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THE INFLUENCE OF THE MATERIAL PROPERTIES OF AN INHOMOGENEOUS PRE-STRESSED HOLLOW CYLINDER CONTAINING AN INVISCID FLUID ON THE DISPERSION OF THE QUASI - SCHOLTE WAVES

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Abstract. The influence of the material properties of the pre-stressed cylinder containing a compressible inviscid fluid on an dispersion of the axisymmetric duasi – Scholte waves is studied. It is supposed that the inhomogeneous pre-stresses in the cylinder are caused by the fluid pressure acting on the internal face surface of the cylinder before the wave propagation. A motion of the cylinder under wave propagation is described by the so-called three-dimensional linearized equations of the theory of elastic waves in bodies with initial stresses and the flow of the fluid - by the linearized Euler equations. For the solving he corresponding eigenvalue problem, the discrete–analytical solution method is employed and the related dispersion equation is solved numerically. The dispersion curves containing the first roots of the dispersion equation refer to the quasi – Scholte waves which play an important role in helping to understand the solid-fluid dynamic interaction. A study of the influence of the cylinder material properties on the character of these curves is the main novelty of the presented paper. The numerical results are presented and discussed on. For this purpose, the steel, aAluminum, lucite, and soft – rubber are selected as the cylinder materials, and water is chosen as the fluid. It is established, in particular, according to the numerical results, that an increase in the shear wave propagation velocity in the cylinder material causes an increase in the propagation velocity of the quasi – Scholte waves.

Key words: inhomogeneously pre-stressed cylinder, axisymmetric wave dispersion, material properties, compressible fluid, quasi-Scholte waves.

1. Introduction.

As is known, the Scholte wave is a non-dispersive wave propagating in a solid-fluid interface and was discovered by Scholte in 1942 and reported in the paper [1]. From that time up to now, many investigations have been made, for instance, in the papers [2 – 5] and others listed therein. This is because Scholte waves are widely applied for the determination of fluid properties and appear after a «certain» frequency. Note that for frequencies that are greater than the «certain» frequency, the Scholte wave obtained is non-dispersive. However, before the «certain» frequency, the generated waves are dispersive ones and these waves are called Quasi – Scholte waves.

Typically, the Quasi – Scholte waves, are generated in a plate-fluid or cylinder-fluid type of hydro-elastic systems. For understanding and application of these waves, it is required to make corresponding theoretical investigations on the influence of the material properties of the cylinder (or plate) on the dispersion of these waves. Moreover, such investigations are also required for study of the influence of the particularities of these hydro-elastic systems on this dispersion. One of these particularities is the inhomogeneous initial stresses which may appear in the cylinder as a result of the fluid pressure acting on its inner surface.

The presented paper is dedicated to the study of the influence of the inhomogeneous pre-stressed cylinder material properties on the dispersion of Quasi – Scholte waves in the cylinder containing a compressible inviscid fluid. Under these investigations, it is assumed that the hollow cylinder has inhomogeneous initial stresses caused by the fluid pressure acting on the internal surface of this cylinder.

It should be noted that the investigations related to the wave dispersion in the inhomogeneous pre-stressed cylinder-fluid systems began in the paper [6] in which the discrete – analytical solution method is developed and employed for the study of the corresponding eigenvalue problem. What is more, in the paper [6], the investigations are made within the scope of the three-dimensional linearized equations and relations of the theory of elastic waves in bodies with initial stresses [7, 8] and within the scope of linearized Euler equations for compressible inviscid fluids [9]. However, in the paper [6], concrete numerical results are obtained for the case where the cylinder material is steel and the fluid is water. Therefore, the numerical results obtained in the paper [6] cannot answer the question of how the cylinder material properties act on the dispersion curves of the longitudinal axisymmetric waves propagating in the hydro-elastic system under consideration. In the present paper, this question is addressed and therein lies the main novelty of the paper. That is, the study is made on the influence of the inhomogeneous pre-stressed cylinder material properties on the dispersion curves of the longitudinal axisymmetric waves propagating in the hydro-elastic system with respect to the corresponding Quasi – Scholte waves.

Finally, note that in the present investigation, the method proposed in the paper [6] will also be used.

2. Formulation of the problem.

Consider the inhomogeneous pre-stressed hollow cylinder with infinite length containing compressible inviscid barotropic fluid, and associate the cylindrical $Or\theta z$ and Cartesian $Ox_1x_2x_3$ ($x_3 = z$) (Fig. 1) systems of coordinates with the central axis of the cylinder. Two states are distinguished in the hydro-elastic system under consideration, i.e. the initial and perturbed states, and the Lagrange and Euler coordinates are introduced for describing the motion of the cylinder and fluid, respectively. Due to the smallness of the deformations and perturbations, the difference between these coordinates will be ignored.

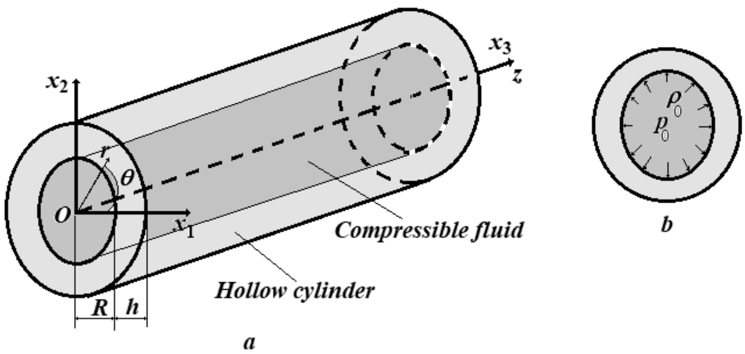

Fig. 1. The sketch of the hydro-elastic system under consideration

Within these frameworks, it is assumed that in the initial state (i.e. before the axisymmetric wave propagation), the fluid pressure with intensity p_0 acts on the interior of the cylinder and causes static stresses in the cylinder, and these stresses are called the pre-stresses. Moreover, assume that in the initial state, the fluid in the interior of the cylinder is at rest.

