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**OPTIMAL BOUNDARY CONTROL OF STRING VIBRATIONS WITH SET
VALUES OF DEFLECTION FUNCTIONS AND POINT VELOCITIES AT
DIFFERENT INTERMEDIATE TIME INSTANTS**

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Abstract. A problem of optimal boundary control of string vibrations is considered with given initial and final conditions as well as the given values of deflection function and velocities of the points at different intermediate moments and with a quality criterion specified over the entire time interval. Using the method of separation of variables and methods of optimal control theory with multipoint intermediate conditions, the optimal boundary controls are constructed for the arbitrary numbers of the first harmonics. As an application of the proposed constructive approach, an optimal boundary control is constructed with the given values of the deflection function and velocities of the points of the string at different intermediate moments.

Key words: string vibrations, boundary control, optimal control of vibrations, intermediate conditions, separation of variables.

Introduction.

The problems of control and optimal control of vibration processes with both distributed and boundary effects have significant theoretical interest and increasing practical importance. These problems are being studied by many researchers [1 – 9, 12, 14 – 16]. In practice, problems of boundary control and optimal control often arise, when it is necessary to generate vibrations with predetermined (desired) intermediate parameters (deflection shape, velocity of string points, and others). Modeling and control of dynamic systems described both by ordinary differential equations and partial differential equations with intermediate conditions is an actively developing direction in modern control theory. Such problems, in particular, are investigated in [3, 4, 10, 14, 15]. In the articles of the author were considered the problems of boundary control and optimal boundary control of string vibrations with a displacement of one end with the other end fixed under given constraints at intermediate time instants. In particular, [4] proposes a constructive approach to build the boundary control of string vibrations with given initial and final conditions, which allows setting the values of the deflection function at intermediate time instants. Also, the problem of optimal boundary control of string vibrations with given values of velocities of points at intermediate time instants is considered. This research is a continuation of [4, 15].

The purpose of this paper is to develop a constructive approach to building an optimal boundary control function for string vibrations with a displacement at two ends with given initial and final conditions and given values of the deflection function and point velocities at different intermediate time instants and with a quality criterion specified over the entire time interval. The problem is reduced to a distributed control problem with zero boundary conditions. Then, using the method of separation of variables, the resulting problem is reduced to

an optimal control problem for ordinary differential equations with given initial, final, and multipoint intermediate conditions. Optimal boundary controls are constructed by the method of moment problems for arbitrary numbers of first harmonics. The obtained results are illustrated with an example.

§1. Problem statement.

Let the state of a distributed vibratory system (small transverse vibrations of a stretched string), i.e., deviations from the equilibrium state, be described by the function $Q(x, t)$, $0 \leq x \leq l$, $0 \leq t \leq T$, which at $0 < x < l$ and $t > 0$ obeys the wave equation

$$\frac{\partial^2 Q}{\partial t^2} = a^2 \frac{\partial^2 Q}{\partial x^2} \quad (1.1)$$

with the initial conditions

$$Q(x, 0) = \varphi_0(x); \quad \left. \frac{\partial Q}{\partial t} \right|_{t=0} = \psi_0(x), \quad 0 \leq x \leq l \quad (1.2)$$

and the boundary conditions

$$Q(0, t) = \mu(t); \quad Q(l, t) = \nu(t), \quad 0 \leq t \leq T, \quad (1.3)$$

where functions $\mu(t)$ and $\nu(t)$ are boundary controls.

In equation (1.1), $a^2 = T_0 / \rho$, where T_0 is tension of a string, ρ is density of a homogeneous string.

Let at some intermediate time instants t_k ($k = 1, \dots, m$),

$$0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$$

the intermediate values of the deflection function and the values of the velocities of the string points be given

$$Q(x, t_i) = \varphi_i(x), \quad 0 \leq x \leq l, \quad i = 2\alpha - 1, \quad \alpha = 1, \dots, m/2; \quad (1.4)$$

$$\left. \frac{\partial Q}{\partial t} \right|_{t=t_j} = \psi_j(x), \quad 0 \leq x \leq l, \quad j = 2\alpha, \quad \alpha = 1, \dots, m/2. \quad (1.5)$$

We assume m to be an even number.

The problem of boundary optimal control of string vibrations with given values of the deflection function and velocities of string points at intermediate time instants is stated as follows: Among the possible controls $\mu(t)$ and $\nu(t)$, $0 \leq t \leq T$, find optimal controls that transfer the system from a given initial state (1.2), satisfying intermediate conditions (1.4) and (1.5), to the final state

$$Q(x, T) = \varphi_T(x) = \varphi_{m+1}(x); \quad \left. \frac{\partial Q}{\partial t} \right|_{t=T} = \psi_T(x) = \psi_{m+1}(x), \quad 0 \leq x \leq l \quad (1.6)$$

and that minimize the functional

$$\int_0^T (\mu^2(t) + \nu^2(t)) dt. \quad (1.7)$$

It is assumed that the function $Q(x, t) \in C^2(\Omega_T)$, where the set $\Omega_T = \{(x, t) : x \in [0, l], t \in [0, T]\}$, functions $\varphi_0(x)$, $\varphi_i(x) \in C^2[0, l]$, $i = 2\alpha + 1$, and $\psi_{m+1}(x)$, $\psi_j(x) \in C^1[0, l]$, $j = 2\alpha$, $\alpha = 0, 1, \dots, m/2$. It is also assumed that all functions are such that the matching conditions are satisfied.

$$\mu(0) = \varphi_0(0); \quad \dot{\mu}(0) = \psi_0(0); \quad v(0) = \varphi_0(l); \quad \dot{v}(0) = \psi_0(l); \quad (1.8)$$

$$\mu(t_i) = \varphi_i(0); \quad \dot{\mu}(t_j) = \psi_j(0); \quad v(t_i) = \varphi_i(l); \quad \dot{v}(t_j) = \psi_j(l); \quad (1.9)$$

$$i = 2\alpha - 1; \quad j = 2\alpha, \quad \alpha = 1, \dots, m/2;$$

$$\mu(T) = \varphi_T(0); \quad \dot{\mu}(T) = \psi_T(0); \quad v(T) = \varphi_T(l); \quad \dot{v}(T) = \psi_T(l). \quad (1.10)$$

Note that in the formulation of the problem, the intermediate values of the deflection function (1.4) and the values of the velocities of the string points (1.5) can be specified in any order for intermediate time instants, and the applied approach does not depend on the order.

It is also worth noting that since at certain intermediate time instants t_k ($k = 1, \dots, m$) either only the values of the deflection function (1.4) or only the values of the derivative of the deflection function (1.5) of the string are given, it is not appropriate to use the approach of solving the optimal control problem stepwise. Therefore, the paper proposes such an approach to solving the considered problem of optimal control, that takes into account the specifics of the intermediate conditions.

§2. Reduction of the problem to a problem with zero boundary conditions.

