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COMBINATORIAL PROPERTIES OF *P*-POSETS OF WIDTH 2

We calculate the coefficient of transitiveness for all nonserial posets of width 2 with positive definite quadratic Tits form.

Introduction. In [6], for a finite quiver (directed graph) Q with the set of vertices Q_0 and the set of arrows Q_1, P. Gabriel introduced a quadratic form $q_Q:\ \mathbb{Z}^n \to \mathbb{Z},\ n = |Q_0|$, called by him the *quadratic Tits form of the quiver* Q:

$$q_Q(z) = q_Q(z_1, \ldots, z_n) := \sum_{i \in Q_0} z_i^2 - \sum_{i \to j} z_i z_j ,$$

where $i \to j$ runs through the set Q_1. He proved that the quiver Q has finite representation type over a field k (i.e., finitely many indecomposable representations, up to isomorphism) if and only if its quadratic Tits form is positive. This Gabriel's work laid the foundations of a new direction in the theory of algebra dealing with the investigation of the relationships between the properties of representations of various objects and the properties of quadratic forms associated with these objects.

The above quadratic form is naturally generalized to a (finite) poset $0 \notin S$:

$$q_S(z) = z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i<j,\, i,j \in S} z_i z_j - z_0 \sum_{i \in S} z_i .$$

In [5] Yu. A. Drozd showed that a poset S has finite representation type if and only if its quadratic Tits form is weakly positive, i.e., takes positive value on any nonzero vector with nonnegative coordinates (representations of posets were introduced by L. A. Nazarova and A. V. Roiter in [7]).

For posets, in contrast to quivers, the sets of those with weakly positive and with positive Tits forms do not coincide. Therefore the investigations of posets with positive Tits form seems to be quite natural; notice that they are analogs of the Dynkin diagrams. Posets of this type were studied by the authors (from different points of view) in many papers (see e.g. [1–4]).

Below we consider only finite posets. A poset with positive Tits form is called simply a *P*-poset.

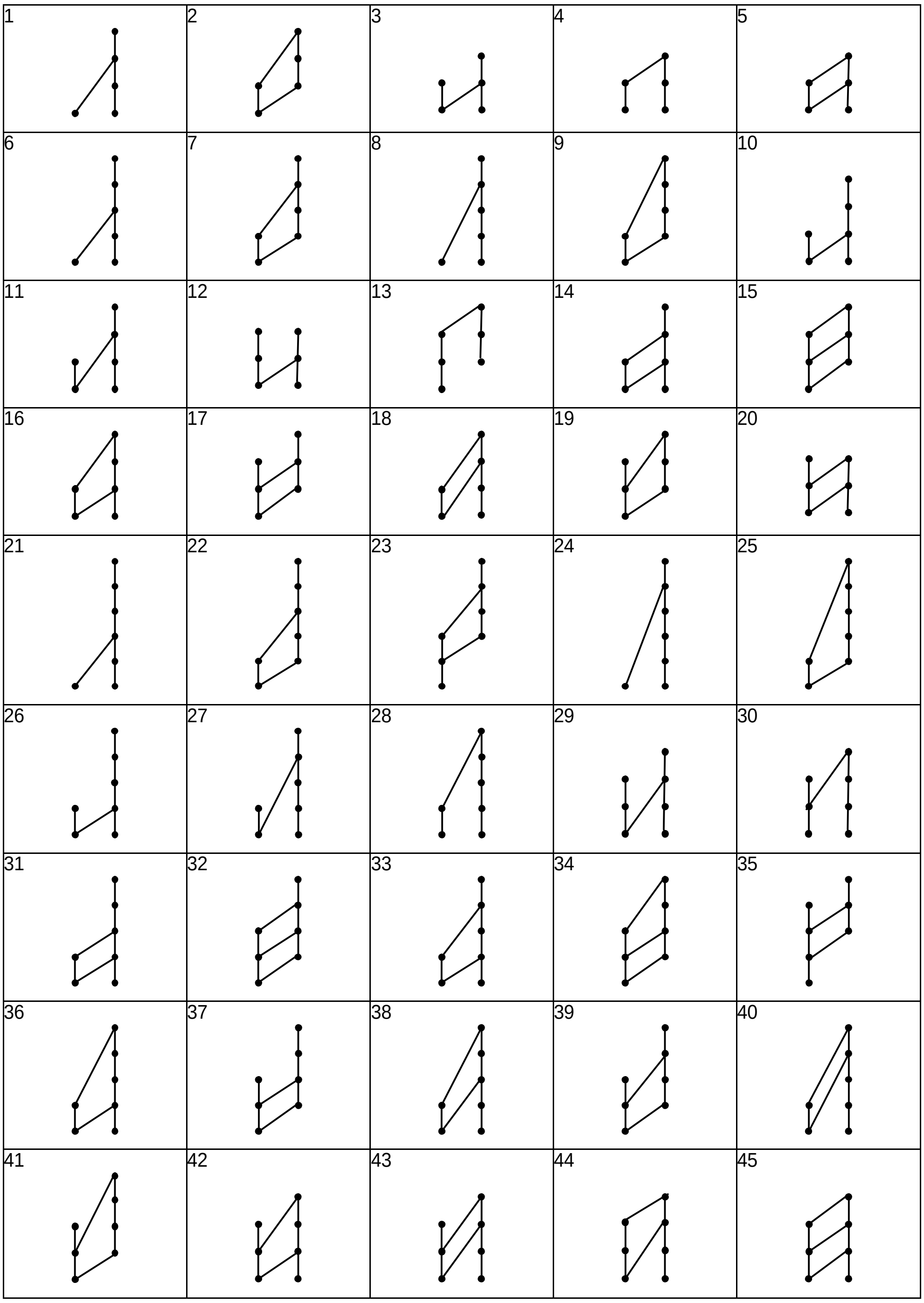
The *P*-posets can be divided into two classes: serial and nonserial [2]. A poset S is called serial if for any $N > |S|$ there exists a poset T_N of order N with positive quadratic. Tits form, which contains S, and nonserial otherwise. Recall that the width of a poset S is defined to be the maximum number of pairwise incomparable elements of S and the height of S the smallest length of a chain joining minimal and maximal elements of S; they are denoted $w(S)$ and $h(S)$, respectively.

