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# SOLVABLE LIE ALGEBRAS OF DERIVATIONS OF POLYNOMIAL RINGS IN THREE VARIABLES

*Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $A = \mathbb{K}[x_1, x_2, x_3]$ be the polynomial ring in three variables and $R = \mathbb{K}(x_1, x_2, x_3)$ be the field of rational functions. If $L$ is a subalgebra of the Lie algebra $W_3(\mathbb{K})$ of all $\mathbb{K}$-derivations of $A$, then $RL$ is a Lie algebra over $\mathbb{K}$ and $dim_R RL$ will be called the rank of $L$ over $R$. We study solvable subalgebras $L$ of $W_3(\mathbb{K})$ of rank 3 over $R$. It is proved that $L$ is isomorphic to a subalgebra of the general affine Lie algebra $aff_3(\mathbb{K})$ if $L$ contains an abelian ideal $I$ of rank 3 over $R$. If $L$ has an ideal $I$ with $rk_R I = 2$, then $L$ is contained in a subalgebra $\bar{L}$ of $\tilde{W}_3(\mathbb{K}) = Der_{\mathbb{K}} R$ such that $\bar{L}$ is an extension of a subalgebra of $aff_2(F)$ by a subalgebra of dimension $\le 2$, where $F$ is the field of constants of $I$ in $R$.*

**Introduction.** Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $A = \mathbb{K}[x_1, x_2, x_3]$ the polynomial ring in three variables and $R = \mathbb{K}(x_1, x_2, x_3)$ the field of rational functions. Recall that a $\mathbb{K}$-linear operator $D : A \to A$ is called a $\mathbb{K}$-derivation on $A$ if $D$ satisfies the Leibniz's rule: $D(fg) = D(f)g + fD(g)$ for all $f, g \in A$. The Lie algebra $W_3(\mathbb{K})$ of all $\mathbb{K}$-derivations on $A$ is a very interesting mathematical object closely connected with groups of symmetries of partial differential equations. In case $\mathbb{K}$ is the field of real or comlpex numbers, all finite dimensional subalgebras of $W_1(\mathbb{K})$ and $W_2(\mathbb{K})$ were described in works of S. Lie, P. Olver, N. Kamran. The natural problem of classification of all finite dimensional subalgebras of $W_3(\mathbb{K})$ remains still open. S. Lie [7] began to study such subalgebras, but his classification even of nilpotent subalgebras is incomplete. U. Amaldi [1, 2] continued study of subalgebras of $W_3(\mathbb{K})$ but his classification is unsatisfactory. Note that the problem of classifying even nilpotent finite-dimensional subalgebras of $W_4(\mathbb{K})$ is wild (i.e. it contains a hopeless problem of classifying pairs of square matrices up to simultaneous similarity [3]).

We study finite dimensional solvable subalgebras of rank 3 over $R$ of the Lie algebra $W_3(\mathbb{K})$ (nilpotent subalgebras of $W_3(\mathbb{K})$ were studied in [10]). The main results of the paper: it is proved in Theorem 1 that a solvable finite dimensional subalgebra $L$ of $W_3(\mathbb{K})$ possessing an abelian ideal of rank 3 over $R$ is isomorphic to a subalgebra of the general affine Lie algebra $aff_3(\mathbb{K})$. If $L$ has an abelian ideal $I$ of rank 2 over $R$, then $L$ can be embedded in a subalgebra $\bar{L}$ of $W_3(\mathbb{K}) = Der_{\mathbb{K}} R$ such that $\bar{L}$ is an extension of a subalgebra of $aff_2(F)$ by a subalgebra of dimension $\le 2$, where $F$ is the field of constants for the ideal $I$ in the field $R$.

