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MODELING THE STRESS STATE OF A SEDIMENT BASIN IN A SUBDUCTION ZONE USING S. P. TIMOSHENKO'S THIN PLATE THEORY

A mathematical model of the stress field in sedimentary layers within a subduction zone is proposed, accounting for lateral displacements, gravity, and frictional forces at the interface with the basement. The necessary assumptions and limitations of the model are discussed, enabling the application of thin-plate theory based on S. P. Tymoshenko's hypotheses. The stress and displacement fields are calculated, and the distribution of principal compressive stress in the plane-strain state is analyzed. Using the Coulomb–Mohr failure criterion, two sets of probable slip lines are constructed, predicting the orientation of thrust faults. These predictions closely correspond to typical thrust structures, particularly those observed in the Ukrainian Carpathians.

Keywords: mathematical modeling in geology, plate theory, stresses in rocks, thrust structures.

Introduction. Studying the stress state at the time of regional and local geological structure formation is crucial for understanding their characterristics in greater detail. The study of deformations in the upper lithosphere involves not only classical geological and geophysical approaches but also modern methods incorporating the mathematical framework of continuum mechanics [e.g. 11, 17, 20, 22–24]. Mathematical and computer modeling hold significant potential for testing both existing and emerging geological hypotheses regarding the formation of structures and the thermodynamic processes within the Earth's crust. Current issues include the geodynamic conditions for the formation of thrust structures, which are common in many mobile areas of the crust, in particular in the Carpathian region [4, 10, 21]. While field tectonophysics methods enable the reconstruction of paleotectonic stresses based on empirical data – such as crack orientations and slip lines – assuming the statistical nature of the measurements [3, 20], an accurate mathematical model provides further insight by clarifying the influence of individual factors and tracing the evolution of the stress-strain state under varying boundary conditions. The initial stages of compression in a geosynclinal sedimentary basin, driven by the subduction-related thrusting of its basement, are of particular interest. This geotectonic interaction likely plays a key role in the formation of fault zones, which can subsequently evolve into thrusts.

The work aims to develop a relatively simple mathematical model, based on the S. P. Tymoshenko plate theory, to analyze stress fields within a sedimentary layer affected by basement subduction. The objectives include adapting continuum mechanics equations for stress-strain state calculations, defining the geometric dimensions and physical-mechanical properties of rocks and massifs, and obtaining and analyzing numerical results. A specific focus is on addressing the little-studied factor of sliding line formation under compression, which may eventually lead to thrust-type faults.

1. Hypotheses and mathematical formulation of the problem. The upper layers of the lithosphere are often treated as a solid body with distinct elastic properties. Additionally, gravity and friction forces play critical roles in influencing and defining the nature of mechanical interactions between geological bodies.

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Quantitative calculations based on the theory of elasticity have a well-established history and substantial results [2, 9, 17], yet this approach remains relevant today [12, 14, 18,]. However, rheologically more complex models that account for viscoplastic deformations, temperature, chemical, and phase transformations typically require numerical methods and advanced computer modeling [e.g., 5, 6, 15].

The choice of approach should be guided by the specific characteristics of the geological object under study, its spatial and temporal context on a geological time scale, the availability of empirical data, and the interpretability of the results.

The study of the mechanisms behind the formation of fold-thrust structures is a significant and productive area in the development of mathematical modeling [1, 8]. A key approach to explaining the formation of fault zones is the 'critical thrust wedge' model (also known as the Mohr–Coulomb wedge model) [7, 22, 19], which incorporates the effects of frictional forces and cohesive strength of rocks along inclined contacts or décollement horizons.