Since the boundary conditions (1.3) are inhomogeneous, we reduce the formulated problem to a problem with zero boundary conditions [13].

We seek the solution to equation (1.1) in the form of the sum

$$Q(x, t) = V(x, t) + W(x, t), \quad (2.1)$$

where $V(x, t)$ is an unknown function with homogeneous boundary conditions

$$V(0, t) = V(l, t) = 0. \quad (2.2)$$

$V(x, t)$ is to be determined, and $W(x, t)$ is a solution to equation (1.1) with inhomogeneous boundary conditions

$$W(0, t) = \mu(t); \quad W(l, t) = v(t). \quad (2.3)$$

Function $W(x, t)$ has the form [13]

$$W(x, t) = (v(t) - \mu(t)) \frac{x}{l} + \mu(t). \quad (2.4)$$

Substituting (2.1) into (1.1) and taking into account (2.4), we obtain an equation for the function $V(x, t)$

$$\frac{\partial^2 V}{\partial t^2} = a^2 \frac{\partial^2 V}{\partial x^2} + F(x, t), \quad (2.5)$$

where

$$F(x, t) = (\ddot{\mu}(t) - \ddot{v}(t)) \frac{x}{l} - \ddot{\mu}(t). \quad (2.6)$$

Due to the initial, intermediate, and boundary conditions, respectively, (1.2), (1.4) – (1.6), function $V(x, t)$ must satisfy the following initial conditions

$$\begin{aligned} V(x, 0) &= \varphi_0(x) - (v(0) - \mu(0)) \frac{x}{l} - \mu(0); \\ \frac{\partial V}{\partial t} \Big|_{t=0} &= \psi_0(x) - (\dot{v}(0) - \dot{\mu}(0)) \frac{x}{l} - \dot{\mu}(0); \end{aligned} \quad (2.7)$$

intermediate conditions

$$V(x, t_i) = \varphi_i(x) - (\nu(t_i) - \mu(t_i)) \frac{x}{l} - \mu(t_i), \quad i = 2\alpha - 1, \quad \alpha = 1, \dots, m/2;$$

$$\left. \frac{\partial V}{\partial t} \right|_{t=t_j} = \psi_j(x) - (\dot{\nu}(t_j) - \dot{\mu}(t_j)) \frac{x}{l} - \dot{\mu}(t_j), \quad j = 2\alpha, \quad \alpha = 1, \dots, m/2 \quad (2.8)$$

and final conditions

$$V(x, T) = \varphi_T(x) - (\nu(T) - \mu(T)) \frac{x}{l} - \mu(T);$$

$$\left. \frac{\partial V}{\partial t} \right|_{t=T} = \psi_T(x) - (\dot{\nu}(T) - \dot{\mu}(T)) \frac{x}{l} - \dot{\mu}(T). \quad (2.9)$$

Therefore, given conditions (2.10) – (2.12), conditions (2.7) – (2.9) are written as follows, respectively:

$$V(x, 0) = \varphi_0(x) - ((l) - \varphi_0(0)) \frac{x}{l} - \varphi_0(0);$$

$$\left. \frac{\partial V}{\partial t} \right|_{t=0} = \psi_0(x) - (\psi_0(l) - \psi_0(0)) \frac{x}{l} - \psi_0(0); \quad (2.10)$$

$$V(x, t_i) = \varphi_i(x) - (\varphi_i(l) - \varphi_i(0)) \frac{x}{l} - \varphi_i(0); \quad i = 2\alpha - 1; \quad \alpha = 1, \dots, m/2;$$

$$\left. \frac{\partial V}{\partial t} \right|_{t=t_j} = \psi_j(x) - (\psi_j(l) - \psi_j(0)) \frac{x}{l} - \psi_j(0); \quad j = 2\alpha; \quad \alpha = 1, \dots, m/2; \quad (2.11)$$

$$V(x, T) = \varphi_T(x) - (\varphi_T(l) - \varphi_T(0)) \frac{x}{l} - \varphi_T(0);$$

$$\left. \frac{\partial V}{\partial t} \right|_{t=T} = \psi_T(x) - (\psi_T(l) - \psi_T(0)) \frac{x}{l} - \psi_T(0). \quad (2.12)$$

Thus, solving the problem is reduced to solving the control problem (2.5), (2.6) with boundary conditions (2.2) and minimized functional (1.7), which is formulated as follows: Find optimal boundary controls $\mu^0(t)$ and $\nu^0(t)$ at $0 \leq t \leq T$, which transfer the vibration described by equation (2.5) with boundary conditions (2.2) from a given initial state (2.10) through intermediate states (2.11) to the final state (2.12) and minimize functional (1.7).

§3. Solving the problem with zero boundary conditions by the method of moments.

The boundary conditions (2.2) in problem (2.5), (2.6) are homogeneous, and assuming that the consistency conditions are satisfied and that the functions used belong to the indicated corresponding spaces, according to the theory of Fourier series, we seek the solution to equation (2.5) in the form

$$V(x, t) = \sum_{k=1}^{\infty} V_k(t) \sin \frac{\pi k}{l} x, \quad \text{where } V_k(t) = \frac{2}{l} \int_0^l V(x, t) \sin \frac{\pi k}{l} x dx. \quad (3.1)$$

Let us present the functions $F(x, t)$, $\varphi_i(x)$, $\psi_j(x)$, $i = 0, m+1$, $i = 2\alpha - 1$, $j = 0, m+1$, $j = 2\alpha$, $\alpha = 1, \dots, m/2$, in the form of Fourier series. By substituting their expressions together with $V(x, t)$ from (3.1) into equations (2.5), (2.6) and into conditions (2.10) – (2.12), we obtain

$$\ddot{V}_k(t) + \lambda_k^2 V_k(t) = F_k(t); \quad \lambda_k^2 = \left(\frac{a\pi k}{l} \right)^2; \quad (3.2)$$

$$V_k(0) = \varphi_k^{(0)} - \frac{2a}{\lambda_k l} [\varphi_0(0) - \varphi_0(l)(-1)^k]; \quad \dot{V}_k(0) = \psi_k^{(0)} - \frac{2a}{\lambda_k l} [\psi_0(0) - \psi_0(l)(-1)^k]; \quad (3.3)$$

$$V_k(t_i) = \varphi_k^{(i)} - \frac{2a}{\lambda_k l} [\varphi_i(0) - \varphi_i(l)(-1)^k], \quad i = 2\alpha - 1, \quad \alpha = 1, \dots, m/2;$$

$$\dot{V}_k(t_j) = \psi_k^{(j)} - \frac{2a}{\lambda_k l} [\psi_j(0) - \psi_j(l)(-1)^k], \quad j = 2\alpha, \quad \alpha = 1, \dots, m/2; \quad (3.4)$$

$$V_k(T) = \varphi_k^{(T)} - \frac{2a}{\lambda_k l} [\varphi_T(0) - \varphi_T(l)(-1)^k]; \quad \dot{V}_k(T) = \psi_k^{(T)} - \frac{2a}{\lambda_k l} [\psi_T(0) - \psi_T(l)(-1)^k], \quad (3.5)$$

where

$$F_k(t) = \frac{2a}{\lambda_k l} [\ddot{v}(t)(-1)^k - \ddot{\mu}(t)]. \quad (3.6)$$

Here, the Fourier coefficients are denoted through $\varphi_k^{(i)}$, $\psi_k^{(j)}$, $\varphi_k^{(0)}$, $\varphi_k^{(T)}$, $\psi_k^{(0)}$ and $\psi_k^{(T)}$ corresponding to the functions $\varphi_i(x)$, $\psi_j(x)$, $\varphi_0(x)$, $\varphi_T(x)$, $\psi_k^{(0)}$ and $\psi_k^{(T)}$, respectively.

