https://doi.org/10.15407/apmm2024.22.123-130

UDC 512.6

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ON DIRECT PRODUCTS OF METACYCLIC MILLER–MORENO p-GROUPS AND CYCLIC p-GROUPS AS ADDITIVE GROUPS OF LOCAL NEARRINGS

A finite group is called a Miller–Moreno group if it is non-abelian and all its proper subgroups are abelian. The direct products of Miller–Moreno p-groups and cyclic p-groups as additive groups of nearrings with identity and local nearrings are considered.

Keywords: nearring, local nearring, Miller–Moreno group

Introduction. A nearring R with an identity is called local if the set of all non-invertible elements of R forms a subgroup of the additive group of R.

In paper [8] it was given a full classification of the metacyclic Miller–Moreno p-groups which appear as the additive groups of finite local nearrings. Moreover, if G is such an additive group, then we describe all possible multiplications "$\cdot$" on G for which the system $(G,+,\cdot)$ is a local nearring.

In the paper we consider the direct products of Miller–Moreno p-groups and cyclic p-groups as additive groups of nearrings with identity and local nearrings.

1. Preliminaries. We also recall, that a finite group is called a Miller–Moreno group if it is non-abelian and all its proper subgroups are abelian.

As a direct consequence of [9] we get the following statement.

Lemma 1. *Metacyclic Miller–Moreno p-groups, where p is a prime number and $p > 2$, are isomorphic to the group $G = \langle a \rangle \rtimes \langle b \rangle$ of order p^{m+n} with $a^{p^m} = b^{p^n} = 1$ and $b^{-1}ab = a^{1+p^{m-1}}$, where $m \geq 2$ and $n \geq 1$.*

Consider the direct products of metacyclic Miller–Moreno p-groups and cyclic p-groups. It is trivially obtain from Lemma 1 the following result.

Lemma 2. *Let G be a direct product of metacyclic Miller–Moreno p-group and cyclic p-group. Then G is a group of the following type: the group $G = (\langle a \rangle \rtimes \langle b \rangle) \times \langle c \rangle$ of order p^{m+n+k} with $a^{p^m} = b^{p^n} = c^{p^k} = 1$, $b^{-1}ab = a^{1+p^{m-1}}$, $ca = ac$ and $cb = bc$, where $m \geq 2$, $n \geq 1$ and $k \geq 1$.*

In what follows we use the following notation: $F(p^m, p^n, p^k)$ denotes an additively written group from Lemma 2 with generators a, b and c of orders p^m, p^n and p^k, respectively, so that $-b + a + b = a(1 + p^{m-1})$, $c + a = a + c$ and $c + b = b + c$, where $m \geq 2$, $n \geq 1$ and $k \geq 1$.

We will give the basic definitions (see, [4], [5]).

Definition 1. A set R with two binary operations "$+$" and "$\cdot$" is called a (left) nearring if the following statements hold:

1) $(R,+) = R^+$ is a (not necessarily abelian) group with neutral element 0;

2) $(R,\cdot)$ is a semigroup;

3) $x(y+z) = xy + xz$ for all x, y, $z \in R$.

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If R is a nearring, then the group R^+ is called the *additive group* of R. As it follows from statement 3), for each subgroup M of R^+ and each element $x \in R$ the set $xM = \{x \cdot y \mid y \in M\}$ is a subgroup of R^+ and, in particular, $x \cdot 0 = 0$. If in addition $0 \cdot x = 0$, then the nearring R is called *zero-symmetric*, and if the semigroup $(R,\cdot)$ is a monoid, i.e. it has an identity element i, then R is a *nearring with identity* i. In the latter case the group R^* of all invertible elements of the monoid $(R,\cdot)$ is called the *multiplicative group* of R. A subgroup M of R^+ is called R^*-*invariant* if $rM \leq M$ for each $r \in R^*$, and M is an (R,R)-*subgroup*, if $xMy \subseteq M$ for arbitrary x, $y \in R$.

Definition 2. A nearring R with identity is said to be local if the set $L = R \setminus R^*$ of all non-invertible elements of R is a subgroup of R^+.