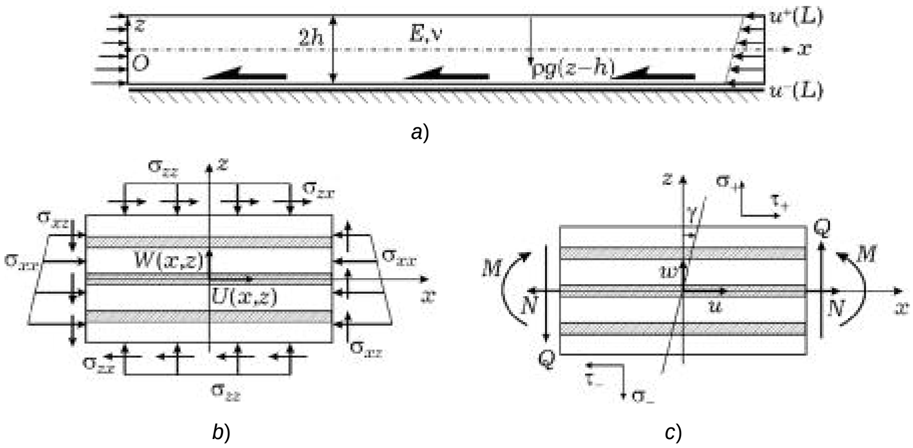

Fig. 1. Plane deformation of a layered sedimentary rock element (2D model) under compression caused by basement moving:
a) Geometric dimensions and loading scheme.
b) Forces acting on the medium's element in the section.
c) Plate analogy based on the S. P. Tymoshenko's theory framework

To substantiate our model, let us define the data and hypotheses. A computational model represents a balance between accurately approximating reality and introducing necessary simplifications. These simplifications allow the focus to remain on the most critical factors influencing a specific process, while secondary factors are neglected to derive analytical relationships or obtain numerical results suitable for analysis and interpretation.

For the problem described above, several assumptions must be made regarding the stress-strain state, geometric dimensions, and loading conditions:

- *Linear Elasticity Assumption*: The deformation process is modeled within the framework of linear elasticity theory, which is appropriate for processes dominated by brittle failure, such as crack and fault formation.
- *Two-Dimensional Approximation*: For sufficiently elongated geological structures, the three-dimensional problem is simplified to a two-dimensional vertical section. This simplification assumes that all cross-sections remain planar and undergo deformation only within their planes (Fig. 1*a, b*).
- *Layer Geometry and Plate Theory Analogy*: The sedimentary layer's length is assumed to be significantly greater than its thickness. This

permits the use of averaged values and assumptions about the displacement and stress through-the-thickness distributions, similar to the approach in plate theory (Fig. 1*c*).

- *Layered Structure Assumption*: The real layered structure of the sedimentary rocks is an orthotropic or transversely isotropic material with effective elastic properties and an average rock density.
- *Lithostatic Pressure Balance*: The lithostatic pressure $p = 2h\rho g$ made by the sedimentary layer in contact with the foundation, where g is the gravitational acceleration, is balanced by the reaction forces from the foundation. Additionally, we neglect the deformation due to transverse compression and rock compaction.
- *Compression Mechanism*: Lateral compression of the sedimentary layer is induced by the foundation moving, modeled by a rigid body. The tangential forces are transmitted through the friction mechanism, with the coefficient of mu sliding friction denoted. The left edge of the sedimentary basin remains stationary while undergoing significant lateral compression strains.
- *Sedimentary Layer Movement*: At a certain distance L from the basin slope, the entire sedimentary layer begins to move together with the underlying platform, without significant deformation. Therefore, the influence of the right part of the sedimentary rocks on the compression of the left part can be neglected, allowing the calculation model to focus on a segment of the layer rather than the entire vertical section.

Based on the aforementioned hypotheses, we formulate a system of equations for the mathematical model using the theory of thin plates as per S. P. Tymoshenko's framework, which accounts for transverse shear strains. Considering a plate of thickness $2h$ and length L in a rectangular coordinate system Oxz, $(x;z) \in [0;L] \times [-h;h]$, subjected to mass forces (lithostatic pressure) $p = 2h\rho g$ and, in the general case, tangential stresses on the surfaces $z = \pm h$, the governing equations are expressed as follows:

– equilibrium equations:

$$\frac{dN}{dx} = -2\tau_2 = -(\tau^+ - \tau^-), \ \frac{dM}{dx} - Q = -2h\tau_1 = -h(\tau^+ + \tau^-), \ \frac{dQ}{dx} = -p; \quad (1)$$