The general way of solving equation (3.2) with initial conditions (3.3) and its time derivative have the form

$$\begin{aligned} V_k(t) &= V_k(0) \cos \lambda_k t + \frac{1}{\lambda_k} \dot{V}_k(0) \sin \lambda_k t + \frac{1}{\lambda_k} \int_0^t F_k(\tau) \sin \lambda_k(t - \tau) d\tau; \\ \dot{V}_k(t) &= -\lambda_k V_k(0) \sin \lambda_k t + \dot{V}_k(0) \cos \lambda_k t + \int_0^t F_k(\tau) \cos \lambda_k(t - \tau) d\tau. \end{aligned} \quad (3.7)$$

Taking into account the intermediate (3.4) and final (3.5) conditions, from (3.7) we obtain that the functions $F_k(\tau)$ for each k should satisfy the following system of equalities:

$$\begin{aligned} \int_0^T F_k(\tau) \sin \lambda_k(T - \tau) d\tau &= \tilde{C}_{1k}(T); \quad \int_0^T F_k(\tau) \cos \lambda_k(T - \tau) d\tau = \tilde{C}_{2k}(T); \\ \int_0^{t_i} F_k(\tau) \sin \lambda_k(t_i - \tau) d\tau &= \tilde{C}_{1k}(t_i), \quad i = 2\alpha - 1, \quad \alpha = 1, \dots, m/2; \\ \int_0^{t_j} F_k(\tau) \cos \lambda_k(t_j - \tau) d\tau &= \tilde{C}_{2k}(t_j), \quad j = 2\alpha, \quad \alpha = 1, \dots, m/2, \end{aligned} \quad (3.8)$$

where

$$\begin{aligned} \tilde{C}_{1k}(T) &= \lambda_k V_k(T) - \lambda_k V_k(0) \cos \lambda_k T - \dot{V}_k(0) \sin \lambda_k T; \\ \tilde{C}_{2k}(T) &= \dot{V}_k(T) + \lambda_k V_k(0) \sin \lambda_k T - \dot{V}_k(0) \cos \lambda_k T; \\ \tilde{C}_{1k}(t_i) &= \lambda_k V_k(t_i) - \lambda_k V_k(0) \cos \lambda_k t_i - \dot{V}_k(0) \sin \lambda_k t_i; \\ \tilde{C}_{2k}(t_j) &= \dot{V}_k(t_j) + \lambda_k V_k(0) \sin \lambda_k t_j - \dot{V}_k(0) \cos \lambda_k t_j. \end{aligned} \quad (3.9)$$

Substituting the expression of function $F_k(t)$ from (3.6) to (3.8) and integrating by parts, given conditions (2.10) – (2.12), we obtain

$$\begin{aligned} \int_0^T \mu(\tau) \sin \lambda_k (T - \tau) d\tau - \int_0^T \nu(\tau) (-1)^k \sin \lambda_k (T - \tau) d\tau &= C_{1k}(T); \\ \int_0^T \mu(\tau) \cos \lambda_k (T - \tau) d\tau - \int_0^T \nu(\tau) (-1)^k \cos \lambda_k (T - \tau) d\tau &= C_{2k}(T); \\ \int_0^T \mu(\tau) h_k^{(i)}(\tau) d\tau - \int_0^T \nu(\tau) (-1)^k h_k^{(i)}(\tau) d\tau &= C_{1k}(t_i), \quad i = 2\alpha - 1, \quad \alpha = 1, \dots, m/2; \\ \int_0^T \mu(\tau) g_k^{(j)}(\tau) d\tau - \int_0^T \nu(\tau) (-1)^k g_k^{(j)}(\tau) d\tau &= C_{2k}(t_j), \quad j = 2\alpha, \quad \alpha = 1, \dots, m/2, \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} C_{1k}(T) &= \frac{1}{\lambda_k^2} \left[\frac{\lambda_k l}{2a} \tilde{C}_{1k}(T) + X_{1k} - (-1)^k Y_{1k} \right]; \quad C_{2k}(T) = \frac{1}{\lambda_k^2} \left[\frac{\lambda_k l}{2a} \tilde{C}_{2k}(T) + X_{2k} - (-1)^k Y_{2k} \right]; \\ C_{1k}(t_i) &= \frac{1}{\lambda_k^2} \left[\frac{\lambda_k l}{2a} \tilde{C}_{1k}(t_i) + X_{1k}^{(i)} - (-1)^k Y_{1k}^{(i)} \right]; \\ C_{2k}(t_j) &= \frac{1}{\lambda_k^2} \left[\frac{\lambda_k l}{2a} \tilde{C}_{2k}(t_j) + X_{2k}^{(j)} - (-1)^k Y_{2k}^{(j)} \right]; \\ X_{1k} &= \lambda_k \varphi_T(0) - \psi_0(0) \sin \lambda_k T - \lambda_k \varphi_0(0) \cos \lambda_k T; \\ X_{2k} &= \psi_T(0) - \psi_0(0) \cos \lambda_k T + \lambda_k \varphi_0(0) \sin \lambda_k T; \\ Y_{1k} &= \lambda_k \varphi_T(l) - \psi_0(l) \sin \lambda_k T - \lambda_k \varphi_0(l) \cos \lambda_k T; \\ Y_{2k} &= \psi_T(l) - \psi_0(l) \cos \lambda_k T + \lambda_k \varphi_0(l) \sin \lambda_k T; \\ X_{1k}^{(i)} &= \lambda_k \varphi_i(0) - \psi_0(0) \sin \lambda_k t_i - \lambda_k \varphi_0(0) \cos \lambda_k t_i; \\ Y_{1k}^{(i)} &= \lambda_k \varphi_i(l) - \psi_0(l) \sin \lambda_k t_i - \lambda_k \varphi_0(l) \cos \lambda_k t_i; \\ X_{2k}^{(j)} &= \psi_j(0) - \psi_0(0) \cos \lambda_k t_j + \lambda_k \varphi_0(0) \sin \lambda_k t_j; \\ Y_{2k}^{(j)} &= \psi_j(l) - \psi_0(l) \cos \lambda_k t_j + \lambda_k \varphi_0(l) \sin \lambda_k t_j; \end{aligned} \quad (3.11)$$

$$h_k^{(i)}(\tau) = \begin{cases} \sin \lambda_k (t_i - \tau), & 0 \leq \tau \leq t_i \\ 0, & t_i < \tau \leq T \end{cases}; \quad g_k^{(j)}(\tau) = \begin{cases} \cos \lambda_k (t_j - \tau), & 0 \leq \tau \leq t_j \\ 0, & t_j < \tau \leq T \end{cases}.$$