Notations in the paper are standard. The ground field $\mathbb{K}$ is algebraically closed of characteristic zero. If $L$ is a subalgebra of the Lie algebra $W_3(\mathbb{K})$, then $F = F(L)$ is the field on constants of $L$ in $R = \mathbb{K}(x_1, x_2, x_3)$ (we consider any derivation $D \in W_3(\mathbb{K})$ as derivation of $R$ in the natural way: $D(f / g) = (D(f)g - fD(g)) / g^2$). If $V$ is an $n$-dimensional vector space over

$\mathbb{K}$ and $\mathfrak{gl}(V)$ the Lie algebra of all linear operators on $V$ we can consider the semidirect product $\mathfrak{gl}(V) \rightthreetimes V$, where $V$ is considered as an abelian Lie algebra. The Lie algebra $\mathfrak{gl}(V) \rightthreetimes V$ will be called the general affine Lie algebra and denoted by $aff_n(\mathbb{K})$ (in case $\mathbb{K} = \mathbb{R}$ the Lie algebra $aff_n(\mathbb{R})$ corresponds to the general affine Lie group $GA_n(\mathbb{R})$).

**Subalgebras with an abelian ideal of rank 3 over $R$.**

The next two lemmas contain standard facts about derivations (see for example, [8]). More information about derivations of polynomial rings can be found in [9].

**Lemma 1.** *Let $D_1, D_2 \in W_3(\mathbb{K})$ and $a, b \in R$. Then*

$$[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1.$$

*If $[D_1, D_2] = 0$, then $[aD_1, bD_2] = aD_1(b)D_2 - bD_2(a)D_1$.*

**Lemma 2.** *If $L \subseteq W_3(\mathbb{K})$ and $F = F(L)$ the field of constants for $L$ in $R$, then $FL$ is a Lie algebra over $F$. If $L$ is abelian, nilpotent or solvable then so is $FL$.*

**Lemma 3.** *Let $D_1, \ldots, D_n$ be a basis of the vector space $W_3(\mathbb{K})$ over the field $R$. Then $\bigcap_{i=1}^{n} Ker D_i = \mathbb{K}$.*

*Proof*. Suppose that $\bigcap_{i=1}^{n} Ker D_i \neq \mathbb{K}$ and let $f_1 \in \bigcap_{i=1}^{n} Ker D_i$, $f_1 \in R \setminus \mathbb{K}$. Then there exists a transcendence basis $\{f_1, \ldots, f_n\}$ of $R$ over $\mathbb{K}$ and the subfield $\mathbb{K}(f_1, \ldots, f_n)$ is isomorphic to the field $\mathbb{K}(x_1, \ldots, x_n)$. The function $f_1$ defines the derivation $S$ of the field $\mathbb{K}(f_1, \ldots, f_n)$ and this derivation can be uniquely extended to the derivation $S$ of $\mathbb{K}(x_1, \ldots, x_n)$ (we keep the same notation for the extended derivation). But $S = \sum_{i=1}^{n} s_i D_i$ for some $s_i \in R$ and therefore $S(f_1) = \sum_{i=1}^{n} s_i D_i(f_1) = 0$ by the choice of the element $f_1$. This is impossible because $S(f_1) = 1$. The obtained contradiction shows that $\bigcap_{i=1}^{n} Ker D_i = \mathbb{K}$.

**Corollary 1.** *If $L$ is an abelian subalgebra of $W_3(\mathbb{K})$ and $rk_R L = n$, then* $\dim_{\mathbb{K}} L = n$.

*Proof*. Let $D_1, \ldots, D_n$ be a basis of $L$ over $R$. Then any element $D \in L$ is of the form $D = \sum_{i=1}^{n} s_i D_i$ for some $s_i \in R$. Since $[D_i, D] = 0 = \sum_{j=1}^{n} D_i(s_j) D_j$ we have that $D_i(s_j) = 0$, $i, j = 1, \ldots, n$. By Lemma 3, $s_i \in \mathbb{K}$ and $D_1, \ldots, D_n$ is a basis of $L$ over $\mathbb{K}$. Thus $\dim_{\mathbb{K}} L = n$.