Some basic properties of local nearrings are described in the following lemma (see [2], Lemmas 3.2, 3.4 and 3.9).

Lemma 3. *Let R be a local nearring with identity i and L its subgroup of all non-invertible elements of R^+. Then the following statements hold:*

1) L is an (R,R)-subgroup of R^+;

2) each proper R^*-invariant subgroup of R^+ is contained in L;

3) if R is finite, then y is a p-group for some prime p, the subgroup L is normal in R^+ and the factor group R^+ / L is elementary abelian.

2. Groups $F(p^m, p^n, p^k)$. Recall that the exponent of a finite p-group is the maximal order of its elements. The following assertion is easily verified.

Lemma 4. The exponent of $F(p^m, p^n, p^k)$ is equal to p^m for $m \geq n$ and $m \geq k$, to p^n for $n > m$ and $n > k$, and p^k for $k > m$ and $k > n$. If x is an element of maximal order in $F(p^m, p^n, p^k)$, then there exist generators a, b, c of this group such that either $a = x$, $b = x$ or $c = x$ and the relations $a^{p^m} = b^{p^n} = c^{p^k} = 1$, $b^{-1}ab = a^{1+p^{m-1}}$, $ca = ac$ and $cb = bc$, where $m \geq 2$, $n \geq 1$ and $k \geq 1$ hold.

Lemma 5. Let a group G be isomorphic to $F(p^m, p^n, p^k)$. Then for any natural numbers r, s, t, u the equalities

$$cu + bs + ar = ar(1 + sp^{m-1}) + bs + cu$$

and

$$(ar + bs + cu)t = ar(t + s\binom{t}{2}p^{m-1}) + bst + cut \text{ hold.}$$

P r o o f. Let $q = 1 + p^{m-1}$. Since $-b + a + b = a(1 - p^{m-1})$, $a + c = c + a$ and $b + c = c + b$, then $b + a = aq + b$, so $bs + ar = arq^s + bs$ for arbitrary integers $r \geq 0$ and $s \geq 0$. Taking into consideration, that

$$q^s = (1 + p^2)^s \equiv 1 + sp^{m-1} \pmod{p^{m-1}}$$

by binomial's formula, giving $cu + bs + ar = ar(1 + sp^{m-1}) + bs + cu$.

Next, $(ar + bs + cu)t = ar(1 + q^s + \cdots + q^{s(t-1)}) + bst + cut$ by induction on t.

Therefore,

$1+q^s+\cdots+q^{s(t-1)} \equiv 1+(1-sp^{m-1})+\cdots+(1-s(t-1)p^{m-1}) =$

$= t-s\binom{t}{2}p^{m-1} \pmod{p^m}$, thus $(ar+bs+cu)t = ar(t+sp^{m-1})+bst+cut$. □

3. Nearrings with identity on groups $F(p^m,p^n,p^k)$. It is clear that the groups $F(p^m,p^n,p^k)$ are the direct product of metacyclic Miller–Moreno groups of order p^{mn} and cyclic groups of order p^k. Obviously, the direct product of nearrings with identity is a nearring with identity. Therefore, in what follows the additive group of R is isomorphic to a group $F(p^m,p^n,p^k)$.

Let the additive group of a nearring R with identity be isomorphic to a group $F(p^m,p^n,p^k)$. Thus $R^+ = <a>+<b>+<c>$ with elements a, b, c one of which coincides with identity element of R and the relations $ap^m = bp^n = cp^k = 0$, $a+b = b+a(1+p^{m-1})$, $a+c = c+a$ and $b+c = c+b$ are valid, where $m \geq 2$, $n \geq 1$ and $k \geq 1$. Moreover, each element $x \in R$ is uniquely written in the form $x = ax_1+bx_2+cx_3$ with coefficients $0 \leq x_1 < p^m$, $0 \leq x_2 < p^n$ and $0 \leq x_3 < p^k$.