According to the well-known Lame problem discussed in almost all textbooks related to the theory of elasticity (see, for instance, the book [10]), the pre-stresses in the cylinder can be presented as follows:

$$\sigma_{rr}^{0}=\frac{p_0}{(1+h/R)^2-1}\left(1-\frac{R^2}{r^2}\left(1+\frac{h}{R}\right)^2\right);$$

$$\sigma_{\theta\theta}^{0}=\frac{p_0}{(1+h/R)^2-1}\left(1+\frac{R^2}{r^2}\left(1+\frac{h}{R}\right)^2\right);\quad \sigma_{zz}^{0}=\nu(\sigma_{rr}^{0}+\sigma_{\theta\theta}^{0})\,. \tag{1}$$

In (1), p_0 is the intensity of the hydrostatic fluid pressure, σ_{rr}^{0}, $\sigma_{\theta\theta}^{0}$ and σ_{zz}^{0} are radial, circumferential, and axial normal stresses acting in the cylinder in the initial state, R and h are the inner radius and thickness of the cylinder. Moreover, in (1) the upper index «0» indicates belonging of the related quantities to the initial stress state.

Assume that after appearing of the initial stress state in the cylinder, the hydro-elastic system under consideration gets a certain dynamic perturbation as a result of which the axisymmetric waves propagate therein. In the present paper, it is required to investigate how the mechanical properties of the cylinder material influence the dispersion of the studied waves in the presence of the initial inhomogeneous stresses determined by the expressions in (1). To undertake this investigation, the 3D linearized theory of elastic waves in initially stressed bodies is used for describing the motion of the cylinder, and linearized Euler equations are used for describing the flow of the inviscid compressible barotropic fluid.

Thus, according to the monographs [7 – 9, 11], the 3D linearized equations and corresponding relations describing the motion of the cylinder are written.

The equations of motion

$$\frac{\partial t_{rr}}{\partial r}+\frac{\partial t_{zr}}{\partial z}+\frac{1}{r}(t_{rr}-t_{\theta\theta})=\rho\,\frac{\partial^2 u_r}{\partial t^2};\quad \frac{\partial t_{rz}}{\partial r}+\frac{1}{r}t_{rz}+\frac{\partial t_{zz}}{\partial z}=\rho\,\frac{\partial^2 u_z}{\partial t^2}, \tag{2}$$

where

$$t_{rr}=\sigma_{rr}+\sigma_{rr}^{0}(r)\frac{\partial u_r}{\partial r};\quad t_{rz}=\sigma_{rz}+\sigma_{rr}^{0}(r)\frac{\partial u_z}{\partial r};\quad t_{\theta\theta}=\sigma_{\theta\theta}+\sigma_{\theta\theta}^{0}(r)\frac{u_r}{r};$$

$$t_{zr}=\sigma_{zr}+\sigma_{zz}^{0}(r)\frac{\partial u_r}{\partial z};\quad t_{zz}=\sigma_{zz}+\sigma_{zz}^{0}(r)\frac{\partial u_z}{\partial z}\,. \tag{3}$$

The elasticity relations

$$\sigma_{(jj)}=\lambda\,(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{zz})+2\mu\,\varepsilon_{(jj)},\ \ (jj)=rr;\theta\theta;zz\,;\quad \sigma_{rz}=2\mu\,\varepsilon_{rz}\,. \tag{4}$$

The strain-displacement relations

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r};\quad \varepsilon_{\theta\theta}=\frac{u_r}{r};\quad \varepsilon_{zz}=\frac{\partial u_z}{\partial z};\quad \varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right). \tag{5}$$

In (2) and (3), the notation t_{rr}, t_{rz}, $t_{\theta\theta}$, t_{zr} and t_{zz} are the perturbations of the components of the non-symmetric Kirchhoff stress tensor, σ_{rr}, σ_{rz}, $\sigma_{\theta\theta}$, σ_{zr} and σ_{zz} are the perturbations of the components of the ordinary symmetric stress tensor, ρ is the density of the cylinder material, λ and μ are Lame constants of the cylinder material, u_r and u_z are the perturbations of the components of the displacement vector, ε_{rr}, ε_{rz}, $\varepsilon_{\theta\theta}$, ε_{zr} and ε_{zz} are the perturbations of the components of the strain tensor.

Thus, the motion in the cylinder with initial stresses expressed in (1) is described by the complete system of linearized equations and relations in (2) – (5).

According to [9], the fluid flow in the cylinder is described by utilizing the following system of linearized Euler equations for barotropic compressible inviscid fluids.

The linearized continuity equation

$$\frac{\partial \rho'}{\partial t} + \rho_0 \left(\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{\partial V_z}{\partial z} \right) = 0 . \tag{6}$$

Linearized equations of the fluid flow:

$$\frac{\partial V_r}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial r} ; \quad \frac{\partial V_z}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} . \tag{7}$$

The linearized state equation

$$p' = a_0^2 \rho' , \tag{8}$$

where a_0 is the sound speed in the fluid and ρ_0 is the density of the fluid in the initial state, V_r and V_z are the perturbations of the components of the velocity vector, ρ' is the perturbation of the density of the fluid, p' is the perturbation of the pressure.