**Theorem 1.** *Let $L$ be a solvable subalgebra of the Lie algebra $W_3(\mathbb{K})$. If $L$ has an abelian ideal $I$ of rank 3 over $R$, then $L$ is isomorphic to a solvable subalgebra of the general affine Lie algebra $aff_3(\mathbb{K})$. In particular* $3 \leq \dim_{\mathbb{K}} L \leq 9$.

*Proof*. Take any basis $D_1, D_2, D_3$ of the ideal $I$ over the field $R$. Then any element $D \in L$ can be written in the form

$$D = s_1 D_1 + s_2 D_2 + s_3 D_3, \quad s_i \in R.$$

Since $[D_i, D] = D_i(s_1)D_1 + D_i(s_2)D_2 + D_i(s_3)D_3 \in I$ we have by Lemma 4 that $D_i(s_j) \in \mathbb{K}, i, j = 1,2,3$. So we can correspond to any element $D \in L$ the matrix

$$B_D = \begin{pmatrix} D_1(s_1) & D_1(s_2) & D_1(s_3) \\ D_2(s_1) & D_2(s_2) & D_2(s_3) \\ D_3(s_1) & D_3(s_2) & D_3(s_3) \end{pmatrix} \in M_3(\mathbb{K}). \quad (1)$$

Denote by $S$ the set of all columns of such matrices $B_D$, where $D$ runs over the subalgebra $L$. Since $S \subseteq \mathbb{K}^3$, the three-dimension vector space over $\mathbb{K}$, we have $d = rk_{\mathbb{K}} S \le 3$. If $d = 0$, then all columns for all $D \in L$ are zero and therefore $s_i \in \mathbb{K}, i = 1,2,3$ by Lemma 3. This means $L = I$. So we can assume that $d \ge 1$.

<u>Case 1.</u> $d = 1$. Then there exists an element $D \in L \setminus I$ which can be written in the form $D = s_1D_1 + s_2D_2 + s_3D_3$ such that all columns of $S$ are proportional to the column $(D_1(s_1), D_2(s_1), D_3(s_1))^T$ (here $\cdot^T$ denotes the transpose of the row) of the corresponding matrix $B_D$. Take any element $(D_1(t), D_2(t), D_3(t))^T \in S$. Then there exists $\gamma \in \mathbb{K}$ such that

$$(D_1(t), D_2(t), D_3(t))^T = \gamma (D_1(s_1), D_2(s_1), D_3(s_1))^T.$$

It follows from the last equality that

$$D_1(t - \gamma s_1) = D_2(t - \gamma s_1) = D_3(t - \gamma s_1) = 0.$$

By Lemma 3 we obtain $t - \gamma s_1 = \delta$ for some $\delta \in \mathbb{K}$, i.e. $t = \gamma s_1 + \delta$. The latter means that for any element $D \in L,\ D = t_1D_1 + t_2D_2 + t_3D_3, t_i \in R$, the corresponding matrix $B_D$ has the columns $(D_1(t_i), D_2(t_i), D_3(t_i))^T$, $i = 1,2,3$, with $t_i = f_i(s), \deg f_i \le 1, f_i \in \mathbb{K}[t]$. Since $(D_1(s_1), D_2(s_1), D_3(s_1))^T$ is nonzero we can assume without loss of generality that $D_1(s_1) = 1, D_2(s_1) = \gamma_2, D_3(s_1) = \gamma_3$ for some $\gamma_2, \gamma_3 \in \mathbb{K}$. Put

$$D_{1'} = D_1, \quad D_{2'} = D_2 - \gamma_2 D_1, \quad D_{3'} = D_3 - \gamma_3 D_1.$$

Then $D_{1'}(s_1) = 1, D_{2'}(s_1) = 0, D_{3'}(s_1) = 0$ and $D_{1'}, D_{2'}, D_{3'}$ form a basis of $I$ over $R$. Let $D = t_1D_1 + t_2D_2 + t_3D_3$ be an arbitrary element in $L$ and $t_i = \gamma_i s_i + \delta_i, i = 1,2,3.$ Then the map $\varphi : L \to aff_3(\mathbb{K})$ which is defined by the rule: $\varphi(D_i) = x_i$, $\varphi(s_1D_i) = x_1x_i$ and further by linearity, is an embedding of $L$ into the Lie algebra $aff_3(\mathbb{K})$.