In the paper we consider the case when a coincides with identity element of R, so that $xa = ax = x$ for each $x \in R$, and $p > 2$. Thus R^+ is of exponent p^m and $m \geq n$ and $m \geq k$. Furthermore, for each $x \in R$ there exist integers $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\phi(x)$, $\psi(x)$ and $\xi(x)$ such that $xb = a\alpha(x)+b\beta(x)+c\gamma(x)$ and $xc = a\phi(x)+b\psi(x)+c\xi(x)$, respectively. It is clear that modulo p^m, p^n, p^k, p^m, p^n and p^k, respectively, these integers are uniquely determined by x and so some mappings $\alpha : R \to Z_{p^m}$, $\beta : R \to Z_{p^n}$, $\gamma : R \to Z_{p^k}$, $\phi : R \to Z_{p^m}$, $\psi : R \to Z_{p^n}$ and $\xi : R \to Z_{p^k}$ are determined.

Lemma 6. *Let* $x = ax_1+bx_2+cx_3$ *and* $y = ay_1+by_2+cy_3$ *be elements of* R. *If* a *coincides with identity element of* R, *then*

$$xy = a(x_1y_1+\alpha(x)y_2+\phi(x)y_3+(x_1x_2\binom{y_1}{2}+\alpha(x)x_1y_1y_2+\alpha(x)\beta(x)\binom{y_2}{2}+$$

$$+x_2\phi(x)y_1y_3+\beta(x)\phi(x)y_2y_3+\phi(x)\psi(x)\binom{y_3}{2}p^{m-1})+$$

$$+b(x_2y_1+\beta(x)y_2+\psi(x)y_3)+c(x_3y_1+\gamma(x)y_2+\xi(x)y_3). \qquad (*)$$

Moreover, for the mappings
$\alpha : R \to Z_{p^m}$, $\beta : R \to Z_{p^n}$, $\gamma : R \to Z_{p^k}$, $\phi : R \to Z_{p^m}$, $\psi : R \to Z_{p^n}$ *and* $\xi : R \to Z_{p^k}$
the following statements hold:

(0) $\alpha(0) = \beta(0) = \gamma(0) = \phi(0) = \psi(0) = \xi(0) = 0$ *if and only if the nearring* R *is zero-symmetric;*

(1) $\alpha(a) = 0$, $\beta(a) = 1$, $\gamma(a) = 0$, $\phi(c) = 0$, $\psi(c) = 0$ *and* $\xi(c) = 1$;

(2) $\alpha(xy) = x\alpha(y)+\alpha(x)\beta(y)+\phi(x)\gamma(y)+(x_1x_2\binom{\alpha(y)}{2}+\alpha(x)x_1\alpha(y)\beta(y)+$

$$+\alpha(x)\beta(x)\binom{\beta(y)}{2}+x_2\phi(x)\alpha(y)\gamma(y)+\beta(x)\phi(x)\beta(y)\gamma(y)+\phi(x)\psi(x)\binom{\gamma(y)}{2})p^{m-1};$$

(3) $\beta(xy)=x_2\alpha(y)+\beta(x)\beta(y)+\psi(x)\gamma(y)$;

(4) $\gamma(xy)=x_2\alpha(y)+\gamma(x)\beta(y)+\xi(x)\gamma(y)$;

(5) $\phi(xy)=x_1\phi(y)+\alpha(x)\psi(y)+\phi(x)\xi(y)+(x_1x_2\binom{\phi(y)}{2}+\alpha(x)x_1\phi(y)\psi(y)+$

$$+\alpha(x)\beta(x)\binom{\psi(y)}{2}+x_2\phi(x)\phi(y)\xi(y)+\beta(x)\phi(x)\psi(y)\xi(y)+\phi(x)\psi(x)\binom{\xi(y)}{2})p^{m-1};$$

(6) $\psi(xy)=x_2\phi(y)+\beta(x)\psi(y)+\psi(x)\xi(y)$;

(7) $\xi(xy)=x_2\phi(y)+\gamma(x)\psi(y)+\xi(x)\xi(y)$.