In this way, the complete system of equations (6) – (8) is obtained, within the scope of which the flow of the fluid in the perturbed state is described. For carrying out the intended investigations, the foregoing systems of equations with the following boundary and compatibility conditions, must be supplied.

The boundary conditions on the external surface of the cylinder:

$$t_{rr}\Big|_{r=R+h} = 0 ; \quad t_{rz}\Big|_{r=R+h} = 0 . \tag{9}$$

The compatibility conditions on the interface surface between the fluid and cylinder, i.e. on the internal surface of the cylinder

$$t_{rr}\Big|_{r=R} = -p' ; \quad t_{rz}\Big|_{r=R} = 0 ; \quad \left.\frac{\partial u_r}{\partial t}\right|_{r=R} = V_r\big|_{r=R} . \tag{10}$$

The condition of boundedness of the quantities related to the fluid at the central axis of the cylinder:

$$\left\{ |p'|, |\rho'|, |V_r|, |V_z| \right\}\Big|_{r=0} < \infty . \tag{11}$$

This completes the mathematical formulation of the problem under consideration.

3. On the solution method.

For the solution to the system of equations (2) – (5), the discrete-analytical method developed and employed in the references [6, 12 – 14] is employed. According to this method, the interval $[R, R+h]$ is divided into the N number of sub-intervals which are determined through the expression $(R+(n-1)h/N) \le r \le (R+nh/N)$, where $1 \le n \le N$. Thus, it is assumed that the inhomogeneous initial stresses determined by expression (1) are homogeneous ones in each sub-interval and the values of these stresses are determined as follows:

$$\sigma_{rr}^0(r) \approx \sigma_{rr}^0(r_n) ; \quad \sigma_{\theta\theta}^0(r) \approx \sigma_{\theta\theta}^0(r_n) ; \quad \sigma_{zz}^0(r) \approx \sigma_{zz}^0(r_n) ;$$
$$r_n = R + (n-1)h/N + h/(2N) . \tag{12}$$

Moreover, it is assumed that the following full contact conditions are satisfied on the interfaces between the sub-intervals:

$$t_{rr}^1\Big|_{r=R} = -p' ; \quad t_{rz}^1\Big|_{r=R} = 0 ; \quad \left.\frac{\partial u_r^1}{\partial t}\right|_{r=R} = V_r\Big|_{r=R} ; \quad t_{rr}^1\Big|_{r=R+h/N} = t_{rr}^2\Big|_{r=R+h/N} ;$$

$$t_{rz}^1\Big|_{r=R+h/N}=t_{rz}^2\Big|_{r=R+h/N}\ ;\quad u_r^1\Big|_{r=R+h/N}=u_r^2\Big|_{r=R+h/N}\ ;\quad u_z^1\Big|_{r=R+h/N}=u_z^2\Big|_{r=R+h/N}\ ;\ \ldots\ ;$$

$$t_{rr}^{n-1}\Big|_{r=R+(n-1)h/N}=t_{rr}^n\Big|_{r=R+(n-1)h/N}\ ;\quad t_{rz}^{n-1}\Big|_{r=R+(n-1)h/N}=t_{rz}^n\Big|_{r=R+(n-1)h/N}\ ; \tag{13}$$

$$u_r^{n-1}\Big|_{r=R+(n-1)h/N}=u_r^n\Big|_{r=R+(n-1)h/N}\ ;\quad u_z^{n-1}\Big|_{r=R+(n-1)h/N}=u_z^n\Big|_{r=R+(n-1)h/N}\ ;\ \ldots\ ;$$

$$t_{rr}^N\Big|_{r=R+h}=0;\quad t_{rz}^N\Big|_{r=R+h}=0\ .$$

In (13) the upper index $1,\ 2,\ ...,\ N$ indicates the number of the sublayer.

By direct verification, it is established that there are $4N+1$ conditions in (13), where the number N is determined from the convergence requirement of the numerical results which will be discussed below. Note that the upper indices in (13) and below indicate the number of the corresponding sub-intervals.

Thus, according to the relations in (12), the following equations of motion are obtained from equations (2) and (3), which are satisfied in each n^{th} sub-interval separately.

$$\frac{\partial\sigma_{rr}^n}{\partial r}+\sigma_{rr}^0(r_n)\frac{\partial^2u_r^n}{\partial r^2}+\frac{\partial\sigma_{zr}^n}{\partial z}+\sigma_{zz}^0(r_n)\frac{\partial^2u_r^n}{\partial z^2}+\frac{1}{r}(\sigma_{rr}^n-\sigma_{\theta\theta}^n)+$$

$$+\sigma_{rr}^0(r_n)\frac{1}{r}\frac{\partial u_r^n}{\partial r}-\sigma_{\theta\theta}^0(r_n)\frac{u_r^n}{r^2}=\rho\frac{\partial^2u_r^n}{\partial t^2}\ ;$$

$$\frac{\partial\sigma_{rz}^n}{\partial r}+\sigma_{rr}^0(r_n)\frac{\partial^2u_z^n}{\partial r^2}+\frac{1}{r}\sigma_{rz}^n+\sigma_{rr}^0(r_n)\frac{1}{r}\frac{\partial u_z^n}{\partial r}+ \tag{14}$$

$$+\frac{\partial\sigma_{zz}^n}{\partial z}+\sigma_{zz}^0(r_n)\frac{\partial^2u_z^n}{\partial z^2}=\rho\frac{\partial^2u_z^n}{\partial t^2}\ .$$

