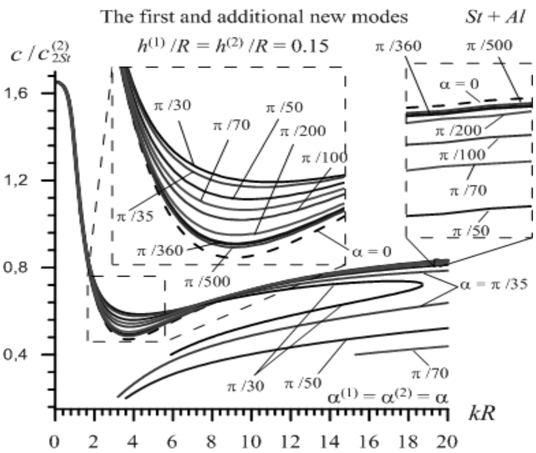
For concretization of the numerical results and discussions, we assume that the materials of the layers are steel (*St*) and aluminum (*Al*) and, according to the monograph [3], the material density, modulus of elasticity, Poisson's coefficients and shear wave propagation velocity of the *St* (*Al*) we select as $\rho_{St} = 7795$ kg/m^3, $E_{St} = 19{,}6$ GPa, ν_{St} and $c_{2St} = 3152$ m/s ($\rho_{Al} = 2770$ kg/m^3, $E_{Al} = 7{,}28$ GPa, $\nu_{Al} = 0{,}3$ and $c_{2Al} = 3179$ m/s), respectively.

In all investigations, we assume that $\alpha^{(1)} = \alpha^{(2)} = \alpha$ and analyze the results obtained for the case where the inner and outer layers are made of steel and aluminum, respectively (denote this case as ($St + Al$)) or vice versa (denote this case as ($Al + St$)).

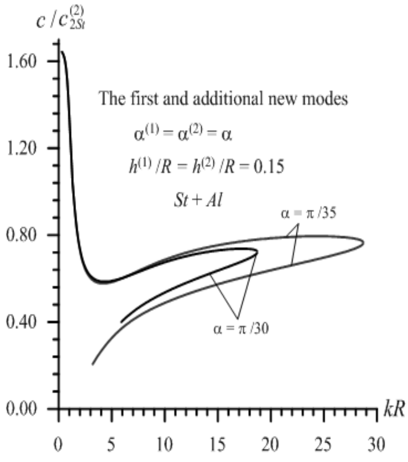
Assume that $h^{(1)}/R = h^{(2)}/R$ and first, we consider the case where the material of the inner layer of the cylinder is steel and the material of the outer cylinder is aluminum. We begin this consideration by analyzing the dispersion curves obtained for the first (fundamental) mode of vibration shown in Fig. 1 for the cases $h^{(1)}/R = h^{(2)}/R = 0{,}1$ (Fig. 1, *a*), 0,15 (Fig. 1, *b*, *c*) and 0,25 (Fig. 1, *d*).


The first and additional new modes
h(1)/R = h(2)/R = 0.10
α(1) = α(2) = α
St + Al
c / c(2)2St
kR
π /200
π /50
π /40
π /35
π /32
π /34
π /33
π /30
π /100
α = 0
0.00
0.30
0.60
0.90
1.20
1.50
1.80
0
2
4
6
8
10
12
14
16
18
20

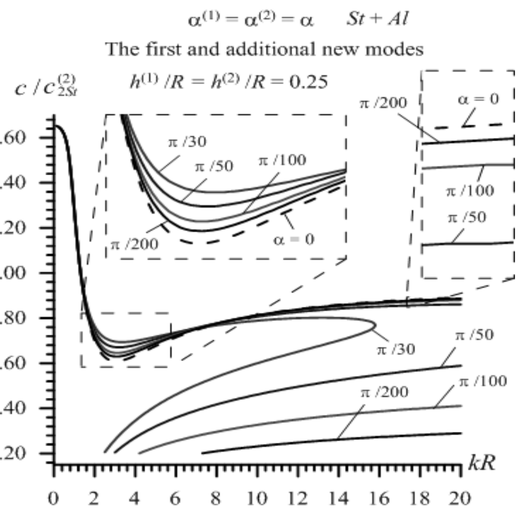
a


The first and additional new modes
St + Al
h(1)/R = h(2)/R = 0.15
c / c(2)2St
π /360
π /500
α = 0
π /30
π /70
π /50
π /200
π /100
π /35
α = π /35
α(1) = α(2) = α
kR
0,4
0,8
1,2
1,6
0
2
4
6
8
10
12
14
16
18
20