P r o o f. By the left distributive law, we have

$$xy=(xa)y_1+(xb)y_2+(xc)y_3=(ax_1+bx_2+cx_3)y_1+$$
$$+(a\alpha(x)+b\beta(x)+c\gamma(x))y_2+(a\phi(x)+b\psi(x)+c\xi(x))y_3.$$

Furthermore, Lemma 5 implies that

$$(ax_1+bx_2+cx_3)y_1=ax_1(y_1x_2\binom{y_1}{2}p^{m-1})+bx_2y_1+cx_3y_1,),$$

$$a\alpha(x)(y_2+\beta(x)\binom{y_1}{2}p^{m-1})+b\beta(x)y_2+c\gamma(x)y_2)$$

and

$$(a\phi(x)+b\psi(x)+c\xi(x))y_3=a\phi(x)(y_3+\psi(x)\binom{y_3}{2}p^{m-1})+b\psi(x)y_3+c\xi(x)y_3.$$

Thus

$$xy=ax_1(y_1+x_2\binom{y_1}{2}p^{m-1})+a\alpha(x)(y_2+\beta(x)\binom{y_1}{2}p^{m-1})(1+x_2y_1p^{m-1})+$$

$$+b(x_2y_1+\beta(x)y_2)+a\phi(x)(y_3+\psi(x)\binom{y_3}{2}p^{m-1})+b\psi(x)y_3+c(x_2y_1+$$

$$+\gamma(x)y_2+\xi(x)y_3)=a(x_1y_1+x_1x_2\binom{y_1}{2}p^{m-1})+a\alpha(x)(y_2+$$

$$+x_1y_1y_2p^{m-1}+\beta(x)\binom{y_2}{2}p^{m-1})+a\phi(x)(y_3+\psi(x)\binom{y_3}{2}p^{m-1})(1+$$

$$+(x_2y_1+\beta(x)y_2)p^{m-1})+b(x_2y_1+\beta(x)y_2)+b\psi(x)y_3+c(x_3y_1+\gamma(x)y_2+$$

$$+\xi(x)y_3)=a(x_1y_1+x_1x_2\binom{y_1}{2}p^{m-1})+a\alpha(x)(y_2+x_1y_1y_2p^{m-1}+$$

$$+\beta(x)\binom{y_2}{2}p^{m-1})+a(\phi(x)y_3+\phi(x)\psi(x)\binom{y_3}{2}p^{m-1})(1+$$

$$+ x_2y_1p^{m-1} + \beta(x)y_2p^{m-1}) + b(x_2y_1 + \beta(x)y_2) + b\psi(x)y_3 + c(x_3y_1 +$$

$$+ \gamma(x)y_2 + \xi(x)y_3) = a(x_1y_1 + x_1x_2\binom{y_1}{2}p^{m-1}) + a\alpha(x)(y_2 +$$

$$+ x_1y_1y_2p^{m-1} + \beta(x)\binom{y_2}{2}p^{m-1}) + a(\phi(x)y_3 + x_2\phi(x)y_1y_3p^{m-1} +$$

$$+ \beta(x)\phi(x)y_2y_3p^{m-1} + \phi(x)\psi(x)\binom{y_3}{2}p^{m-1}) + b(x_2y_1 +$$

$$+ \beta(x)y_2) + b\psi(x)y_3 + c(x_3y_1 + \gamma(x)y_2 + \xi(x)y_3).$$

Finally,

$$(a(x_1y_1 + \alpha(x)y_2 + \phi(x)y_3 + (x_1x_2\binom{y_1}{2} + \alpha(x)x_1y_1y_2 + \alpha(x)\beta(x)\binom{y_2}{2} +$$

$$+ x_2\phi(x)y_1y_3 + \beta(x)\phi(x)y_2y_3 + \phi(x)\psi(x)\binom{y_3}{2}p^{m-1}) +$$

$$+ b(x_2y_1 + \beta(x)y_2 + \psi(x)y_3) + c(x_3y_1 + \gamma(x)y_2 + \xi(x)y_3).$$