It follows from relation (3.10) that for each harmonic (i.e., for each $k = 1, 2, \dots$), the motion described by equation (3.2), (3.6) with conditions (3.3) – (3.5) is completely control-

lable if and only if for any given values of constants $C_{1k}(T)$, $C_{2k}(T)$, $C_{1k}(t_i)$, $C_{2k}(t_j)$, $i = 2\alpha - 1$, $j = 2\alpha$, $\alpha = 1, \dots, m/2$, it is possible to find in (3.11) controls $\mu(t)$ and $\nu(t)$, $0 \leq t \leq T$, which satisfy the condition (3.10).

Thus, solving the stated problem of optimal control is reduced to finding the boundary controls $\mu(t)$ and $\nu(t)$, $0 \leq t \leq T$, which for each $k = 1, 2, \dots$ satisfy the integral relations (3.10) and minimize functional (1.7). The optimal control problem for functional (1.7) with integral conditions (3.10) can be considered as a conditional extremum problem from the calculus of variations.

§4. Problem solving.

Since the functional (1.7) is the squared norm of a normed linear space, and the integral relations (3.10) generated by the functions $\mu(t)$ and $\nu(t)$, are linear, the problem of determining the optimal control for each $k = 1, 2, \dots$ can be considered as the problem of moments [5, 11]. Therefore, the solution can be constructed using the algorithm for solving the problem of moments.

In practice, the first few n harmonics of elastic vibrations are usually chosen and the problem of control synthesis is solved using the methods of control theory for finite-dimensional systems. Therefore, we will solve the problem (1.7) and (3.10) at $k = 1, 2, \dots$ with the aid of an algorithm for solving the moment problem. To solve the finite-dimensional (at $k = 1, 2, \dots$) moment problem (1.7) and (3.10), following [11], we need to find the values $p_k, q_k, \gamma_{ik}, \gamma_{jk}$, $k = 1, 2, \dots$, $i = 2\alpha - 1$, $j = 2\alpha$, $\alpha = 1, \dots, m/2$, bound by the condition

$$\sum_{k=1}^n \left[p_k C_{1k}(T) + q_k C_{2k}(T) + \sum_{\substack{i=2\alpha-1 \\ \alpha=1}}^{m/2} \gamma_{ik} C_{1k}(t_i) + \sum_{\substack{j=2\alpha \\ \alpha=1}}^{m/2} \gamma_{jk} C_{2k}(t_j) \right] = 1, \quad (4.1)$$

for which

$$(\rho_n^0)^2 = \min_{(4.1)} \int_0^T [h_{1n}^2(\tau) + h_{2n}^2(\tau)] d\tau, \quad (4.2)$$

where

$$\begin{aligned} h_{1n}(\tau) &= \sum_{k=1}^n \left[p_k \sin \lambda_k (T - \tau) + q_k \cos \lambda_k (T - \tau) + \sum_{\substack{i=2\alpha-1 \\ \alpha=1}}^{m/2} \gamma_{ik} h_k^{(i)}(\tau) + \sum_{\substack{j=2\alpha \\ \alpha=1}}^{m/2} \gamma_{jk} g_k^{(j)}(\tau) \right]; \\ h_{2n}(\tau) &= \sum_{k=1}^n (-1)^{k+1} \left[p_k \sin \lambda_k (T - \tau) + q_k \cos \lambda_k (T - \tau) + \sum_{\substack{i=2\alpha-1 \\ \alpha=1}}^{m/2} \gamma_{ik} h_k^{(i)}(\tau) + \sum_{\substack{j=2\alpha \\ \alpha=1}}^{m/2} \gamma_{jk} g_k^{(j)}(\tau) \right]. \end{aligned} \quad (4.3)$$

To determine the values $p_k^0, q_k^0, \gamma_{ik}^0, \gamma_{jk}^0$, $k = 1, \dots, n$, $i = 2\alpha - 1$, $j = 2\alpha$, $\alpha = 1, \dots, m/2$ that minimize (4.2), we apply the method of indefinite Lagrange multipliers. The following function is introduced

$$\begin{aligned} f_n &= \int_0^T [(h_{1n}(\tau))^2 + (h_{2n}(\tau))^2] d\tau + \\ &+ \beta_n \left[\sum_{k=1}^n \left(p_k C_{1k}(T) + q_k C_{2k}(T) + \sum_{\substack{i=2\alpha-1 \\ \alpha=1}}^{m/2} \gamma_{ik} C_{1k}(t_i) + \sum_{\substack{j=2\alpha \\ \alpha=1}}^{m/2} \gamma_{jk} C_{2k}(t_j) \right) - 1 \right], \end{aligned}$$

where β_n is an indefinite Lagrange multiplier. Based on this method, by calculating derivatives of function f_n with respect to $p_k, q_k, \gamma_{ik}, \gamma_{jk}, k=1, \dots, n, i=2\alpha-1, j=2\alpha, \alpha=1, \dots, m/2$ and equating to zero, given notation (4.3), we obtain the following system of integral relations