b


c / c(2)2St
The first and additional new modes
α(1) = α(2) = α
h(1)/R = h(2)/R = 0.15
St + Al
α = π /35
α = π /30
kR
0.00
0.40
0.80
1.20
1.60
0
5
10
15
20
25
30

c


α(1) = α(2) = α
St + Al
The first and additional new modes
h(1)/R = h(2)/R = 0.25
c / c(2)2St
π /200
α = 0
π /30
π /50
π /100
kR
0.20
0.40
0.60
0.80
1.00
1.20
1.40
1.60
0
2
4
6
8
10
12
14
16
18
20

d

Fig. 1

From the results presented in Fig. 1, it is clear that additional new types of dispersion curves appear, and these curves approach the dispersion curves of the first mode with increasing kR. To distinguish these additional modes related to the same angle, we introduce the term «maximum additional mode». This means that the wave propagation velocity in this additional mode is larger than the corresponding velocity in the other additional modes constructed for the same angle α

At the same time, these results show that there are cases where this approaching is completed with a contact of the dispersion curves of the first mode with the dispersion curve of the «maximum additional mode». For example, it can be seen from Fig. 1, *a* that for relatively large values of the angle α (i.e., for the angle $\alpha = \pi / 30$, $\pi / 32$, $\pi / 33$) the dispersion curves related to the «maximum additional mode» are close to the corresponding dispersion curves related to the first mode. However, in the cases where $\alpha < \pi / 34$, this proximity does not occur, although the convergence of the dispersion curves to each other takes place before a certain value of kR, after which these curves move away from each other.

Note that at the docking point (we denote the dimensionless wavenumber at this point by $(kR)_1^*$) of the dispersion curves, the relation $\left(d(c/c_2)/d(kR)\right)\big|_{kR=(kR)_1^*} = \infty$ exists, and in this way, a new kind of mode is formed. After $kR = (kR)_1^*$, i.e. in the cases when $kR > (kR)_1^*$, before a certain value of kR (denote it by $(kR)_2^*$), a «band gap» appears for the propagation of the considered wave in the first mode and from the point $kR = (kR)_2^*$, two new branches of the dispersion curves appear docking at this point and the relation $\left(d(c/c_2)/d(kR)\right)\big|_{kR=(kR)_2^*} = \infty$ exists. Consequently, the interval $(kR)_1^* < kR < (kR)_2^*$ is the «band gap» and the values of $(kR)_1^*$ and $(kR)_2^*$ increase with decreasing angle α.

It should be noted that the above conclusions about the «bandgap» apply to relatively large values of the angle α, i.e., above a certain value of α. Before this certain α (denoted by α^*), the dispersion curves related to the first mode and the dispersion curves related to the «maximum additional mode» are not in contact with each other. Analysis of the graphs in Figs. 1, *a*, *b* and 1, *d* show that the values of α^* depend on the ratio $h^{(1)} / R (= h^{(2)} / R)$ and that these values increase with $h^{(1)} / R (= h^{(2)} / R)$. Moreover, the analysis shows that the values of $(kR)_1^*$ and $(kR)_2^*$ also increase with $h^{(1)} / R (= h^{(2)} / R)$.

In Fig. 1, the «band gaps» can be seen only in Fig. 1, *a, b* and Fig. 1, *d*, these «band gaps» are not visible. This is because the «band gaps» in the cases where $h^{(1)} / R (= h^{(2)} / R) = 0{,}15$ and 0,25 occur in the region for which $kR > 20$, and this region is not shown in Fig. 1. As an example of this conclusion, Fig. 1, *c* shows the occurrence of the beginning of the «band gaps», which is outside the range $0 < kR < 20$.