Since $0 \cdot a = a \cdot 0 = 0$, it follows that R is a zero-symmetric nearring if and only if $0 = 0 \cdot b = a\alpha(0) + b\beta(0) + c\gamma(0)$ and $0 = 0 \cdot c = a\phi(0) + b\psi(0) + c\xi(0)$. Equivalently we have $\alpha(0) \equiv 0 \pmod{p^m}$, $0 \le x_3 < p^k$, $\gamma(0) \equiv 0 \pmod{p^k}$, $\phi(0) \equiv 0 \pmod{p^m}$, $\psi(0) \equiv 0 \pmod{p^n}$, $\xi(0) \equiv 0 \pmod{p^k}$.

The associativity of multiplication in R implies that for all x, $y \in R$

$$(xy)b = x(yb) \qquad 1)$$

and

$$(xy)c = x(yc). \qquad 2)$$

According to $xb = a\alpha(x) + b\beta(x) + c\gamma(x)$, we obtain

$$(xy)b = a\alpha(xy) + b\beta(xy) + c\gamma(xy) \qquad 3)$$

and $yb = a\alpha(y) + b\beta(y) + c\gamma(y)$. Substituting the last equation to the right part of equality 1), we also have

$$x(yb) = a(x_1\alpha(y) + \alpha(x)\beta(y) + \phi(x)\gamma(y) + (x_1x_2\binom{\alpha(y)}{2} + \alpha(x)x_1\alpha(y)\beta(y) +$$

$$+ \alpha(x)\beta(x)\binom{\beta(y)}{2} + x_2\phi(x)\alpha(y)\gamma(y) + \beta(x)\phi(x)\beta(y)\gamma(y) +$$

$$+ \phi(x)\psi(x)\binom{\gamma(y)}{2})p^{m-1}) + b(x_2\alpha(y) + \beta(x)\beta(y) + \psi(x)\gamma(y)) +$$

$$+ c(x_2\alpha(y) + \gamma(x)\beta(y) + \xi(x)\gamma(y)). \qquad 4)$$

Since equality 1) implies the congruence of the corresponding coefficients in formulas 3) and 4), we obtain statements (2)–(4).

$$\alpha(xy) = x\alpha(y) + \alpha(x)\beta(y) + \phi(x)\gamma(y) + (x_1x_2\binom{\alpha(y)}{2} + \alpha(x)x_1\alpha(y)\beta(y) +$$

$$+\alpha(x)\beta(x)\binom{\beta(y)}{2}+x_2\phi(x)\alpha(y)\gamma(y)+\beta(x)\phi(x)\beta(y)\gamma(y)+\phi(x)\psi(x)\binom{\gamma(y)}{2})p^{m-1};$$

$$\beta(xy)=x_2\alpha(y)+\beta(x)\beta(y)+\psi(x)\gamma(y);$$

$$\gamma(xy)=x_2\alpha(y)+\gamma(x)\beta(y)+\xi(x)\gamma(y).$$

Next, according to $\phi(x)=0\,(\mathrm{mod}\,p^m)$ instead of y in equality 2), we get

$$(xy)c=a\phi(xy)+b\psi(xy)+c\xi(xy) \qquad 5)$$

and $yc=a\phi(y)+b\psi(y)+c\xi(y)$. Substituting the last equation to the right part of equality 2), we also have

$$x(yc)=a(x_1\phi(y)+\alpha(x)\psi(y)+\phi(x)\xi(y)+(x_1x_2\binom{\phi(y)}{2}+$$

$$+\alpha(x)x_1\phi(y)\psi(y)+\alpha(x)\beta(x)\binom{\psi(y)}{2}+x_2\phi(x)\phi(y)\xi(y)+$$

$$+\beta(x)\phi(x)\psi(y)\xi(y)+\phi(x)\psi(x)\binom{\xi(y)}{2})p^{m-1})+b(x_2\phi(y)+$$

$$+\beta(x)\psi(y)+\psi(x)\xi(y))+c(x_3\phi(y)+\gamma(x)\psi(y)+\xi(x)\xi(y)). \qquad 6)$$

