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p-GROUPS WITH CYCLIC SUBGROUP OF INDEX p AND LOCAL NEARRINGS

p-Groups with cyclic subgroup of index p as the additive groups and the multiplicative groups of local nearrings are investigated. Furthermore, the classifications of such nearrings are given.

Keywords: p-group, nearring, local nearring, additive group, multiplicative group.

Introduction. Clearly every associative ring is a nearring and each group is the additive group of a nearring, but not necessarily of a nearring with identity. The question what group can be the additive group of a nearring with identity is far from solution.

In [5] it was shown that on each cyclic group there is a unique nearring with identity which is in fact a commutative ring with identity.

Furthermore, it was proved that there exists no nearring with identity on the quaternion group and found all nearrings with identity on the dihedral group of order 8 in [4]. It was shown that the generalized quaternion groups cannot be the additive groups of nearrings with identity in [6].

The dihedral groups D_n of order $2n$ as additive groups of a nearring with identity were investigated in [11]. It was proved that D_n is such a group if and only if $n = 2p$ for a prime number p and in the case, when p is odd, there exists, up to isomorphism, a unique nearring with identity on D_n.

In [3] it is classified all non-abelian groups of order less than 32 that can be the additive groups of a nearring with identity and found the number of non-isomorphic nearrings with identity on such groups (see also GAP package SONATA [1]).

Recall that a nearring R is called local, if the set L of all non-invertible elements of R forms a subgroup of its additive group R^+.

A study of local nearrings was initiated by Maxson [13] who defined a number of their basic properties and proved in particular that the additive group of a finite zero-symmetric local nearring is a p-group. It follows from [5] that local nearrings with cyclic additive group are commutative local rings.

In [14] it is described all non-isomorphic zero-symmetric local nearrings with non-cyclic additive groups of order p^2 which are not nearfields. He also shown in [15] that every non-cyclic abelian p-group of order $p^n > 4$ is the additive group of a zero-symmetric local nearring which is not a ring. For instance, neither a generalized quaternion group nor a non-abelian group of order 8 can be the additive group of a local nearring [16] (see also [6]).

The split metacyclic groups which are the additive groups of finite local nearrings were classified in [22].

However, it is not true that any finite group is the additive group of a nearring with identity. Therefore the determination of the non-abelian finite p-groups which are the additive groups of local nearrings is an open problem (see [7]).

Local nearrings with abelian multiplicative groups that are not rings are studied in [9]. In particular, it was proved that the set of all non-invertible

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elements of this nearring forms an abelian subgroup of index 2 in its additive group. Also, it was obtained the full classification of such nearrings with cyclic multiplicative groups.

The local nearrings of order 2^n whose multiplicative group is minimal non-abelian or, in a different terminology, a Miller–Moreno group, are investigated in [25]. Moreover, it was proved that order of local nearrings with metacyclic multiplicative 2-group does not exceed 32.

In present paper we study -groups with cyclic subgroup of index p as the additive groups and the multiplicative groups of local nearrings. Furthermore, the classifications of such nearrings are given.

1. Preliminaries. In [10, Theorem 12.5.1] it was proved that there exist seven types of p-groups with cyclic subgroup of index p.

Theorem 1. *The groups of order p^n which contain a cyclic subgroup of index p are of the following types:*

abelian, $n \geq 1$, cyclic:

1) $a^{p^n} = 1$;

$n \geq 2$:

2) $a^{p^{n-1}} = 1$, $b^p = 1$, $ba = ab$;

non-abelian, p is odd, $n \geq 3$:

3) $a^{p^{n-1}} = 1$, $b^p = 1$, $ba = a^{1+p^{n-2}}b$;

$p = 2$, $n \geq 3$:

4) generalized quaternion group:

$a^{2^{n-1}} = 1$, $b^2 = a^{2^{n-2}}$, $ba = a^{-1}b$;

$p = 2$, $n \geq 3$:

5) dihedral group:

$a^{2^{n-1}} = 1$, $b^2 = 1$, $ba = a^{-1}b$;

$p = 2$, $n \geq 4$:

6) $a^{2^{n-1}} = 1$, $b^2 = 1$, $ba = a^{1+2^{n-2}}b$;

$p = 2$, $n \geq 4$:

7) $a^{2^{n-1}} = 1$, $b^2 = 1$, $ba = a^{-1+2^{n-2}}b$.

We recall some notions and facts concerning nearrings.

Definition 1. A (left) nearring is a set $R = (R, +, \cdot)$ with two binary operations, addition $+$ and $\cdot$ multiplication, such that

1. $(R, +)$ is a group with neutral element 0,
2. $(R, \cdot)$ is a semigroup, and
3. $x(y+z) = xy + xz$ for all x, y, $z \in R$.

The group $(R, +)$ of a nearring R is denoted by R^+ and called the *additive group* of R. If in addition $0 \cdot x = 0$ for all $x \in R$, then the nearring R is called *zero-symmetric*. Furthermore, R is a *nearring with an identity i* if the semigroup $(R, \cdot)$ is a monoid with identity element i. In the latter case the group of all invertible elements of the monoid $(R, \cdot)$ is denoted by R^* and called the *multiplicative group* of R.

Definition 2 [13]. *A nearring R with identity is said to be local if the set $L = R \setminus R^*$ of all non-invertible elements of R is a subgroup of R^+.*

As it was shown in [13, Theorem 7.4], the additive group of a finite local nearring is a p -group for a prime p.

We have the following result.