The present paper is devoted to the investigation of combinatorial properties of *P*-posets S in the case $w(S) = 2$.

Let S be a finite poset and $S_<^2 := \{(x, y) \mid x, y \in S, x < y\}$. If $(x, y) \in S_<^2$ and there is no z satisfying $x < z < y$, then one says that x and y are *neighboring*. We put $n_w = n_w(S) := |S_<^2|$ and denote by $n_e = n_e(S)$ the number of pairs of neighboring elements. On the language of the Hasse diagram $H(S)$ (that represents S in the plane), n_e is equal to the number of all its edges and n_w to the number of all its paths, up to parallelity, going bottom-up (two path is called parallel if they start and terminate at the same vertices). The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and n_w we call *the coefficient of transitiveness of S*. If $n_w = 0$ (then $n_e = 0$), we assume $k_t = 0$.

The aim of this paper is to calculate k_t for all nonserial P-posets of width 2.

Preliminary. Indicate a part of the table from [2], in which are written all P-posets. They are given up to isomorphism and anti-isomorphism.



Main result. We write all the coefficients of transitiveness k_t up to the second decimal place. The following theorem holds.

Theorem 1. *The following holds for P-posets* 1–45:

N	n_e	n_w	k_t	N	n_e	n_w	k_t	N	n_e	n_w	k_t
1	4	8	0,50	16	6	11	0,45	31	7	18	0,61
2	5	8	0,38	17	6	11	0,45	32	8	18	0,55
3	4	6	0,33	18	6	10	0,40	33	7	17	0,59
4	4	6	0,33	19	6	10	0,40	34	8	17	0,53
5	5	7	0,29	20	6	9	0,33	35	7	17	0,59
6	5	13	0,62	21	6	19	0,68	36	7	16	0,56
7	6	13	0,54	22	7	19	0,63	37	7	16	0,56
8	5	12	0,58	23	7	19	0,63	38	7	15	0,53
9	6	12	0,50	24	6	17	0,65	39	7	15	0,53
10	5	10	0,50	25	7	17	0,59	40	7	14	0,50
11	5	9	0,44	26	6	15	0,60	41	7	14	0,50
12	5	8	0,38	27	6	16	0,54	42	7	13	0,46
13	5	9	0,44	28	6	12	0,50	43	7	12	0,42
14	6	12	0,50	29	6	11	0,45	44	7	13	0,46
15	7	12	0,42	30	6	11	0,45	45	8	15	0,47

The proof is carried out by direct calculations.

Now we formulate one corollary from the theorem (for finite posets).

Corollary 1. *Let S be a nonserial P-poset of width 2 such that* $k_t(S)$ ³ $k_t(T)$ *for all other such posets T. Then* $h(S)$ ³ $h(T)$.

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КОМБІНАТОРНІ ВЛАСТИВОСТІ *P*-ЧАСТКОВО ВПОРЯДКОВАНИХ МНОЖИН ШИРИНИ 2

Обчислено коефіцієнт транзитивності для всіх несерійних частково впорядкованих множин ширини 2 з додатно визначеною квадратичною формою Тітса.

КОМБИНАТОРНЫЕ СВОЙСТВА *P*-ЧАСТИЧНО УПОРЯДОЧЕННЫХ МНОЖЕСТВ ШИРИНЫ 2

Рассчитан коэффициент транзитивности для всех несерийных частично упорядоченных множеств ширины 2 с положительно определенной квадратичной формой Титса.

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