Finally, comparing the coefficients under a, b and c in formulas 5) and 6), we derive statements (5)–(7) of the lemma.

$$\psi(x)=0\,(\mathrm{mod}\,p^m),\qquad \psi(xy)=x_2\phi(y)+\beta(x)\psi(y)+\psi(x)\xi(y),$$

$$\xi(xy)=x_2\phi(y)+\gamma(x)\psi(y)+\xi(x)\xi(y).$$

□

4. Local nearrings whose additive groups are isomorphic to $F(p^m,p^n,p^k)$. Let R be a local nearring whose additive group R^+ is isomorphic to $\alpha(x)=0\,(\mathrm{mod}\,p^m)$. Then $R^+=\langle a\rangle+\langle b\rangle+\langle c\rangle$ with elements a, b and c, where a coincides with identity element of R and the relations $\gamma(x)=0\,(\mathrm{mod}\,p^k)$, $b^{-1}ab=a^{1+p^{m-1}}$, $ca=ac$ and $cb=bc$, where $m\ge 2$, $n\ge 1$ and $k\ge 1$, are valid. Moreover, each element $\xi(x)=1\,(\mathrm{mod}\,p^k)$ is uniquely written in the form $x=ax_1+bx_2+cx_3$ with coefficients $0\le x_1<p^m$, $0\le x_2<p^n$ and $0\le x_3<p^k$.

Through this section let R be a local nearring with $|R:L|=p$.

Consider a coincides with identity element of R, so that $xa=ax=x$ for each $x\in R$, $m\ge n$ and $m\ge k$. Furthermore, for each $x\in R$ there exist integers $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\phi(x)$, $\psi(x)$ and $\xi(x)$ such that $xb=a\alpha(x)+b\beta(x)+c\gamma(x)$ and $xc=a\phi(x)+b\psi(x)+c\xi(x)$. It is clear that modulo p^m, p^n, p^k and p^m, p^n p^k, respectively, these integers are uniquely determined by x and so some mappings $\alpha:R\to Z_{p^m}$, $\beta:R\to Z_{p^n}$, $\gamma:R\to Z_{p^k}$, $\phi:R\to Z_{p^m}$, $\psi:R\to Z_{p^n}$ and $\xi:R\to Z_{p^k}$ are determined.

If $|R:L|=p$, then $L=\langle ap\rangle+\langle b\rangle+\langle c\rangle$. Since $R^*=R\setminus L$ it follows that

$$R^*=ax_1+bx_2+cx_3 \mid x_1\ne 0\,(\mathrm{mod}\,p)$$

and $x=ax_1+bx_2+cx_3$ is invertible if and only if $x_1\ne 0\,(\mathrm{mod}\,p)$. Since L is the (R,R)-subgroup in R^+ by statement 1) of Lemma 3 it follows that $xb\in L$ and

$xc \in L$, hence $a\alpha(x) \in L$ and $a\phi(x) \in L$ for each $x \in R$. Thus $\alpha(x) \equiv 0 \pmod p$ and $\phi(x) \equiv 0 \pmod p$. Therefore, for local nearrings R we have the same multiplication as for nearrings with identity, i.e. multiplication (*). □

Lemma 7. *Let $x = ax_1 + bx_2 + cx_3$ and $y = ay_1 + by_2 + cy_3$ be elements of R and $|R:L| = p$. If a coincides with identity element of R, then $m \ge n$, $m \ge k$ and multiplication (*) holds for the mappings from Lemma 6.*

Next, we will give examples of local nearrings.