To these equations, the elasticity relation (4) must be added, and the relation between the deformations and displacements (5) must be rewritten for each sub-interval separately. For the solution to the system of equations (14), (4), and (5), the classical Lame decomposition (see, for instance, the monograph [11]) is employed, which for the axisymmetric problems can be written as follows:

$$u_r^n=\frac{\partial\Phi^n}{\partial r}+\frac{\partial^2\Psi^n}{\partial r\partial z}\ ;\quad u_z^n=\frac{\partial\Phi^n}{\partial z}-\frac{\partial^2\Psi^n}{\partial r^2}-\frac{\partial\Psi^n}{r\partial r}\ . \tag{15}$$

Substituting the expressions in (15) into the system of equations (5), (4), and (14), and doing the corresponding cumbersome mathematical calculations, the following equations for the potential functions Φ^n and Ψ^n in (15) are obtained:

$$\left(1+\frac{\sigma_{rr}^0(r_n)}{\lambda+2\mu}\right)\frac{\partial^2\Phi^n}{\partial r^2}+\left(1+\frac{\sigma_{\theta\theta}^0(r_n)}{\lambda+2\mu}\right)\frac{\partial\Phi^n}{r\partial r}+\left(1+\frac{\sigma_{zz}^0(r_n)}{\lambda+2\mu}\right)\frac{\partial^2\Phi^n}{\partial z^2}=\frac{1}{(c_1)^2}\frac{\partial^2\Phi^n}{\partial t^2}\ ;$$

$$\left(1+\frac{\sigma_{rr}^0(r_n)}{\mu}\right)\frac{\partial^2\Psi^n}{\partial r^2}+\left(1+\frac{\sigma_{\theta\theta}^0(r_n)}{\mu}\right)\frac{\partial\Psi^n}{r\partial r}+\left(1+\frac{\sigma_{zz}^0(r_n)}{\mu}\right)\frac{\partial^2\Psi^n}{\partial z^2}=\frac{1}{(c_2)^2}\frac{\partial^2\Psi^n}{\partial t^2}\ . \tag{16}$$

In (16), the notation $c_1=\sqrt{(\lambda+2\mu)/\rho}$ and $c_2=\sqrt{\mu/\rho}$ is used. Moreover, in the cases, where $\sigma_{zz}^0(r_n)=0,\ \sigma_{rr}^0(r_n)=0$ and $\sigma_{\theta\theta}^0(r_n)=0$, the equations in (16) coincide with the corresponding equations of classical elastodynamics [11].

Representing the functions Φ^n, u_r^n, σ_{rr}^n, $\sigma_{\theta\theta}^n$ and σ_{zz}^n with the multiplying $\sin(kz-\omega t)$ and the functions Ψ^n, u_z^n and σ_{rz}^n with the multiplying $\cos(kz-\omega t)$, and denoting the amplitudes of the corresponding quantities with the same symbols, the following equations for the amplitudes of the potentials Φ^n and Ψ^n are obtained:

$$\frac{d^2\Phi^n}{d(r_2)^2}+\frac{\alpha_1(r_n)}{r_2}\frac{d\Phi^n}{dr_2}+\Phi^n=0; \quad \frac{d^2\Psi^n}{d(r_1)^2}+\frac{\alpha(r_n)}{r_1}\frac{d\Psi^n}{dr_1}+\Psi^n=0, \tag{17}$$

where

$$\alpha(r_n)=\frac{1+\sigma_{\theta\theta}^0(r_n)/\mu}{1+\sigma_{rr}^0(r_n)/\mu}; \quad \beta(r_n)=\frac{1+\sigma_{zz}^0(r_n)/\mu}{1+\sigma_{rr}^0(r_n)/\mu};$$

$$r_1^n=kr\sqrt{\frac{c^2}{(c_2)^2(1+\sigma_{rr}^0(r_n)/\mu)}-(\beta(r_n))^2}; \quad c=\omega/\kappa; \quad \alpha_1(r_n)=\frac{1+\sigma_{\theta\theta}^0(r_n)/(\lambda+2\mu)}{1+\sigma_{rr}^0(r_n)/(\lambda+2\mu)};$$

$$\beta_1(r_n)=\frac{1+\sigma_{zz}^0(r_n)/(\lambda+2\mu)}{1+\sigma_{rr}^0(r_n)/(\lambda+2\mu)}; \quad r_2^n=kr\sqrt{\frac{c^2}{(c_1)^2(1+\sigma_{rr}^0(r_n)/(\lambda+2\mu))}-(\beta_1(r_n))^2}. \tag{18}$$