<u>Case 2.</u> $d = rk_{\mathbb{K}} S = 2$. Then there exist linearly independent columns on the set $S$ of the form

$$(D_1(s_1), D_2(s_1), D_3(s_1))^T, (D_1(s_2), D_2(s_2), D_3(s_2))^T \quad (2)$$

(these columns can belong to different matrices $B_D, D \in L$). Therefore any column $(D_1(t), D_2(t), D_3(t))^T \in S$ is a linear combination of columns in (2). One can easily show that $t = f(s_1, s_2)$ for some polynomial $f \in \mathbb{K}[u, v]$, $\deg f \le 1$. Note that the rank of the matrix

$$\begin{pmatrix} D_1(s_1) & D_1(s_2) \\ D_2(s_1) & D_2(s_2) \\ D_3(s_1) & D_3(s_2) \end{pmatrix} \quad (3)$$

is equal to $2$. Without loss of generality one can assume that the first and second rows of this matrix are linearly independent. But then there exist $\gamma_1, \gamma_2 \in \mathbb{K}$ such that

$$(1,0) = \gamma_1 \left(D_1(s_1), D_1(s_2)\right) + \gamma_2 \left(D_2(s_1), D_2(s_2)\right). \tag{4}$$

Denoting $D_{1'} = \gamma_1 D_1 + \gamma_2 D_2$ we have $D_{1'}(s_1) = 1,\ D_{1'}(s_2) = 0$. Analogously one can find $\delta_1, \delta_2 \in \mathbb{K}$ such that the element $D_{2'} = \delta_1 D_1 + \delta_2 D_2$ has properties $D_{2'}(s_1) = 0,\ D_{2'}(s_2) = 1$.

Further, the third row of the matrix (3) is a linear combination of the first and second rows and therefore $(D_3 - \mu_1 D_1 - \mu_2 D_2)(s_i) = 0,\ i = 1,2$. Denoting $D_{3'} = D_3 - \mu_1 D_1 - \mu_2 D_2$ we obtain $D_{i'}(s_j) = \delta_{ij},\ i = 1,2,3,\ j = 1,2$. If $D \in L$ is an arbitrary element, then $D = t_1 D_1 + t_2 D_2 + t_3 D_3$ for some $t_1, t_2, t_3 \in R$. Since $t_i = f_i(s_1, s_2),\ \deg f_i \le 1$ we see that $L$ can be embedded in the Lie algebra $aff_3(\mathbb{K})$.

Case 3. $rk_{\mathbb{K}} S = 3$ can be considered analogously.

**Subalgebras with abelian ideals of $rk \le 2$ over $R$.**

**Lemma 4.** *Let $L$ be a subalgebra of the Lie algebra $W_n(\mathbb{K})$ and $I$ be an ideal of $L$. If $F = F(I)$ is the field of constants for $I$ in $R$, then $D(F) \subseteq F$ for any element $D \in L$.*

*Proof*. Let $D \in L$ and $r \in F$ be arbitrarily chosen. Then for any $D_1 \in I$ we have $D_1(r) = 0$ and therefore

$$0 = D(D_1(r)) = D_1(D(r)) + [D, D_1](r).$$

Since $[D, D_1] \in I$ we have $[D, D_1](r) = 0$ and consequently $D_1(D(r)) = 0$. The latter means that $D(r) \in F$ because the element $D_1$ was arbitrarily chosen in the ideal $I$. Thus $D(F) \subseteq F$.