Note that all the above discussions of the results presented in Fig. 1 refer to the qualitative influences of the considered type of inhomogeneous residual stresses on the dispersion curves obtained for the first (fundamental) and the new «maximum additional» modes. At the same time, these results also illustrate the quantitative influence of residual stresses on the dispersion curves for the first mode. Thus, a comparison of the dispersion curves obtained in the cases when $\alpha > 0$ with the corresponding propagation curves obtained in the case when $\alpha = 0$ shows that the character of the influence depends on the values of the dimensionless wavenumber kR. Thus, before a certain value of kR, the residual stresses cause an increase in the wave propagation velocity, but after this value of kR, the residual stresses cause a decrease in this velocity. Note that the magnitude of the above increase becomes more substantial around the wavenumber kR, where the dispersion curves obtained at $\alpha = 0$ exhibit a velocity minimum. Also note that the magnitude of the above decrease becomes more considerable for relatively large values of the dimensionless wavenumber kR. In all these cases, the magnitude of the influences of the residual stresses on the wave propagation velocity increases monotonically with the angle α.

So far, we have analysed the numerical results with respect to the two-layer hollow cylinder $St + Al$, that is, with respect to the two-layer hollow cylinder in which the materials of the inner and outer layers are steel (*St*) and aluminium (*Al*), respectively. We now analyse the numerical results for the two-layer hollow cylinder in which the materials of the inner and outer layers are aluminium and steel, respectively, and refer to this case as $Al + St$. Consider only the case in which $h^{(1)} / R (= h^{(2)} / R) = 0,25$, and analyse the graphs in Fig. 2, which show the dispersion curves in the first mode. From these results, it is clear that the arrangement of the materials of the layers in the cylinder can significantly change the character of the influence of the residual stresses in the cylinder on the dispersion curves not only in a quantitative but also in a qualitative sense. Thus, a comparison of the graphs in Fig. 2 with the corresponding graphs in Fig. 1, *d* shows that, in contrast to the $St + Al$ cylinder, no contact between the «maximum additional mode» and the first mode is observed in the $Al + St$ cylinder, and that in the interval $0 < kR < 20$ the residual stresses lead to an increase in the wave propagation velocity in the first mode, and the magnitude of this increase grows with the angle α.

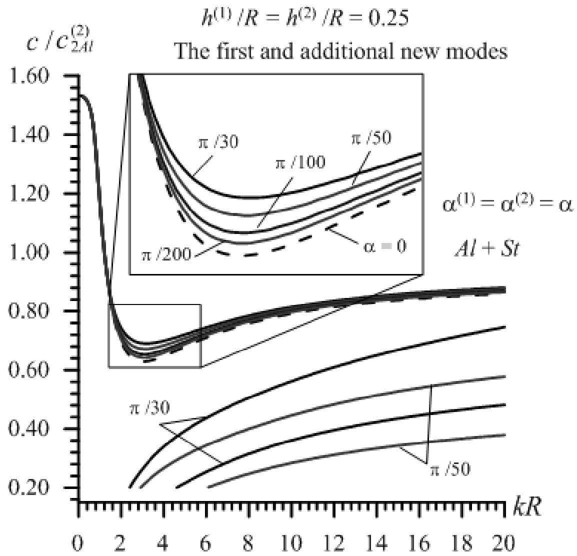


Fig. 2

With this, we limit ourselves to the analysis of the numerical results, recalling that these results are obtained in the case where $N_1 = N_2 = 27$. In order to justify the choice of the number of sublayers in each layer of the two-layer cylinder, we consider the numerical results illustrating the convergence of these results with respect to the numbers N_1 and N_2. For this purpose, we consider the numerical results in Fig. 3, which illustrate the convergence for the two-layer hollow cylinder $St + Al$ in the first mode for the case $h^{(1)} / R = 0,25$ and $\alpha = \pi / 30$. From these results, it follows that convergence around the point kR, where the wave propagation velocity on the dispersion curve constructed for the case $\alpha = 0$ has its minimum, requires smaller numbers of N_1 and N_2 than around the point $kR = (kR)_1^*$. Therefore, to obtain more accurate numerical results around the point $kR = (kR)_1^*$, it is necessary to choose the numbers N_1 and N_2 that satisfy the relation $N_1 = N_2 \geq 27$.