$$\begin{aligned}
& \sum_{\sigma=1}^n I_{\sigma} \int_0^T \left[G_{\sigma}(p_{\sigma}, q_{\sigma}, \lambda_{\sigma}, T, \tau) + \sum_{\substack{s=2\alpha-1 \\ \alpha=1}}^{m/2} \gamma_{s\sigma} h_{\sigma}^{(s)}(\tau) + \sum_{\substack{r=2\alpha \\ \alpha=1}}^{m/2} \gamma_{r\sigma} g_{\sigma}^{(r)}(\tau) \right] \times \\
& \quad \times \sin \lambda_k(T-\tau) d\tau = -\frac{\beta_n}{2} C_{1k}(T); \\
& \sum_{\sigma=1}^n I_{\sigma} \int_0^T \left[G_{\sigma}(p_{\sigma}, q_{\sigma}, \lambda_{\sigma}, T, \tau) + \sum_{\substack{s=2\alpha-1 \\ \alpha=1}}^{m/2} \gamma_{s\sigma} h_{\sigma}^{(s)}(\tau) + \sum_{\substack{r=2\alpha \\ \alpha=1}}^{m/2} \gamma_{r\sigma} g_{\sigma}^{(r)}(\tau) \right] \times \\
& \quad \times \cos \lambda_k(T-\tau) d\tau = -\frac{\beta_n}{2} C_{2k}(T); \\
& \sum_{\sigma=1}^n I_{\sigma} \int_0^T \left[G_{\sigma}(p_{\sigma}, q_{\sigma}, \lambda_{\sigma}, T, \tau) + \sum_{\substack{s=2\alpha-1 \\ \alpha=1}}^{m/2} \gamma_{s\sigma} h_{\sigma}^{(s)}(\tau) + \sum_{\substack{r=2\alpha \\ \alpha=1}}^{m/2} \gamma_{r\sigma} g_{\sigma}^{(r)}(\tau) \right] \times \\
& \quad \times h_k^{(i)}(\tau) d\tau = -\frac{\beta_n}{2} C_{1k}(t_i); \\
& \sum_{\sigma=1}^n I_{\sigma} \int_0^T \left[G_{\sigma}(p_{\sigma}, q_{\sigma}, \lambda_{\sigma}, T, \tau) + \sum_{\substack{s=2\alpha-1 \\ \alpha=1}}^{m/2} \gamma_{s\sigma} h_{\sigma}^{(s)}(\tau) + \sum_{\substack{r=2\alpha \\ \alpha=1}}^{m/2} \gamma_{r\sigma} g_{\sigma}^{(r)}(\tau) \right] \times \\
& \quad \times g_k^{(j)}(\tau) d\tau = -\frac{\beta_n}{2} C_{2k}(t_j), \quad k=1, \dots, n, \quad i=2\alpha-1, \quad j=2\alpha, \quad \alpha=1, \dots, m/2,
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
I_{\sigma} &= 1 + (-1)^{\sigma+1}; \quad G_{\sigma}(p_{\sigma}, q_{\sigma}, \lambda_{\sigma}, T, \tau) = p_{\sigma} \sin \lambda_{\sigma}(T-\tau) + q_{\sigma} \cos \lambda_{\sigma}(T-\tau); \\
&\sigma = 1, \dots, n.
\end{aligned}$$

By calculating the integrals on the left-hand sides of equations (4.4), factoring in notation (3.12) and adding condition (4.1) to the resulting equations, we obtain a closed system of $4n+m+1$ algebraic equations for the same number of unknown values $p_k, q_k, \gamma_{ik}, \gamma_{jk}, k=1, \dots, n, i=2\alpha-1, j=2\alpha, \alpha=1, \dots, m/2$ and β_n .

$$\begin{aligned}
& \sum_{\sigma=1}^n I_{\sigma} \left[a_{\sigma k} p_{\sigma} + b_{\sigma k} q_{\sigma} + \sum_{\substack{s=2\alpha-1 \\ \alpha=1}}^{m/2} c_{1\sigma k}^{(s)} \gamma_{s\sigma} + \sum_{\substack{r=2\alpha \\ \alpha=1}}^{m/2} c_{2\sigma k}^{(r)} \gamma_{r\sigma} \right] = -\frac{\beta_n}{2} C_{1k}(T); \\
& \sum_{\sigma=1}^n I_{\sigma} \left[d_{\sigma k} p_{\sigma} + e_{\sigma k} q_{\sigma} + \sum_{\substack{s=2\alpha-1 \\ \alpha=1}}^{m/2} f_{1\sigma k}^{(s)} \gamma_{s\sigma} + \sum_{\substack{r=2\alpha \\ \alpha=1}}^{m/2} f_{2\sigma k}^{(r)} \gamma_{r\sigma} \right] = -\frac{\beta_n}{2} C_{2k}(T);
\end{aligned}$$

$$\begin{aligned}
\sum_{\sigma=1}^n I_{\sigma} \left[a_{\sigma k}^{(i)} p_{\sigma} + b_{\sigma k}^{(i)} q_{\sigma} + \sum_{\substack{s=2\alpha-1 \\ \alpha=1}}^{m/2} c_{1\sigma k}^{(si)} \gamma_{s\sigma} + \sum_{\substack{r=2\alpha \\ \alpha=1}}^{m/2} c_{2\sigma k}^{(ri)} \gamma_{r\sigma} \right] &= -\frac{\beta_n}{2} C_{1k}(t_i) ; \\
\sum_{\sigma=1}^n I_{\sigma} \left[d_{\sigma k}^{(j)} p_{\sigma} + e_{\sigma k}^{(j)} q_{\sigma} + \sum_{\substack{s=2\alpha-1 \\ \alpha=1}}^{m/2} f_{1\sigma k}^{(sj)} \gamma_{s\sigma} + \sum_{\substack{r=2\alpha \\ \alpha=1}}^{m/2} f_{2\sigma k}^{(rj)} \gamma_{r\sigma} \right] &= -\frac{\beta_n}{2} C_{2k}(t_j) ; \quad (4.5) \\
\sum_{k=1}^n \left[p_k C_{1k}(T) + q_k C_{2k}(T) + \sum_{\substack{i=2\alpha-1 \\ \alpha=1}}^{m/2} \gamma_{ik} C_{1k}(t_i) + \sum_{\substack{j=2\alpha \\ \alpha=1}}^{m/2} \gamma_{jk} C_{2k}(t_j) \right] &= 1,
\end{aligned}$$

$$k = 1, \dots, n, \quad i = 2\alpha - 1, \quad j = 2\alpha, \quad \alpha = 1, \dots, m/2,$$

where

$$\begin{aligned}
a_{\sigma k} &= \int_0^T \sin \lambda_{\sigma}(T-\tau) \sin \lambda_k(T-\tau) d\tau ; \quad b_{\sigma k} = \int_0^T \cos \lambda_{\sigma}(T-\tau) \sin \lambda_k(T-\tau) d\tau ; \\
c_{1\sigma k}^{(s)} &= \int_0^T h_{\sigma}^{(s)}(\tau) \sin \lambda_k(T-\tau) d\tau ; \quad c_{2\sigma k}^{(r)} = \int_0^T g_{\sigma}^{(r)}(\tau) \sin \lambda_k(T-\tau) d\tau ; \\
d_{\sigma k} &= \int_0^T \sin \lambda_{\sigma}(T-\tau) \cos \lambda_k(T-\tau) d\tau ; \quad e_{\sigma k} = \int_0^T \cos \lambda_{\sigma}(T-\tau) \cos \lambda_k(T-\tau) d\tau ; \\
f_{1\sigma k}^{(s)} &= \int_0^T h_{\sigma}^{(s)}(\tau) \cos \lambda_k(T-\tau) d\tau ; \quad f_{2\sigma k}^{(r)} = \int_0^T g_{\sigma}^{(r)}(\tau) \cos \lambda_k(T-\tau) d\tau ; \quad (4.6) \\
a_{\sigma k}^{(i)} &= \int_0^T h_k^{(i)}(\tau) \sin \lambda_{\sigma}(T-\tau) d\tau ; \quad b_{\sigma k}^{(i)} = \int_0^T h_k^{(i)}(\tau) \cos \lambda_{\sigma}(T-\tau) d\tau ; \\
c_{1\sigma k}^{(si)} &= \int_0^T h_{\sigma}^{(s)}(\tau) h_k^{(i)}(\tau) d\tau ; \quad c_{2\sigma k}^{(ri)} = \int_0^T g_{\sigma}^{(r)}(\tau) h_k^{(i)}(\tau) d\tau ; \\
d_{\sigma k}^{(j)} &= \int_0^T g_k^{(j)}(\tau) \sin \lambda_{\sigma}(T-\tau) d\tau ; \quad e_{\sigma k}^{(j)} = \int_0^T g_k^{(j)}(\tau) \cos \lambda_{\sigma}(T-\tau) d\tau ; \\
f_{1\sigma k}^{(sj)} &= \int_0^T h_{\sigma}^{(s)}(\tau) g_k^{(j)}(\tau) d\tau ; \quad f_{2\sigma k}^{(rj)} = \int_0^T g_{\sigma}^{(r)}(\tau) g_k^{(j)}(\tau) d\tau .
\end{aligned}$$