**Theorem 2.** *Let $L$ be a solvable finite dimensional subalgebra of the Lie algebra $W_3(\mathbb{K})$ with $\mathrm{r}k_R L = 3$. If $L$ has an ideal $I$ of rank 2 over $R$ and $F = F(L)$ is the field of constants of $I$ in $R$, then the Lie algebra $L$ is contained in the subalgebra $\bar{L} = F\bar{I} + L$ of $W_3(\mathbb{K})$ where $\bar{I} = (RI) \cap L$. The Lie algebra $\bar{L}$ is solvable, $F\bar{I}$ is its ideal of rank 2 over $R$ which is isomorphic to a subalgebra of $aff_2(F)$. The Lie algebra $\bar{L}$ is an extension of the ideal $F\bar{I}$ by a Lie algebra of dimension 1 or 2 over $\mathbb{K}$.*

*Proof*. The intersection $\bar{I} = (RI) \cap L$ is an ideal of the Lie algebra $L$ with $\mathrm{r}k_R \bar{L} = 2$ and $\dim_{\mathbb{K}} L / \bar{I} \le 2$ (see [8]). Let $F$ be the field of constants for $I$ in $R$. Since $D(F) \subseteq F$ for any $D \in L$ (by Lemma 4), the subalgebra $F\bar{I}$ of the algebra $W_3(\mathbb{K})$ is an ideal of the Lie algebra $F\bar{I} + L$. One can easily show that $\mathrm{r}k_{\mathbb{R}} \bar{I} = 2$. By Theorem 1 of the paper [6], the Lie algebra $F\bar{I}$ (as a Lie algebra over the field $F$) is isomorphic to a subalgebra of the Lie algebra $aff_2(F)$. Since $\dim_{\mathbb{K}} L / \bar{I} \le 2$, it holds obviously $\dim_{\mathbb{K}} L + F\bar{I} / F\bar{I} \le 2$. Note that the Lie algebra $L + F\bar{I}$ is in general case of infinite dimension over $\mathbb{K}$ although $\dim_F F\bar{I} \le 7$ (the sum $F\bar{I} + L$ is not in general a Lie algebra over $F$ but only over the field $\mathbb{K}$). The proof is complete.

Further notations are taken from Theorem 2. Let $I_1 = \mathbb{K}D_1$ be a one-dimensional ideal of $L$ lying in $I$ and $\mathbb{K}D_2 + I_1$ be an ideal of the quotient algebra $L/I_1$ lying in $I/I_1$ (such ideals do exist because $L$ is solvable and $\mathbb{K}$ is algebraically closed). Let $\mathbb{K}D_3+\bar{I}$ be one-dimensional ideal of the Lie algebra $L/\bar{I}$. Then $D_1, D_2, D_3$ are linearly independent over $R$ and form a basis of $RL$ over $R$. By the choice of $D_1$ and $D_2$ there exist $\lambda_1, \lambda_2 \in K$ and $g_2 \in F$ such that

$$[D_3, D_1] = \lambda_1 D_1, \quad [D_3, D_2] = \lambda_2 D_2 + g_2 D_1.$$

The next statement gives more detailed description of the Lie algebra $\bar{L} = F\bar{I} + L$.

**Proposition 1.** *Let $L \subseteq W_3(\mathbb{K})$ be a solvable finite dimensional subalgebra of rank 3 over $R$ with $\dim L > 6$. Under conditions of Theorem 2 either there exist $r_1, r_2 \in R$ with $D_i(r_j) = \delta_{ij}$, $i, j = 1, 2$, and every element $D \in F\bar{I}$ is of the form $D = f_1(r_1, r_2)D_1 + f_2(r_1, r_2)D_2$, $f_i \in \mathbb{K}[t_1, t_2]$, $\deg f_i \le 1$, or there exists $r_i \in R$, $i = 1$ or $i = 2$, with $D_i(r_j) = \delta_{ij}$ and every element $D \in F\bar{I}$ is of the form $D = g_1(r_i)D_1 + g_2(r_i)D_2$, $\deg g_j \le 1$. Then $D_3(r_1) = -\lambda_1 r_1 - g_2 r_2$, $D_3(r_2) = -\lambda_2 r_2$. If $\dim_{\mathbb{K}} L/\bar{I} = 2$, then there exists $\bar{D} \in L \setminus (\mathbb{K}D_3 + \bar{I})$ such that $\bar{D} = r_3 D_3 + s_2 D_2$, $r_3 \in R$, $D_3(r_3) = 1$, $D_1(r_3) = D_2(r_3) = 0$, $D_1(s_2) = 0$, and in this case $\lambda_1 = 0$, $g_2 = 0$, $s_2 = \lambda_2 r_2 r_3 + f$, $f \in \mathbb{K}$.*