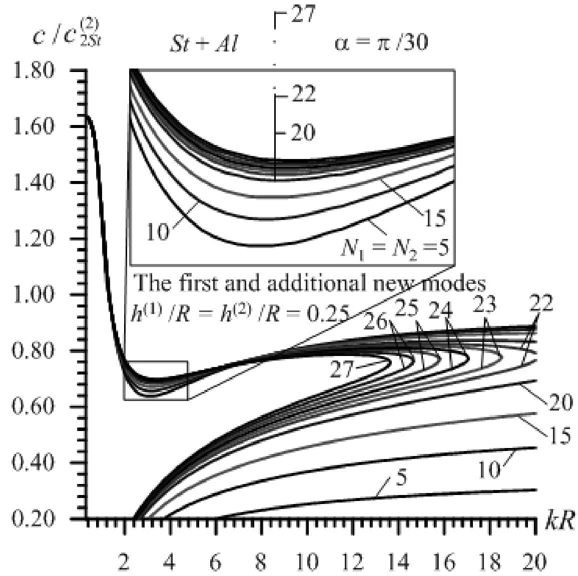


Fig. 3

Note that the convergence of the numerical results shown in Fig. 3 also show the validity of the algorithm and the PC programs used in the framework in which these results are obtained. Moreover, using the same algorithm and the programs PC and with the same number of sublayers, dispersion curves are obtained for the case when $\alpha = 0$, i.e., for the case when the residual stresses in the cylinder are absent, and these dispersion curves agree with the classical results listed in many references (see, for example, the monograph [2] and many other references listed in the monographs [3] and [5]). This statement also confirms the reliability of the programs and algorithms of PC, with which we obtained the discussed results. Unfortunately, we have not found related results from other researchers that we could use for comparison with the present results.

Conclusions.

In the present paper, within the framework of the second version of the small initial deformation theory of the 3D linearized theory of elastic waves in bodies with initial stresses, the dispersion of axisymmetric longitudinal waves propagating in a two-layer hollow cylinder with inhomogeneous residual stresses is studied.

According to the analyzes of the obtained dispersion curves, the following concrete conclusions can be drawn regarding the influence of the considered type of residual stresses on the dispersion of axisymmetric waves propagating in the two-layer hollow cylinder:

– in the $St + Al$ hollow cylinder case, a contact of the dispersion curve of the «maximum additional mode» with the dispersion curve of the first mode can occur, and for the values of kR (denoted by $(kR)_1^*$) at which this contact occurs, it satisfies the relation $d(c/c_{2St}^{(2)})/d(kR)\Big|_{kR=(kR)_1^*} = \infty$;

– a «bandgap» also occurs in the range $(kR)_1^* < kR < (kR)_2^*$, and from $kR = (kR)_2^*$ new branches of the first mode begin, and the relation $d(c/c_{2St}^{(2)})/d(kR)\Big|_{kR=(kR)_2^*} = \infty$ occurs;

– the values of $(kR)_1^*$ and $(kR)_2^*$ increase with decreasing cutting angle and with thickness of layers;

– the magnitude of the influence of the residual stresses on the dispersion curves increases with the cutting angle;

– the arrangement of the materials of the layers in the cylinder can significantly affect the influence of residual stresses on the wave propagation velocity in all the modes considered.

Р Е З Ю М Е . Досліджено вплив неоднорідних залишкових напружень, спричинених «контактом» розрізу, на дисперсію осесиметричних поздовжних хвиль у двошаровому порожнистому циліндрі. Відповідно до класифікації Гузя, дане дослідження виконано з використанням другого варіанту малої початкової деформації лінеаризованої 3D теорії пружних хвиль у тілах з початковими напруженнями. Використано розроблений авторами дискретний аналітичний метод для розв'язування рівнянь відповідного поля. Наведено та обговорено числові результати, що ілюструють вплив залишкових напружень на дисперсійні криві. Величина залишкових напружень оцінюється кутом перерізу (зрізу) в результаті контактування, через який виникають залишкові напруження в шарах циліндра.

К Л Ю Ч О В І С Л О В А : неоднорідні залишкові напруження, дисперсія хвиль, двошаровий порожнистий циліндр, заборонена зона, осесиметричні поздовжні хвилі.

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