Lemma 8. *Let R be a local nearring whose additive group of R^+ is isomorphic to $F(p^m, p^n, p^k)$, $|R:L| = p$, $m \ge n$ and $m \ge k$. If $x = ax_1 + bx_2 + cx_3$, $y = ay_1 + by_2 + cy_3 \in R$, then the mappings $\alpha: R \to Z_{p^m}$, $\beta: R \to Z_{p^n}$, $\gamma: R \to Z_{p^k}$, $\phi: R \to Z_{p^m}$, $\psi: R \to Z_{p^n}$ and $\xi: R \to Z_{p^k}$ can be the following:*

$$\phi(x) = 0 \pmod{p^m},\ \psi(x) = 0 \pmod{p^m},\ \alpha(x) = 0 \pmod{p^m},$$

$$\gamma(x) = 0 \pmod{p^k},\ \xi(x) = 1 \pmod{p^k},\ \beta(x) = \begin{cases} 1, & if\ x_1 \not\equiv 0 \pmod p; \\ 0, & if\ x_1 \equiv 0 \pmod p. \end{cases}$$

P r o o f. It is easy to check that the functions from the statement of the lemma satisfy Lemma 7. □

As a consequence of Lemma 8 we have the following result.

Theorem 1. *For each odd prime p, $m \ge n$ and $m \ge k$ there exists a local nearring R whose additive group R^+ is isomorphic to $F(p^m, p^n, p^k)$.*

Let $[n, i]$ be the i-th group of order n in the SmallGroups library in GAP [3].

Example 1. Let $G \cong (C_{25} \rtimes C_5) \times C_5$ [625,13]. If $x = ax_1 + bx_2 + cx_3$ and $y = ay_1 + by_2 + cy_3 \in G$ and $(G, +, \cdot)$ is a local nearring, then as above "$\cdot$" can be the following multiplication:

$$x \cdot y = a(x_1y_1 + \beta(x)x_1x_2 \left(\binom{y_3}{2} * 5\right) + b(x_2y_1 + \beta(x)y_2) + c(x_3y_1 + y_3),$$

where $\beta(x) = \begin{cases} 1, & if\ x_1 \not\equiv 0 \pmod 5; \\ 0, & if\ x_1 \equiv 0 \pmod 5. \end{cases}$

The following computer program verified that the nearring obtained in Example 1 is indeed a local nearring.

```
G := SmallGroup(625,13);
gen := MinimalGeneratingSet(G);
List(gen,Order);
a := gen[1];
b := gen[2];
c := gen[3];
mulGR := function(x,y)
local x1,x2,x3,y1,y2,y3;
for x1 in [0..24] do
for x2 in [0..4] do
for x3 in [0..4] do
```

$for\ y1\ in\ [0..24]\ do$

$for\ y2\ in\ [0..4]\ do$

$for\ y3\ in\ [0..4]\ do$

$if\ x = a^{x1} * b^{x2} * c^{x3}\ and\ y = a^{y1} * b^{y2} * c^{y3}\ then\ return$

$a^{(x1*y1+x1*x2*Binomial(y1,2)*5)} * b^{(x2*y1+y2)} * c^{(x3*y1+y3)}; fi;$

$od; od; od; od; od; od; end;$

$n := ExplicitMultiplicationNearRingNC(G, mulGR);$

$M := MultiplicationTable(n);$

$muR := NearRingMultiplicationByOperationTable(G, M, AsSortedList(G));$

$n := ExplicitMultiplicationNearRing(G, muR);$

$IsLocalNearRing(n);$

From [1], [7] and [6] we have the following number of all non-isomorphic zero-symmetric local nearrings on the group G from Example 1.

$IdGroup(R^+)$	$StructureDescription(R^+)$	$n(R^+)$
[625, 13]	$(C_{25} \rtimes C_5) \times C_5$	630

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ПРО ПРЯМІ ДОБУТКИ МЕТАЦИКЛІЧНИХ p-ГРУП МІЛЛЕРА–МОРЕНО ТА ЦИКЛІЧНИХ p-ГРУП ЯК АДИТИВНИХ ГРУП ЛОКАЛЬНИХ МАЙЖЕ-КІЛЕЦЬ

Скінченна група називається групою Міллера–Морено, якщо вона неабелева і всі її власні підгрупи є абелевими. Розглядаються прямі добутки p-груп Міллера–Морено та циклічних p-груп як адитивних груп майже-кілець з одиницею та локальних майже-кілець.

Ключові слова: *майже-кільце, локальне майже-кільце, група Міллера–Морено*

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Obtained 27.11.24