According to [13, 15], the solution to the equations in (17) is found as follows:

$$\Phi^n=A_1^n(r_2)^{\gamma_1(r_n)}E_{\gamma_1(r_n)}(r_2^{n_i})+A_2^n(r_2)^{\gamma_1(r_n)}F_{\gamma_1(r_n)}(r_2^n); \tag{19}$$

$$\Psi^n=B_1^n(r_1)^{\gamma(r_n)}E_{\gamma(r_n)}(r_1^n)+B_2^n(r_1)^{\gamma(r_n)}F_{\gamma(r_n)}(r_1^n), \tag{20}$$

where A_1^n, A_2^n, B_1^n and B_2^n are unknown constants and

$$\gamma_1(r_n)=(1-\alpha_1(r_n))/2; \quad \gamma(r_n)=\left(1-\alpha(r_n)\right)/2;$$

$$E_{\gamma_1(r_n)}(r_2^n)=\begin{cases}J_{\gamma_1(r_n)}(r_2^n), \text{ if } (r_2^n)^2/r^2>0;\\ I_{\gamma_1(r_n)}(r_2^n), \text{ if } (r_2^n)^2/r^2<0;\end{cases} \quad F_{\gamma_1(r_n)}(r_2^n)=\begin{cases}Y_{\gamma_1(r_n)}(r_2^n), \text{ if } (r_2^n)^2/r^2>0;\\ K_{\gamma_1(r_n)}(r_2^n), \text{ if } (r_2^n)^2/r^2<0;\end{cases}$$

$$E_{\gamma(r_n)}(r_1^n)=\begin{cases}J_{\gamma(r_n)}(r_1^n), \text{ if } (r_1^n)^2/r^2>0;\\ I_{\gamma(r_n)}(r_1^n), \text{ if } (r_1^n)^2/r^2<0;\end{cases} \quad F_{\gamma(r_n)}(r_1^{n_i})=\begin{cases}Y_{\gamma(r_{n_i})}(r_1^n), \text{ if } (r_1^n)^2/r^2>0;\\ K_{\gamma(r_n)}(r_1^n), \text{ if } (r_1^{n_i})^2/r^2<0.\end{cases} \tag{21}$$

In (21), $J_\delta(x)$ and $I_\delta(x)$ are the Bessel and modified Bessel functions of the first kind, respectively, however, $Y_\delta(x)$ and $K_\delta(x)$ are the Bessel and Modified Bessel functions of the second kind, respectively.

Substituting these solutions into the presentation (15), the expressions for the displacements are determined, and then using the relations in (5) and (4), the expressions for the stresses within each sub-interval are obtained. In this way, the analytic expressions related to the motion of the cylinder within each sub-interval into which the region $\left[R,\, R+h\right]$ is divided, are determined.

To reduce the volume of the paper, the explicit forms of these expressions for the displacements and stresses which enter the conditions in (13) are not presented here.

Consider also determination of the quantities related to the flow of the fluid, and for this purpose, according to [9], the following presentations for the general solution to equations in (6) – (8) are used:

$$\rho' = -a_0^{-2}\rho_0 \frac{\partial}{\partial t}\Phi_f\,; \quad p' = -\rho_0 \frac{\partial}{\partial t}\Phi_f\,; \quad V_r = \frac{\partial}{\partial r}\Phi_f\,; \quad V_z = \frac{\partial}{\partial z}\Phi_f\,, \tag{22}$$

where

$$\left[\Delta - \frac{1}{a_0^2}\frac{\partial^2}{\partial t^2}\right]\Phi_f = 0\,; \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\,. \tag{23}$$

Representing the functions V_z, p' and ρ' by the multiplying $\sin(kz - \omega t)$, and the functions Φ_f and V_r by the multiplying $\cos(kz - \omega t)$, the following equation from (23) for Φ_{f_1} (where $\Phi = \Phi_{f_1}(r)\cos(kz - \omega t)$) is obtained

$$\left(\frac{d^2}{dr_3^2} + \frac{1}{r_3}\frac{d}{dr_3} + 1\right)\Phi_{f_1}(r) = 0; \quad r_3 = kr\sqrt{\left(\frac{c}{a_0}\right)^2 - 1}\,. \tag{24}$$

According to the conditions in (11), the solution to equation (24) is found as follows:

$$\Phi_{f_1}(r) = \begin{cases} FJ_0(r_3), \text{ if } r_3^2 > 0; \\ FI_0(r_3), \text{ if } r_3^2 < 0, \end{cases} \tag{25}$$

where $J_0(r_3)$ ($I_0(r_3)$) is the first kind of Bessel (modified Bessel) function of the zeroth order and F is an unknown constant.

Using the expression (25) and substituting $\Phi = \Phi_{f_1}(r)\cos(kz - \omega t)$ into the equations in (23), the following expressions for the sought values related to the fluid are obtained:

$$p' = \rho_0\left(V_z^0 k + \omega\right)\sin(kz - \omega t)\begin{cases} FJ_0(r_3), \text{ if } r_3^2 > 0; \\ FI_0(r_3), \text{ if } r_3^2 < 0; \end{cases}$$

$$\rho' = a_0^{-2}\rho_0\left(V_z^0 k + \omega\right)\sin(kz - \omega t)\begin{cases} FJ_0(r_3), \text{ if } r_3^2 > 0; \\ FI_0(r_3), \text{ if } r_3^2 < 0; \end{cases} \tag{26}$$

$$V_r = k\frac{dr_3}{dr}\cos(kz - \omega t)\begin{cases} -FJ_1(r_3), \text{ if } r_3^2 > 0; \\ FI_1(r_3), \text{ if } r_3^2 < 0; \end{cases} \quad V_z = -k\sin(kz - \omega t)\begin{cases} FJ_0(r_3), \text{ if } r_3^2 > 0; \\ FI_0(r_3), \text{ if } r_3^2 < 0. \end{cases}.$$