– the elasticity law:

$$N(x) = 2h\bar{E}\frac{du}{dx}, \ M(x) = \frac{2h^3\bar{E}}{3}\frac{d\gamma}{dx}, \ Q(x) = \frac{5h}{3}G_S\left(\gamma + \frac{dw}{dx}\right); \quad (2)$$

– linear trough-the-thickness distributions:

$$U(x,z) = u + z\gamma, \ W(x,z) = w(x). \quad (3)$$

In this context (Fig. 3*b*):

- $N(x)$ and $Q(x)$ represent the longitudinal and transverse forces, respectively.
- $M(x)$ denotes the bending moment.
- $u(x)$ and $w(x)$ are the longitudinal and vertical displacements, respectively.
- $\gamma(x)$ is the rotation angle of a linear element normal to the middle surface of the plate.

The elastic constants are denoted by $\bar{E} = E/(1-\nu^2)$, where E is the averaged Young's modulus of the sedimentary rocks, ν is Poisson's ratio, and G_S is the shear modulus, respectively.

We assume that the upper surface ($z = h$) of the sedimentary layer is stress-free:

$$\sigma_{zz}(h) = \sigma^{+} = 0\,,\ \ \sigma_{xz}(h) = \tau^{+} = 0\,, \tag{4}$$

and the lower surface ($z = -h$) is subjected to lithostatic pressure p and tangential stresses arising from the friction mechanism, as described by Amonton's law [11]:

$$\sigma_{zz}(-h) = \sigma^{-} = -p\,,\ \ \sigma_{xz}(-h) = \tau_{-} = \mu|p| = 2\mu h\rho g\,. \tag{5}$$

This model assumes linear, or quadratic thru-the-thickness distributions of compressive and shear stresses, respectively [16].

$$\sigma_{xx}(x,z) = \frac{N(x)}{2h} + \frac{3M(x)}{2h^3}z\,,$$

$$\sigma_{zz}(z) = \frac{\sigma^{+}+\sigma^{-}}{2} + \frac{\sigma^{+}-\sigma^{-}}{2}\left(\frac{z}{h}\right) = \frac{\sigma^{-}}{2h}(h-z) = \rho g(z-h)\,, \tag{6}$$

$$\sigma_{xz}(x,z) = \frac{3Q(x)}{2h}\left(1-\frac{z^2}{h^2}\right) + \frac{3\tau^{-}}{4}\left(\frac{z^2}{h^2}-\frac{1}{3}\right) + -\frac{\tau^{-}z}{2h}\,.$$

A feature of this model is that it is statically determinate. At the right edge ($x = L$), it is assumed that $N(x) = 0$, $M(x) = 0$, and $Q(x) = 0$, indicating the absence of additional force factors causing deformation in the sedimentary layer. For this edge, kinematic conditions are established to simulate the activation of the subduction process, driven by the mechanism of basement rocks sliding beneath the sedimentary basin, which is transmitted to the upper layer.

Given that the basement is considered a completely rigid body, the model assumes no deflections ($w = 0$) and no transverse shear forces ($Q = 0$). The boundary conditions on the right incorporate a combination of displacements and rotations:

$$u(L) = -u_L\,, \qquad \gamma(L) = \gamma_L\,. \tag{7}$$

The calculation formulas for forces, and displacements are expressed as follows:

$$N(x) = -\tau^{-}(x-L) = -\mu p(L-x) \le 0\,,$$

$$M(x) = \mu p h(L-x) \ge 0\,,$$

$$u(x) = \frac{-\tau^{-}}{4h\bar{E}}(L-x)^2 + u_L = \frac{\mu p}{4h\bar{E}}(L-x)^2 + u_L\,, \tag{8}$$

$$\gamma(x) = \frac{-\mu h p}{2D}(L-x)^2 + \gamma_L = \frac{-\mu p}{3h^2\bar{E}}(L-x)^2 + \gamma_L\,.$$

On the outer surfaces, following the linear distribution between the displacements of the upper and lower surfaces, we obtain:

$$u(L) = -\frac{U_L^{(+)} + U_L^{(-)}}{2} = -u_L\,,$$

$$\gamma(L) = -\frac{U_L^{(+)} - U_L^{(-)}}{2h} = \gamma_L\,. \tag{9}$$

It is reasonable to assume that $U_L^{(-)} < U_L^{(+)} \le 0$, indicating that, from the perspective of tectonic movement interpretation, the lower layers are “pulled” more intensively, as shown in Fig. 2.