Let values $p_k^0, q_k^0, \gamma_{ik}^0, \gamma_{jk}^0, k=1, \dots, n, i=2\alpha-1, j=2\alpha, \alpha=1, \dots, m/2$ and β_n^0 , be the solution to the closed system of algebraic equations (4.5). Then, according to (4.3), (4.2), we have

$$h_{1n}^0(\tau) = \sum_{k=1}^n \left[G_k(p_k^0, q_k^0, \lambda_k, T, \tau) + \sum_{\substack{i=2\alpha-1 \\ \alpha=1}}^{m/2} \gamma_{ik}^0 h_k^{(i)}(\tau) + \sum_{\substack{j=2\alpha \\ \alpha=1}}^{m/2} \gamma_{jk}^0 g_k^{(j)}(\tau) \right] ;$$

$$h_{2n}^0(\tau) = \sum_{k=1}^n (-1)^{k+1} \left[G_k(p_k^0, q_k^0, \lambda_k, T, \tau) + \sum_{\substack{i=2\alpha-1 \\ \alpha=1}}^{m/2} \gamma_{ik}^0 h_k^{(i)}(\tau) + \sum_{\substack{j=2\alpha \\ \alpha=1}}^{m/2} \gamma_{jk}^0 g_k^{(j)}(\tau) \right]; \quad (4.7)$$

$$(\rho_n^0)^2 = \int_0^T \left[\left(h_{1n}^0(\tau) \right)^2 + \left(h_{2n}^0(\tau) \right)^2 \right] d\tau,$$

where $G_k(p_k^0, q_k^0, \lambda_k, T, \tau) = p_k^0 \sin \lambda_k (T - \tau) + q_k^0 \cos \lambda_k (T - \tau)$.

Following [11], optimal boundary controls $\mu_n^0(\tau)$ and $\nu_n^0(\tau)$ for any $n = 1, 2, \dots$ are represented as:

$$\mu_n^0(\tau) = \frac{1}{(\rho_n^0)^2} h_{1n}^0(\tau); \quad \nu_n^0(\tau) = \frac{1}{(\rho_n^0)^2} h_{2n}^0(\tau).$$

Thus, the optimal controls $\mu_n^0(\tau)$ and $\nu_n^0(\tau)$ $\tau \in [0, T]$, according to formulas (3.12) and (4.7), are written as:

$$\begin{aligned} \mu_n^0(\tau) &= \\ &= \begin{cases} \frac{1}{(\rho_n^0)^2} \sum_{k=1}^n \left[G_k(p_k^0, q_k^0, \lambda_k, T, \tau) + \sum_{\substack{i=2\alpha-1 \\ \alpha=1}}^{m/2} \gamma_{ik}^0 \sin \lambda_k (t_i - \tau) + \sum_{\substack{j=2\alpha \\ \alpha=1}}^{m/2} \gamma_{jk}^0 \cos \lambda_k (t_j - \tau) \right], & 0 \leq \tau \leq t_1; \\ \frac{1}{(\rho_n^0)^2} \sum_{k=1}^n \left[G_k(p_k^0, q_k^0, \lambda_k, T, \tau) + \sum_{\substack{i=2\alpha-1 \\ \alpha=2}}^{m/2} \gamma_{ik}^0 \sin \lambda_k (t_i - \tau) + \sum_{\substack{j=2\alpha \\ \alpha=1}}^{m/2} \gamma_{jk}^0 \cos \lambda_k (t_j - \tau) \right], & t_1 < \tau \leq t_2; \\ \dots \\ \frac{1}{(\rho_n^0)^2} \sum_{k=1}^n \left[G_k(p_k^0, q_k^0, \lambda_k, T, \tau) + \gamma_{mk}^0 \cos \lambda_k (t_m - \tau) \right], & t_{m-1} < \tau \leq t_m; \\ \frac{1}{(\rho_n^0)^2} \sum_{k=1}^n G_k(p_k^0, q_k^0, \lambda_k, T, \tau), & t_m < \tau \leq t_{m+1} = T; \end{cases} \\ \\ \nu_n^0(\tau) &= \\ &= \begin{cases} \frac{1}{(\rho_n^0)^2} \sum_{k=1}^n (-1)^{k+1} \left[G_k(p_k^0, q_k^0, \lambda_k, T, \tau) + \sum_{\substack{i=2\alpha-1 \\ \alpha=1}}^{m/2} \gamma_{ik}^0 \sin \lambda_k (t_i - \tau) + \sum_{\substack{j=2\alpha \\ \alpha=1}}^{m/2} \gamma_{jk}^0 \cos \lambda_k (t_j - \tau) \right], & 0 \leq \tau \leq t_1; \\ \frac{1}{(\rho_n^0)^2} \sum_{k=1}^n (-1)^{k+1} \left[G_k(p_k^0, q_k^0, \lambda_k, T, \tau) + \sum_{\substack{i=2\alpha-1 \\ \alpha=2}}^{m/2} \gamma_{ik}^0 \sin \lambda_k (t_i - \tau) + \sum_{\substack{j=2\alpha \\ \alpha=1}}^{m/2} \gamma_{jk}^0 \cos \lambda_k (t_j - \tau) \right], & t_1 < \tau \leq t_2; \\ \dots \\ \frac{1}{(\rho_n^0)^2} \sum_{k=1}^n (-1)^{k+1} \left[G_k(p_k^0, q_k^0, \lambda_k, T, \tau) + \gamma_{mk}^0 \cos \lambda_k (t_m - \tau) \right], & t_{m-1} < \tau \leq t_m; \\ \frac{1}{(\rho_n^0)^2} \sum_{k=1}^n (-1)^{k+1} G_k(p_k^0, q_k^0, \lambda_k, T, \tau), & t_m < \tau \leq t_{m+1} = T. \end{cases} \end{aligned}$$