After this determination, according to the foregoing discussions, it is established that the analytical expressions of the sought values contain $4N+1$ number of unknown constants and these constants are A_1^n, A_2^n, B_1^n, B_2^n $(n = 1, 2, ..., N)$) and F. Using the $4N+1$ number of the conditions in (13), the system of homogeneous algebraic equations with respect to the unknown constants is obtained. According to the well-known procedure, by equating to zero the determinant of the coefficient matrix of this system, the dispersion equation is obtained. This equation can be formally presented as follows:

$$\det\left[a_{nm}(c/c_2, kR, p_0/\mu, \rho/\rho_0, h/R, a_0/c_2)\right] = 0\,,\ n; m = 1, 2, ..., 4N+1\,. \tag{27}$$

The dispersion equation (27) is solved numerically by employing the «bi-section» method. According to this method, for the selected problem parameters, a value for the dimensionless wavenumber kR is fixed and then used to calculate the sign of the determinant in (27) giving the subsequent values for the ratio c/c_2 with a certain difference. During this calculation, each two subsequent values of the ratio c/c_2 are determined, for which the sign of the determinant changes. The first of these two values of the ratio c/c_2 is selected as a root of the dispersion equation (27). The accuracy of the results is improved by decreasing the value of the difference between the subsequent values of the ratio c/c_2 which in the present investigation is selected as 10^{-5}.

This completes the consideration of the solution method.

4. Numerical results and discussions.

As noted above, the aim of the present investigation is to determine how the material properties of the cylinder influence the dispersion curves of the axisymmetric Quasi – Scholte waves propagating in the hydro-elastic system under consideration. For this purpose, for the cylinder materials, Steel, Aluminum, Lucite, and Soft-rubber are selected, the mechanical properties of which, according to [7 – 9, 16], are given in Table.

Table. The values of the mechanical constants of the cylinder material

Mechanical properties	The selected materials			
	Steel	Aluminum	Lucite	Soft-rubber
Lame constants μ (upper number) and λ (lower number)	$\frac{79\times10^9\,\text{Pa}}{94,4\times10^9\,\text{Pa}}$	$\frac{28\times10^9\,\text{Pa}}{43,1\times10^9\,\text{Pa}}$	$\frac{1,86\times10^9\,\text{Pa}}{3,96\times10^9\,\text{Pa}}$	$\frac{1,2\times10^6\,\text{Pa}}{57,6\times10^6\,\text{Pa}}$
Material density ρ	$7790\,\text{kg/m}^3$	$2770\,\text{kg/m}^3$	$1160\,\text{kg/m}^3$	$1200\,\text{kg/m}^3$
Shear wave propagation velocity $c_2=\sqrt{\mu/\rho}$	$3184\,\text{m/s}$	$3179\,\text{m/s}$	$1266\,\text{m/s}$	$31,6\,\text{m/s}$

In the present paper, the fluid is taken as water with density $\rho_0 = 1000\,\text{kg/m}^3$ and sound speed $a_0 = 1459,5\,\text{m/s}$ [8, 16].

First, to validate the PC programs and solution method employed in the present paper, the numerical results obtained in the paper [17] are used. Note that in the paper [17], dispersion of the axisymmetric waves propagating in the hollow cylinder containing an inviscid compressible fluid is also considered, and under obtaining the numerical results, steel is taken as the material of the cylinder and water is taken as the fluid. Moreover, in the paper [17], it is assumed that $h/R = 0,2$ and the dispersion diagrams, i.e. the graphs of the dependencies between $\omega h/c_2$ and kh are constructed for some modes and also for Quasi-Scholte wave modes. It is evident that in the «steel + water» case under $h/R = 0,2$ and $p_0/\mu = 0,0$, the corresponding results obtained by the present PC programs and algorithm must coincide with those obtained in the paper [17]. As in the present paper, the dispersion diagrams of the Quasi-Scholte waves (i.e. the dispersion diagrams related to the so-called zeroth mode) are considered, and therefore this comparison must be made only for the dispersion curves related to this mode. This comparison is made in Fig. 2 which shows the dispersion diagrams of the Quasi-Scholte waves constructed for various values of the ratio p_0/μ, through which the magnitude of the initial stresses in the cylinder is characterized. Note that in Fig. 2, the dispersion diagram obtained in the case where $p_0/\mu = 0,0$ is drawn by the dashed line which coincides with the corresponding line constructed in the paper [17]. At the same time, it follows from the graphs illustrated in Fig. 2 and constructed under $N = 50$ (where N is the number of sub-intervals into which the interval $[R, R+h]$ is divided) that for various values of the ratio p_0/μ, an increase in the magnitude of the internal pressure of the fluid causes to increase the wave propagation velocity in the hydro-elastic system under consideration. Note that the increase of the wave propagation velocity agrees with well-known physicomechanical considerations. Accordingly, it can be concluded that the solution method and PC programs used under obtaining the presented results are valid.

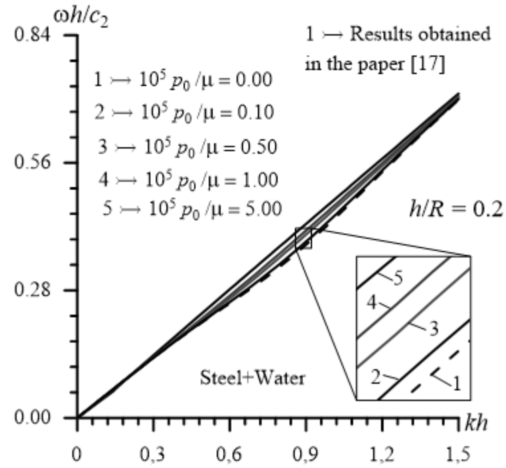

Fig. 2. Dispersion diagrams obtained for the «steel + water» system under various values of the ratio p_0 / μ in the case where $h / R = 0.2$.