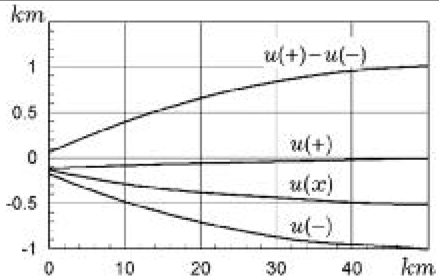

Fig. 2. The displacements on the middle surface $u(x)$ and the bottom/top surfaces $U(-)$, and $U(+)$ along the section, relatively. The bold line indicates the difference in the displacements of the upper layers of the sedimentary stratum relative to the lower layers

2. Stress field analysis. A feature of such a model is that the stresses vary linearly with length and have the greatest value on the left edge, and its value depends only on the geometric parameters adopted in the model (length and thickness), as well as the averaged physical properties of the rocks (elastic moduli and density). We also note that the stresses in each of the vertical sections are the sum of the compressive and bending stresses. In addition to the components of the stress tensor, the structural analysis [17] includes the principal stresses

$$\sigma_{\max,\min} = 0,5(\sigma_{xx} + \sigma_{zz}) \pm \sqrt{0,25(\sigma_{xx} - \sigma_{zz})^2 + \tau_{xz}^2} \quad (10)$$

and the corresponding mutually perpendicular principal axes α_1 and α_2. These are related by the equation:

$$\tan 2\alpha_1 = \frac{2\tau_{xz}}{\sigma_{xx} - \sigma_{zz}}, \ \alpha_2 = \alpha_1 \pm 90^\circ, \quad (11)$$

where α_1 is the angle between the x-axis and the normal to the plane where tangential stresses are absent.

Based on the known orientations of the principal stresses, it is possible to construct stress trajectories, which are important for geological interpretation [17, 20]. In particular, interpolation using Hermitian splines at given nodes can be applied [13].

Finally, according to the Coulomb–Mohr heuristic criterion, in the case of brittle fracture, faults can arise at an angle $\pm\varphi$ to the principal axis, corresponding to the larger compressive principal stress $\sigma_{\min}$ [2, 11, 20]. Note that tensile stresses are considered positive in this context.

The angle φ characterizes the internal friction within rocks or massives and is a property of the medium that must be experimentally determined (based on the envelope of the Mohr circles for the critical pre-fracture state). For sedimentary rocks, it is known that mainly $25^\circ < \varphi < 43^\circ$, with a mean of around 30° [20].

To perform the numerical calculations (computer modeling), minimal computational resources are required. In particular, MS EXCEL spreadsheets are used, making it feasible to apply such models in the educational process. Specifically, we calculate the components of the stress tensor at a given set of points, the principal stresses and their orientation within the sedimentary layer, as well as the orientation of probable sliding lines for brittle fracture (critical elastic state). The following numerical parameters are used: $E = 40$ GPa;

$\nu = 1/4$, $\mu = 4/10$; $2h =$ 4 km; $L =$ 50 km; $\rho = 2500$ kg/m^3; $p = 120$ MPa; $\varphi = 30°$; $U_L^{(+)} = 0$ i $U_L^{(-)} = -1$ km.

Horizontal compressive stresses σ_{xx} dominate this model, which determines the orientation of the principal stresses, $\sigma_{\min}$ (Fig. 3*a*). Notably, the smaller of the principal stresses undergoes a reorientation – from vertical (in the upper layers) to horizontal (in the lower layers). The principal stresses $\sigma_{\min}$ across the entire model are negative, meaning the rocks are in compression. Using the Coulomb–Mohr failure criterion, the orientation of these stresses helps determine the most likely direction for the initiation of conjugate faults [2].