*Proof*. Repeating considerations from the proof of Theorem 1 one can find either elements $r_1, r_2$ with $D_i(r_j) = \delta_{ij}, i, j = 1, 2$, or an element $r \in R$ such that either $D_1(r) = 1, D_2(r) = \gamma$ or $D_1(r) = \delta, D_2(r) = 1$ using only transformations of columns of the matrix $B_D = \begin{pmatrix} D_1(s_1) & D_1(s_2) \\ D_2(s_1) & D_2(s_2) \end{pmatrix}$. If $\delta \ne 0$ we can consider elements $D_{2'} = D_2 - \delta D_1$, $D_{1'} = D_1$ and in this case $D_{1'}(r) = 0$, $D_{2'}(r) = 1$. So we can assume that either $D_1(r) = 1$, $D_2(r) = 0$ or $D_1(r) = 0$, $D_2(r) = 1$ and $r$ is either $r_1$ or $r_2$.

Let us consider the action of elements $D_i$ on $r_i$, $s_j$, $i, j = 1, 2, 3$.

Since $D_1(r_1) = 1$ we have $D_3(D_1(r_1)) = 0$ and therefore

$$D_1(D_3(r_1)) = D_3(D_1(r_1)) - [D_3, D_1](r_1) = 0 - \lambda_1 D_1(r_1) = -\lambda_1.$$

It follows from the equalities $D_1(D_3(r_1)) = -\lambda_1$ and $D_1(-\lambda_1 r_1) = -\lambda_1$ that $D_1(D_3(r_1) + \lambda_1 r_1) = 0$, i.e. $D_3(r_1) = -\lambda_1 r_1 + s'$ for some $s' \in KerD_1$. Analogously the equality

$$D_2(d_3(r_1)) = D_3(D_2(r_1)) - [D_3, D_2](r_1)$$

implies $D_3(r_1) = -g_2 r_2 + s''$ for some $s'' \in KerD_2$. Applying $D_1$ to both sides of the obtained equality $-\lambda_1 r_1 + s' = -g_2 r_2 + s''$ we get $-\lambda_1 = D_1(s'')$. After applying $D_2$ to the same equality we get $D_2(s') = -g_2$. But then $s'' + \lambda_1 r_1 \in KerD_1$. Since $s'' + \lambda_1 r_1 \in KerD_2$ we have $s'' + \lambda_1 r_1 \in KerD_1 \cap KerD_2 = F$. Thus $s'' = -\lambda_1 r_1 + v_1$ for some $v_1 \in F$. It follows from the equality $-\lambda_1 r_1 + s' = -g_2 - \lambda_1 r_1 + v_1$ that $s' = -g_2 r_2 + v_1$. Finally we get

$$D_3(r_1) = -\lambda_1 r_1 - g_2 r_2 + v_1, \; v_1 \in F.$$

Analogously it follows from the equalities

$$D_2(D_3(r_2)) = D_3(D_2(r_2)) - [D_3, D_2](r_2) = 0 - (\lambda_2 D_2 + g_2 D_1)(r_2) = -\lambda_2$$

that $D_3(r_2) = -\lambda_2 r_2 + t'$ for some $t' \in Ker D_2$ and finally

$$D_3(r_2) = -\lambda_2 r_2 + v_2, v_2 \in F.$$

Without loss of generality we can change $D_3$ by $D_{3'} = D_3 - v_1 D_1 - v_2 D_2$. Then $D_{3'}(r_1) = -\lambda_1 r_1 - g_2 r_2$, $D_{3'}(r_2) = -\lambda_2 r_2$. Returning to the old notation we have $D_3(r_1) = -\lambda_1 r_1 - g_2 r_2, D_3(r_2) = -\lambda_2 r_2$.