Now we construct the deflection function corresponding to the optimal controls $\mu_n^0(\tau)$ and $\nu_n^0(\tau)$. Substituting the resulting expressions for optimal controls $\mu_n^0(\tau)$ and $\nu_n^0(\tau)$ into (3.6), and the expression obtained for $F_k^0(t)$ – into (3.7), we obtain the function $V_k^0(t)$, $t \in [0, T]$, $k = 1, \dots, n$. Further, from formula (3.1) we will have

$$V_n^0(x, t) = \sum_{k=1}^n V_k^0(t) \sin \frac{\pi k}{l} x, \quad (4.8)$$

and from (2.4), function $W_n^0(x, t)$ has the form

$$W_n^0(x, t) = \left(\nu_n^0(t) - \mu_n^0(t) \right) \frac{x}{l} + \mu_n^0(t). \quad (4.9)$$

Thus, according to (2.1), for the first n harmonics, the optimal string deflection function $Q_n^0(x, t)$, taking into account (4.8), (4.9), will be written in the form

$$Q_n^0(x, t) = V_n^0(x, t) + W_n^0(x, t). \quad (4.10)$$

Note that the proposed approach allows examining the problem of optimal boundary control for the cases where, at certain intermediate times, only either the values of string deflection or velocity of string points are given for various sequences. In particular, if only intermediate values of the velocities of string points (1.5) are given at time instants t_k ($k = 1, \dots, m$), then we obtain the problem already considered by the author.

§5. Problem solving for the case $m = 2$.

To illustrate the foregoing, suppose that under the boundary conditions (1.3) $Q(l, t) = 0$, $0 \leq t \leq T$ (i.e., $\nu(t) = 0$), and at some intermediate time instants t_1 and t_2 ($0 = t_0 < t_1 < t_2 < t_3 = T$), the intermediate state (deflection) and the values of the velocities of string points are given in the form:

$$Q(x, t_1) = \varphi_1(x), \quad 0 \leq x \leq l; \quad (5.1)$$

$$\left. \frac{\partial Q}{\partial t} \right|_{t=t_2} = \psi_2(x), \quad 0 \leq x \leq l. \quad (5.2)$$

In this case, formula (3.6) implies $F_k(t) = -(2a) / (\lambda_k l) \ddot{u}(t)$, and according to formulas (3.10) we will have the following integral relations

$$\begin{aligned} \int_0^T u(\tau) \sin \lambda_k (T - \tau) d\tau &= C_{1k}(T); \quad \int_0^T u(\tau) \cos \lambda_k (T - \tau) d\tau = C_{2k}(T); \\ \int_0^T u(\tau) h_k^{(1)}(\tau) d\tau &= C_{1k}(t_1); \quad \int_0^T u(\tau) g_k^{(2)}(\tau) d\tau = C_{2k}(t_2), \quad k = 1, 2, \dots, \end{aligned}$$

where

$$h_k^{(1)}(\tau) = \begin{cases} \sin \lambda_k (t_1 - \tau), & 0 \leq \tau \leq t_1; \\ 0, & t_1 < \tau \leq T; \end{cases} \quad g_k^{(2)}(\tau) = \begin{cases} \cos \lambda_k (t_2 - \tau), & 0 \leq \tau \leq t_2; \\ 0, & t_2 < \tau \leq T; \end{cases}$$

$$C_{1k}(T) = \frac{1}{\lambda_k^2} \left[\frac{\lambda_k l}{2a} \tilde{C}_{1k}(T) + X_{1k} \right]; \quad C_{2k}(T) = \frac{1}{\lambda_k^2} \left[\frac{\lambda_k l}{2a} \tilde{C}_{2k}(T) + X_{2k} \right];$$

$$C_{1k}(t_1) = \frac{1}{\lambda_k^2} \left[\frac{\lambda_k l}{2a} \tilde{C}_{1k}(t_1) + X_{1k}^{(1)} \right]; \quad C_{2k}(t_2) = \frac{1}{\lambda_k^2} \left[\frac{\lambda_k l}{2a} \tilde{C}_{2k}(t_2) + X_{2k}^{(2)} \right].$$

Constants $\tilde{C}_{1k}(T)$, $\tilde{C}_{2k}(T)$, $\tilde{C}_{1k}(t_1)$, $\tilde{C}_{2k}(t_2)$ are determined from formula (3.9), and X_{1k} , X_{2k} , $X_{1k}^{(1)}$, $X_{2k}^{(2)}$ – from (3.12).

By applying the above approach, we construct the optimal boundary control $\mu_n^0(\tau)$ at $n=1$ (hence, $k=1$). To determine the values of p_1 , q_1 , γ_{11} , γ_{21} , and β_1 , according to (4.5) and (4.6), we have the following system of algebraic equations

$$\begin{aligned} a_{11}p_1 + b_{11}q_1 + c_{111}^{(1)}\gamma_{11} + c_{211}^{(2)}\gamma_{21} &= -\frac{\beta_1}{2}C_{11}(T); \\ d_{11}p_1 + e_{11}q_1 + f_{111}^{(1)}\gamma_{11} + f_{211}^{(2)}\gamma_{21} &= -\frac{\beta_1}{2}C_{21}(T); \\ a_{11}^{(1)}p_1 + b_{11}^{(1)}q_1 + c_{111}^{(11)}\gamma_{11} + c_{211}^{(21)}\gamma_{21} &= -\frac{\beta_1}{2}C_{11}(t_1); \end{aligned} \quad (5.3)$$

$$d_{11}^{(2)}p_1 + e_{11}^{(2)}q_1 + f_{111}^{(12)}\gamma_{11} + f_{211}^{(22)}\gamma_{21} = -\frac{\beta_1}{2}C_{21}(t_2);$$

$$p_1C_{11}(T) + q_1C_{21}(T) + \gamma_{11}C_{11}(t_1) + \gamma_{21}C_{21}(t_2) = 1,$$

where

$$a_{11} = \frac{T}{2} - \frac{1}{4\lambda_1} \sin 2\lambda_1 T; \quad b_{11} = d_{11} = \frac{1}{2\lambda_1} \sin^2 \lambda_1 T; \quad e_{11} = \frac{T}{2} + \frac{1}{4\lambda_1} \sin 2\lambda_1 T;$$

$$a_{11}^{(1)} = c_{111}^{(1)} = \frac{t_1}{2} \cos \lambda_1 (T - t_1) - \frac{1}{2\lambda_1} \sin \lambda_1 t_1 \cos \lambda_1 T;$$

$$c_{211}^{(2)} = d_{11}^{(2)} = \frac{1}{2\lambda_1} \sin \lambda_1 t_2 \sin \lambda_1 T + \frac{t_2}{2} \sin \lambda_1 (T - t_2);$$