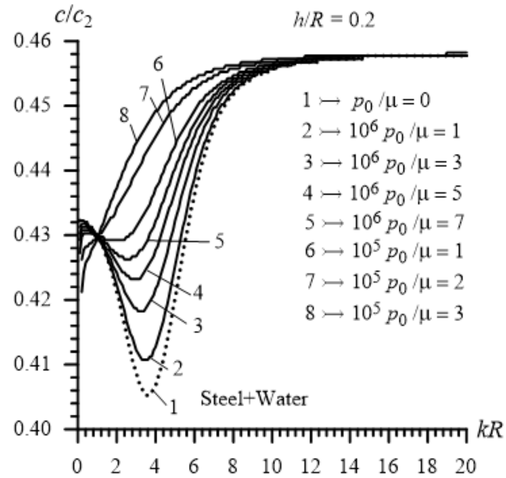

Fig. 3. Dispersion curves of the Quasi-Scholte waves related t to the Steel + Water system

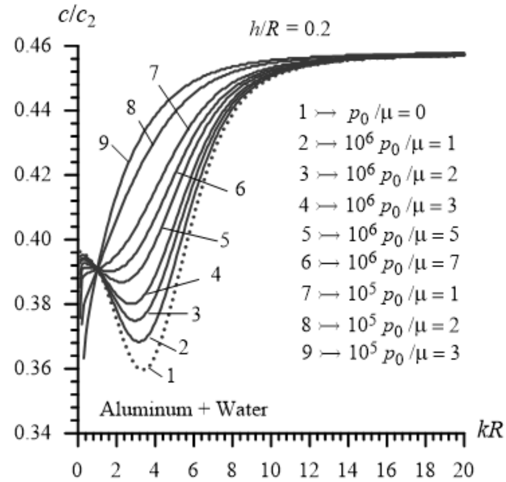

Fig. 4. Dispersion curves of the Quasi-Scholte waves related to the Aluminum + Water system

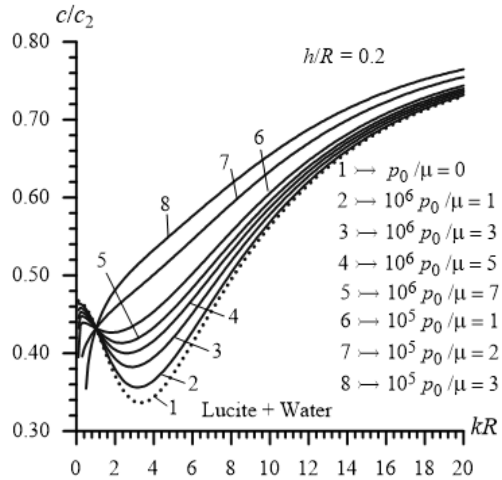

Fig. 5. Dispersion curves of the Quasi-Scholte waves related to the Lucite + Water system

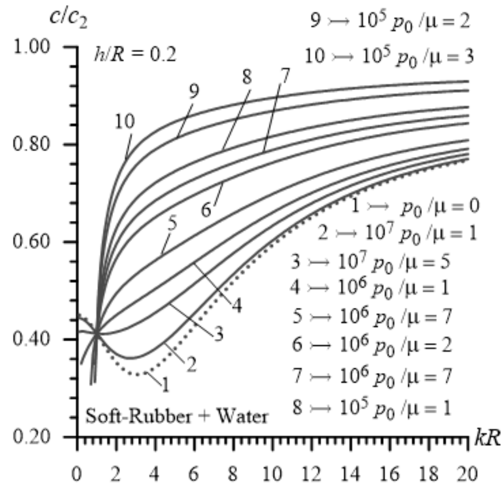

Fig. 6. Dispersion curves of the Quasi-Scholte waves related to the Soft-rubber + Water system

Now consider the dispersion curves of the Quasi – Scholte waves related to each selected cylinder material, for which the values of the mechanical constants are given in Table. These curves are presented in Figs. 3, 4, 5, and 6 for the Steel + Water, Aluminum + Water, Lucite + Water, and Soft-rubber + Water systems, respectively. All these results are obtained for various values of the ratio p_0 / μ in the case where $h / R = 0,2$ and $N = 50$. Note that in these figures the dispersion curves drawn by the dashed lines relate to the case where $p_0 / \mu = 0$.

Thus, it follows from the obtained results that for each selected cylinder material there exists such a value of the dimensionless wavenumber kR (denote this value through $(kR)^*$) under which the influence of the initial stresses on the propagation velocity of the Quasi – Scholte waves disappears. It also follows that in the cases where $kR > (kR)^*$, the propagation velocity of the Quasi – Scholte wave increases monotonically with kR and with the ratio p_0 / μ. However, in the cases where $kR < (kR)^*$, the wave propagation velocity decreases with the ratio p_0 / μ. Moreover, in the cases where $kR < (kR)^*$, the character of the dependence between c / c_2 and kR depends on the value of the ratio p_0 / μ, and an increase in the values of p_0 / μ causes the cut off wavelength to appear, the values of which decrease with p_0 / μ.