We distinguish two sets (or generations) of such directions. The first set forms an angle (counterclockwise rotation) with the maximum compressive stress, while the second set forms the same angle but clockwise (Fig. 3*b*, and *c*). One of these sets aligns with the shape of the boundaries between individual thrusts, while the other set of faults could lead to discontinuous failures in the upper layers, oriented in the opposite direction (mirror reflection relative to the vertical line). Since the reorientation of these directions occurs at relatively shallow depths (above the middle surface of the sedimentary layer), the occurrence of a fault extending through the entire rock thickness is unlikely. This suggests that the model reliably simulates the orientation of faults and the direction of thrust propagation under the given loading conditions of the sedimentary basin.

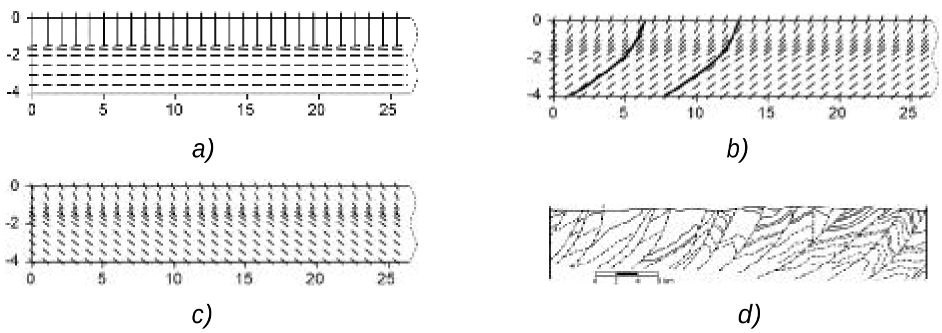

Fig. 3. Comparison of potential sliding lines with the fold-thrust structure of the Carpathians:
a) Orientation of principal compressive stresses.
b) and *c*) Two possible families of sliding lines determined by the Coulomb–Mohr criterion.
d) A fragment of a typical geological cross-section of the Ukrainian Carpathians, illustrating a characteristic thrust structure (with simplifications) [10, 4, 21], where the dash-dotted line represents the faults

Conclusions. The proposed model offers simplicity in its ability to yield analytical results. However, the inherent simplifications and hypotheses represent notable limitations, highlighting the need for more precise calculations using numerical methods, especially for spatial or generalized 2D models. Consequently, the results obtained are qualitative approximations of real geological processes and structures.

Despite these limitations, the model provides valuable insights into the heterogeneity of the stress state and key features of thrust structures, such as the shape of boundaries, the direction of thrust initiation and propagation, and structural formation characteristics.

Future work will focus on extending this approach to related problems in geological mathematical modeling, aiming to further enhance our understanding of complex geological processes.

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МОДЕЛЮВАННЯ НАПРУЖЕНОГО СТАНУ ОСАДОВОГО БАСЕЙНУ В ЗОНІ СУБДУКЦІЇ В МЕЖАХ ТЕОРІЇ ТОНКИХ ПЛАСТИН С. П. ТИМОШЕНКА

Запропоновано математичну модель поля напружень шаруватої товщі осадових порід у зоні субдукції, що враховує дію латеральних переміщень, гравітації та сил тертя на контакті з фундаментом. Обговорено необхідні гіпотези та обмеження цієї моделі, які дають змогу застосувати теорію тонких пластин з використанням гіпотез С. П. Тимошенка. Обчислено поля напружень та переміщень, проаналізовано розподіл головних напружень стиску для плоского деформованого стану. На основі критерію руйнування Кулона–Мора побудовано два сімейства вірогідних ліній ковзання і передбачено орієнтацію зсувних розривних порушень, які добре узгоджуються з типовими насувними структурами в регіоні Українських Карпат.

Ключові слова: *математичне моделювання в геології, теорія пластин, напруження в гірських масивах, насувні структури.*

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