Let now $\dim_{\mathbb{K}} L / \bar{I} = 2$ and $\bar{D} = r_3 D_3 + s_1 D_1 + s_2 D_2$ be any element of $L \setminus (\mathbb{K} D_3 + I)$. Then

$$[\bar{D}, D_3] = [r_3 D_3 + s_1 D_1 + s_2 D_2, D_3] =$$
$$= -D_3(r_3) D_3 - D_3(s_1) D_1 - s_1 [D_1, D_3] - D_3(s_2) D_2 - s_2 [D_2, D_3] =$$
$$= -D_3(r_3) D_3 + (-D_3(s_1) + \lambda_1 s_1 + s_2 g_2) D_1 + (-D_3(s_2) + \lambda_2 s_2) D_2.$$

It follows from these equalities that $D_3(r_3) = -\gamma$, where $\gamma$ is taken from the equality $[\bar{D}, D_3] = \gamma D_3 + D$, where $D \in \bar{I}$. Analogously the equality

$$[r_3 D_3 + s_1 D_1 + s_2 D_2, D_1] = \mu D_1$$

for some $\mu \in \mathbb{K}$ implies $D_1(r_3) = 0, D_1(s_2) = 0$. The equality

$$[r_3 D_3 + s_1 D_1 + s_2 D_2, D_2] = f_1 D_1 + f_2 D_2$$

for some $f_1, f_2 \in F$ yields $D_3(r_3) = 0$. Summarizing we get

$$D_1(r_3) = D_2(r_3) = 0, \quad D_3(r_3) = 1, \quad D_1(s_2) = 0. \tag{5}$$

Since $[\bar{D}, D_1] = \theta D_1$ for some $\theta \in \mathbb{K}$ we have

$$[r_3 D_3 + s_1 D_1 + s_2 D_2, D_3] = (\lambda_1 r_3 - D_1(s_1)) D_1$$

and therefore $\lambda_1 r_3 - D_1(s_1) = \theta$. Thus $D_1(s_1) = \lambda_1 r_3 + \theta$, $\theta \in \mathbb{K}$. Further $[\bar{D}, D_2] = f_1 D_1 + f_2 D_2$ for some $f_1, f_2 \in F$. Analogously $[r_3 D_3 + s_1 D_1 + s_2 D_2, D_2] =$ $= (r_3 g_2 - D_2(s_1)) D_1 + (\lambda_2 r_2 - D_2(s_2)) D_2$ and therefore

$$D_2(s_1) = g_2 r_3 - f_2, \quad D_2(s_2) = \lambda_2 r_3 - f_2. \tag{6}$$

But we have

$$s_1 = g_2 r_2 r_3 - r_2 f_2 + f_3, \quad s_2 = \lambda_2 r_2 r_3 - r_2 f_2 + f_4$$

for some $f_3, f_4 \in F$. It was proved early that $D_1(s_1) = \lambda_1 r_3 + \theta$, $\theta \in \mathbb{K}$, so we have $s_1 = \lambda_1 r_1 r_3 + \theta r_1 + f_5$ for some $f_5 \in F$. Applying $D_2$ to the both sides of the equality

$$\lambda_1 r_1 r_3 + \theta r_1 + f_5 = g_2 r_2 r_3 - r_2 f_2 + f_3 \tag{7}$$

we get $g_2 r_3 - f_2 = 0$. But $r_1, r_2, r_3$ are linearly independent over $F$, so the last equality yields $g_2 = 0$. The equality (7) is now of the form

$$\lambda_1 r_1 r_3 + \theta r_1 + f_5 = -r_2 f_2 + f_3.$$

Applying $D_2$ to the both sides of this equality we get $f_2 = 0$. Therefore $\lambda_1 r_1 r_3 + \theta r_1 + f_5 = f_3$. Applying $D_1$ to the both sides of the last equality we get $\lambda_1 r_3 + \theta = 0$. Since $r_3 \not\in \mathbb{K}$ we have $\lambda_1 = 0$ and therefore $s_1 = 0$. Analogously we can assume that $f_4 = 0$ and $s_2 = \lambda_2 r_2 r_3$. So we have

$$s_1 = 0, \quad s_2 = \lambda_2 r_2 r_3, \quad g_2 = 0, \quad f_2 = 0, \quad \lambda_1 = 0.$$