Comparison of the results obtained for the Steel, Aluminum, Lucite, and Soft-rubber cases with each other shows that the values of the ratio c / c_2 decrease with a decrease of the shear wave propagation velocity c_2 in the selected materials. Thus, according to the numerical results illustrated in Figs. 3, 4, 5, and 6, the following relations can be written:

$$c^{St} / c_2^{St} < c^{Al} / c_2^{Al} < c^{Luc} / c_2^{Luc} < c^{Sof-rub} / c_2^{Sof-rub}, \tag{28}$$

where the upper indices St, Al, Luc, and $Sof-rub$ denote belonging of the corresponding ratio to the Steel, Aluminum, Lucite, and Soft-rubber cases, respectively. However, taking into consideration the values of c_2^{St}, c_2^{Al}, c_2^{Luc}, and $c_2^{Sof-rub}$ given in Table and the values of the ratio c / c_2 obtained for the same fixed value of p_0 / μ and illustrated in Figs. 3, 4, 5, and 6, it can be concluded that the Quasi-Scholte wave propagation velocity in the considered cylinder materials satisfies the following relations:

$$c^{St} > c^{Al} > c^{Luc} > c^{Sof-rub}. \tag{29}$$

Thus, it can also be concluded that for the selected cylinder materials, an increase in the shear wave propagation velocity causes an increase in the Quasi-Scholte wave propagation velocity in the hydro-elastic system under consideration.

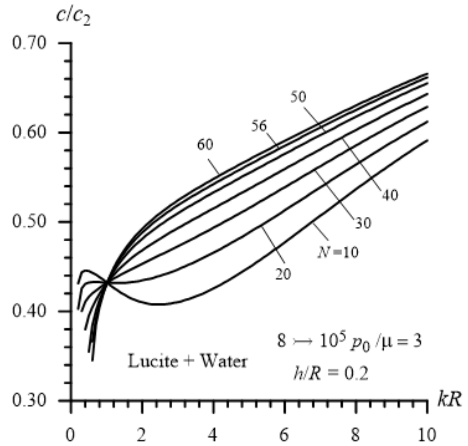

Fig. 7. Convergence of the numerical results with respect to the number N for the Lucite + Water system

This completes the analyses of the numerical results given in Figs. 3, 4, 5, and 6 which are obtained in the case where $N = 50$. For illustration of the suitability of the selected value of N, the results given in Fig. 7 are considered, which show the dispersion curves obtained for the Lucite+Water system under various values of the number N in the case where $10^5 p_0 / \mu = 3$. It follows from these results that the value of $(kR)^*$ does not depend on the number N. At the same time, an increase in the values of the number N causes not only quantitative, but also qualitative change in the dispersion curves. Comparison of the dispersion curves obtained for various values of the number N shows that the value $N = 50$ allows for obtaining results with accuracy 10^{-4}.

Conclusions.

Thus, in the present paper, study is made of the influence of the material properties of the inhomogeneously pre-stressed hollow cylinder containing a compressible inviscid fluid on the dispersion of the Quasi – Scholte axisymmetric waves propagating therein. The corresponding eigenvalue problem is formulated within the scope of the three-dimensional linearized theory of elastic waves in bodies with initial stresses, and of the linearized Euler equations for inviscid compressible fluids. The discrete-analytical solution method is employed for the solution to the corresponding eigenvalue problem. Numerical results are presented for the cases where Steel, Aluminum, Lucite, and Soft-rubber are selected as the cylinder material and Water is selected as the fluid contained in the cylinder.

Analyses of these numerical results allows the following concrete conclusions to be drawn:

– According to the relations in (28), an increase in the values of the shear wave propagation in the cylinder material causes to decrease the values of the ratio c/c_2 , where c is the propagation velocity of the Quasi-Scholte wave, and c_2 is the shear wave propagation in the cylinder material;

– However, according to the relations in (29), an increase in the shear wave propagation velocity in the cylinder material causes to increase the values of the Quasi-Scholte wave propagation velocity c;

– The influence of the material properties of the cylinder on the value of $((kR)^*)$ at which the influence of the initial inhomogeneous stresses in the cylinder on the Quasi-Scholte wave propagation velocity disappear, is insignificant.

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Р Е З Ю М Е. Досліджено вплив властивостей матеріалу попередньо напруженого циліндра, що містить стисливу нев'язку рідину, на дисперсію осесиметричних квазі-хвиль Шольте. Вважається, що неоднорідні попередні напруження в циліндрі викликані тиском рідини, який діє на внутрішню поверхню циліндра перед поширенням хвилі. Рух циліндра при поширенні хвилі описується так званими тривимірними лінеаризованими рівняннями теорії пружних хвиль у тілах з початковими напруженнями, а течія рідини – лінеаризованими рівняннями Ейлера. Для розв'язування відповідної задачі на власні значення використовується дискретно-аналітичний метод розв'язування. Відповідне дисперсійне рівняння розв'язується чисельно. Дисперсійні криві, що містять перші корені дисперсійного рівняння, відносяться до квазі –хвиль Шольте, які відіграють важливу роль у допомозі зрозуміти динамічну взаємодію твердого тіла та рідини. Дослідження впливу властивостей матеріалу циліндра на характер цих кривих є головною новизною даної роботи. Наведено та обговорено числові результати цього впливу. Як матеріал циліндра вибрано сталь, алюміній, люцит і м'яку гуму, а як рідину – воду. Відповідно до числових результатів встановлено, що збільшення швидкості поширення зсувної хвилі в матеріалі циліндра спричиняє збільшення швидкості поширення квазі –хвиль Шольте.

К Л Ю Ч О В І С Л О В А: попередньо напружений неоднорідний циліндр, осесиметрична дисперсія хвилі, властивості матеріалу, стислива рідина, квазі-хвиля Шольте.

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