Proposition 1. *Let R be a local nearring of order p^n and L be of order p^k. The multiplicative group R^* has a prime power order if and only if $p = 2$ and $k = n - 1$, i.e. $|R^*| = 2^{n-1}$.*

P r o o f. Since $R = R^* \cup L$ by Definition 2 it implies $|R| = |R^*| + |L|$. So $|R^*| = |R| - |L| = p^n - p^k = p^k(p^{n-k} - 1)$. Therefore $|R^*|$ can be power of prime if and only if $p = 2$ and $k = n - 1$, i.e. $|R^*| = 2^{n-1}$ as desired. □

In [12] the investigation was focused on some finite groups which are the additive groups of nearrings with identity. These groups have a presentation of the form

$$H = \left\langle a, b \mid a^{p^{n-1}} = 1, b^p = 1, a^b = a^{1+p^{n-2}} \right\rangle \text{ for a prime } p \text{ and } n \geq 3. \quad (1)$$

It is clear that these groups are Miller–Moreno groups, i.e. it is non-abelian and all its proper subgroups are abelian.

We will assume always that $p + n > 5$, since the case $p + n = 5$ is the dihedral group D_4 already considered by Clay [4].

Denote by $n(G)$ the number of all non-isomorphic zero-symmetric nearrings with identity R whose additive group R^+ is isomorphic to the group G.

Theorem 2 [12, Theorem 7.1].

(i) If $p = 2$ and $n = 4$, then $n(H) = 32$.

(ii) If $p = 2$ and $n > 4$, then $n(H) = 2^{n+2}$.

(iii) If $p = 3$, then $n(H) = 3^{n-2}$.

(iv) If $p > 3$, then $n(H) = p^{n-3}$.

The following result immediately follows from [12, Theorem 4.2].

Proposition 2. *There exists exactly one non-isomorphic non-zero-symmetric nearring with identity on each group of presentation (1).*

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Theorem 3. *Let G be a group of Theorem 1. G is the additive group of a nearring with identity if and only if one of the following statement holds:*

1. G is of type 1);
2. G is of type 2);
3. G is of type 3);
4. G is of type 5) and $n = 3$;
5. G is of type 6).

P r o o f. As it was shown in [5, Theorem 1], the groups of type 1) is the additive group of nearrings with identity, that is commutative rings.

As a direct consequence of [6, Theorem] we get statement 2 of the theorem.

Groups of types 3) and 6) are metacyclic Miller–Moreno groups. Therefore from [20, Theorem 3] we obtain statements 3 and 5 of the theorem.

Consider the case of groups of type 4) cannot be the additive groups of nearrings with identity by [6].

Since groups of type 5) are dihedral groups with and from results of paper [11] we get statement 4 of the theorem.

As corollary of [22, Lemma 14] there cannot be a nearring with identity defined on groups of type 7). □

It is clear that groups of types 3) and 6) are of presentation (1). So using Theorem 2 and [4] we can easily conclude the following result:

Proposition 3. *Let G be a non-abelian group of Theorem 3. Then the following statements hold:*

(i) If $p = 2$ and $n = 3$, then $n(G) = 7$.

(ii) If $p = 2$ and $n = 4$, then $n(G) = 32$.

(iii) If $p = 2$ and $n > 4$, then $n(G) = 2^{n+4}$.

(iv) If $p = 3$, then $n(G) = 3^n$.

(v) If $p > 3$, then $n(G) = p^{n-1}$.

It is interested to know what happens with statement of Theorem 3 in case when nearring is local.

Corollary 1. *Let G be a group of Theorem 1. G is the additive group of a local nearring if and only if one of the following statement holds:*

1. G is of type 1);
2. G is of type 2);
3. G is of type 3);
4. G is of type 6).

It was mentioned that if G is of type 1) then R is a local commutative ring and verse vice. So we consider groups which are not cyclic.

Proposition 4. *Let G be a non-cyclic group from Theorem 3. If L is cyclic then G is an elementary abelian group of order p^2.*

Proof. It is immediately follows from [19, Theorem 1]. □

As direct consequence from [24, Lemma 6] we get the following result.

Proposition 5. *Let G be a group of type* 3) and 6) *of Theorem 1. Then $|R : L| = p$ or $|R : L| = 2$ respectively.*

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Theorem 4. *Let G be a group of Theorem 1. G is the multiplicative group of a local nearring R if and only if one of the following statement holds:*

1. G is of type 1) and $p = 2$;
2. G is of type 2), $p = 2$, $n = 3$ or $n = 4$;
3. G is of type 4) and $n = 3$ or $n = 4$;
4. G is of type 5) and $n = 3$;
5. G is of type 6) and $n = 4$;
6. G is of type 7) and $n = 4$.

Proof. Statements 1 and 2 are direct consequences of [9, Theorem 1] and Proposition 1. By the last proposition groups of type 3) cannot support local nearrings.

Groups of types 6) and 7) are metacyclic. So by [25, Lemma 9] statements 5 and 5) hold.

From [26] and [2] we get statement 3 and 4. □

Since local nearrings with cyclic multiplicative group was completely studied by Gorodnik we focus on non-cyclic case. From [21], [1] and [23] we can conclude the following lemma.

Proposition 6. *Let G be a non-cyclic group of Theorem 4. There exist* 501 *local nearrings whose multiplicative group is isomorphic to G.*

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p-ГРУПИ З ЦИКЛІЧНОЮ ПІДГРУПОЮ ІНДЕКСУ p ТА ЛОКАЛЬНІ МАЙЖЕ-КІЛЬЦЯ

Досліджено p-групи з циклічною підгрупою індексу p як адитивні групи та мультиплікативні групи локальних майже-кілець. Крім того, наведено класифікацію таких майже-кілець.

Ключові слова: *p-група, майже-кільце, локальне майже-кільце, адитивна група, мультиплікативна група.*

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