These equalities means that

$$[D_3, D_1] = 0, \quad [D_3, D_2] = \lambda_2 D_2, \quad \overline{D} = r_3 D_3 + s_2 D_2,$$

where $s_2 = \lambda_2 r_2 r_3$, $D_i(r_j) = \delta_{ij}$, $i, j = 1, 2, 3$. The proof is complete.


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## РОЗВ'ЯЗНІ АЛГЕБРИ ЛІ ДИФЕРЕНЦІЮВАНЬ КІЛЕЦЬ МНОГОЧЛЕНІВ ВІД ТРЬОХ ЗМІННИХ


*Нехай $\mathbb{K}$ – алгебраїчно замкнене поле характеристики нуль, $A = \mathbb{K}[x_1, x_2, x_3]$ – кільце многочленів від трьох змінних і $R = \mathbb{K}(x_1, x_2, x_3)$ – поле раціональних функцій. Якщо $L$ – підалгебра алгебри Лі $W_3(\mathbb{K})$ всіх $\mathbb{K}$-диференціювань кільця $A$, то $RL$ є алгеброю Лі над $\mathbb{K}$ і $dim_R RL$ називається рангом алгебри $L$ над $R$. Вивчаються підалгебри $L$ рангу 3 над $R$ алгебри Лі $W_3(\mathbb{K})$. Доведено, що якщо $L$ містить абелевий ідеал $I$ рангу 3 над $R$, то $L$ ізоморфна підалгебрі загальної афінної алгебри Лі $aff_3(\mathbb{K})$. Якщо $L$ має ідеал $I$ з $rk_R I = 2$, то $L$ міститься в підалгебрі $\bar{L}$ алгебри $\tilde{W}_3(\mathbb{K}) = Der_{\mathbb{K}} R$, де $\bar{L}$ – розширення деякої підалгебри з $aff_2(F)$ за допомогою підалгебри розмірності $\leq 2$, $F$ – поле констант для $I$ в $R$.*


## РАЗРЕШИМЫЕ АЛГЕБРЫ ЛИ ДИФФЕРЕНЦИРОВАНИЙ КОЛЕЦ МНОГОЧЛЕНОВ ОТ ТРЕХ ПЕРЕМЕННЫХ


*Пусть $\mathbb{K}$ – алгебраически замкнутое поле характеристики нуль, $A = \mathbb{K}[x_1, x_2, x_3]$ – кольцо многочленов от трех переменных и $R = \mathbb{K}(x_1, x_2, x_3)$ – поле рациональных функций. Если $L$-подалгебра алгебры Ли $W_3(\mathbb{K})$ всех $\mathbb{K}$-дифференцирований кольца $A$, то $RL$ является алгеброй Ли над $\mathbb{K}$ и $dim_R RL$ называется рангом алгебры $L$ над $R$. Исследуются подалгебры $L$ ранга 3 над $R$ алгебры Ли $W_3(\mathbb{K})$. Доказано, что если $L$ содержит абелев идеал $I$ ранга 3 над $R$, то $L$ изоморфна подалгебре общей афинной алгебры Ли $aff_3(\mathbb{K})$. Если $L$ содержит идеал $I$ с $rk_R I = 2$, то $L$ содержится в подалгебре $\bar{L}$ алгебры $\tilde{W}_3(\mathbb{K}) = Der_{\mathbb{K}} R$, где $\bar{L}$ – расширение некоторой подалгебры из $aff_2(F)$ с помощью подалгебры размерности $\leq 2$, а $F$ – поле констант для $I$ в $R$.*



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