$$b_{11}^{(1)} = f_{111}^{(1)} = \frac{1}{2\lambda_1} \sin \lambda_1 t_1 \sin \lambda_1 T - \frac{t_1}{2} \sin \lambda_1 (T - t_1); \quad c_{111}^{(11)} = \frac{t_1}{2} - \frac{1}{4\lambda_1} \sin 2\lambda_1 t_1;$$

$$e_{11}^{(2)} = f_{211}^{(2)} = \frac{1}{2\lambda_1} \sin \lambda_1 t_2 \sin \lambda_1 T + \frac{t_2}{2} \cos \lambda_1 (T - t_2);$$

$$c_{211}^{(21)} = f_{111}^{(12)} = \frac{1}{2\lambda_1} \sin \lambda_1 t_1 \sin \lambda_1 t_2 - \frac{t_1}{2} \sin \lambda_1 (t_2 - t_1); \quad f_{211}^{(22)} = \frac{t_2}{2} + \frac{1}{4\lambda_1} \sin 2\lambda_1 t_2.$$

For simplicity, let us assume that $t_1 = 2l/a$; $t_2 = 4l/a$; $T = 6l/a$. Then, assuming $\lambda_1 = a\pi/l$, we obtain $t_1\lambda_1 = 2\pi$; $t_2\lambda_1 = 4\pi$; $T\lambda_1 = 6\pi$; $\lambda_1(T - t_1) = 4\pi$; $\lambda_1(T - t_2) = 2\pi$; $\lambda_1(t_2 - t_1) = 2\pi$, and, thus, we will have

$$a_{11} = e_{11} = 3l/a; \quad b_{11} = d_{11} = d_{11}^{(2)} = c_{211}^{(2)} = b_{11}^{(1)} = f_{111}^{(1)} = c_{211}^{(21)} = f_{111}^{(12)} = 0;$$

$$e_{11}^{(2)} = f_{211}^{(2)} = f_{211}^{(22)} = \frac{2l}{a}; \quad a_{11}^{(1)} = c_{111}^{(1)} = c_{111}^{(11)} = \frac{l}{a}.$$

In this case, by solving the system of equations (5.3) for $p_1^0, q_1^0, \gamma_{11}^0, \gamma_{21}^0$, we will obtain

$$p_1^0 = A[C_{11}(T) - C_{11}(t_1)]; \quad q_1^0 = 2A[C_{21}(T) - C_{21}(t_2)];$$

$$\gamma_{11}^0 = A[C_{11}(T) + C_{11}(t_1)]; \quad \gamma_{21}^0 = A[3C_{21}(t_2) - 2C_{21}(T)],$$

where

$$A^{-1} = C_{11}(T)[C_{11}(T) - C_{11}(t_1)] + 2C_{21}(T)[C_{21}(T) - C_{21}(t_2)] +$$

$$+ C_{11}(t_1)[C_{11}(T) + C_{11}(t_1)] + C_{21}(t_2)[3C_{21}(t_2) - 2C_{21}(T)];$$

$$C_{11}(T) = \frac{l}{2a}(\varphi_1^{(T)} - \varphi_1^{(0)}); \quad C_{21}(T) = \frac{l}{2a\lambda_1}(\psi_1^{(T)} - \psi_1^{(0)});$$

$$C_{11}(t_1) = \frac{l}{2a}(\varphi_1^{(1)} - \varphi_1^{(0)}); \quad C_{21}(t_2) = \frac{l}{2a\lambda_1}(\psi_1^{(2)} - \psi_1^{(0)}).$$

Therefore, the optimal boundary control $\mu_1^0(\tau)$ is written in the form:

$$\mu_1^0(\tau) = \frac{1}{(\rho_1^0)^2} h_{11}^0(\tau),$$

where

$$h_{11}^0(\tau) = \begin{cases} p_1^0 \sin \lambda_1 (T - \tau) + q_1^0 \cos \lambda_1 (T - \tau) + \gamma_{11}^0 \sin \lambda_1 (t_1 - \tau) + \gamma_{21}^0 \cos \lambda_1 (t_2 - \tau), \\ \quad 0 \leq \tau \leq t_1; \\ p_1^0 \sin \lambda_1 (T - \tau) + q_1^0 \cos \lambda_1 (T - \tau) + \gamma_{21}^0 \cos \lambda_1 (t_2 - \tau), \quad t_1 < \tau \leq t_2; \\ p_1^0 \sin \lambda_1 (T - \tau) + q_1^0 \cos \lambda_1 (T - \tau), \quad t_2 < \tau \leq T; \end{cases}$$

$$\begin{aligned} (\rho_1^0)^2 = & \int_0^{t_1} [p_1^0 \sin \lambda_1 (T - \tau) + q_1^0 \cos \lambda_1 (T - \tau) + \gamma_{11}^0 \sin \lambda_1 (t_1 - \tau) + \gamma_{21}^0 \cos \lambda_1 (t_2 - \tau)]^2 d\tau + \\ & + \int_{t_1}^{t_2} [p_1^0 \sin \lambda_1 (T - \tau) + q_1^0 \cos \lambda_1 (T - \tau) + \gamma_{21}^0 \cos \lambda_1 (t_2 - \tau)]^2 d\tau + \\ & + \int_{t_2}^T [p_1^0 \sin \lambda_1 (T - \tau) + q_1^0 \cos \lambda_1 (T - \tau)]^2 d\tau. \end{aligned}$$

Further, according to the above-presented formulas (4.8) – (4.10), we will have

$$Q_1^0(x, t) = V_1^0(x, t) + W_1^0(x, t) = V_1^0(t) \sin \frac{\pi}{l} x + \left(1 - \frac{x}{l}\right) \mu_1^0(t).$$

Conclusion.

A constructive method is proposed to build an optimal boundary control for the vibration process of a homogeneous string with a given velocity of string points at intermediate time instants and with a quality criterion given over the entire time interval. The proposed approach of optimal boundary control of string vibrations, using the Fourier method, allows extension to other non-one-dimensional vibrating systems.

РЕЗЮМЕ. Розглянуто задачу оптимального граничного керування коливаннями струни із заданими початковими і кінцевими умовами та із заданими значеннями функції відхилення та швидкостей точок у різні проміжні моменти часу та з критерієм якості, заданим на всьому проміжку часу. Використовуючи метод розділення змінних та методи теорії оптимального керування з багатоточковими проміжними умовами, для довільних значень перших гармонік побудовано оптимальні граничні керування. Як застосування запропонованого конструктивного підходу побудовано граничне оптимальне керування із заданим значенням функції відхилення та швидкостей точок струни у різні проміжні моменти часу.

КЛЮЧОВІ СЛОВА: коливання струни, граничне керування, оптимальне керування коливаннями, проміжні умови, розділення змінних.

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