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NEW TRENDS IN CONTROL THEORY

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The article outlines the conceptual foundations of new trends in control theory that have been intensively developing in recent years. Unlike classical control theory, which was formed in the last century and is based on well-known mathematical models of controlled processes in the form of local equations, new approaches to linear stationary systems use input-output relations that follow directly from the Cauchy formula for both continuous and discrete systems. On the basis of the same description, it is possible to substantiate and obtain the so-called data-based models, which are directly linked to data that form, at the observation intervals, the trajectories of already implemented past processes and future ones, for which control is to be synthesized. This approach is focused primarily on finding control from the prediction model. At the same time, the current measurements carried out at the plant make it possible to implement feedback and, in case of discrepancies between the forecast and the real process, to correct the predictive control, i.e. such a way to stabilize it. Control by trajectory prediction model allows to exclude model identification by trajectory data, and control directly on their base. Since the data contain errors, the most important issue in the considered approach is the robustness of the chosen control. A large number of published works are dedicated to this problem, where the guaranteed approach, focused on the worst-case in the data, is the most in demand. In most cases, control synthesis is reduced to solving various optimization problems, mainly on the finite prediction horizon. Considerable attention in the article is paid to methods for solving synthesis problems based on SVD decomposition. To reduce the complexity of the tasks to be solved, it is proposed to reduce it to terminal control on the horizon of a short duration. Then an iterative control strategy is implemented, which, due to feedback, ensures the feasibility of the global control goal.

Keywords: control theory, trajectory model, data-driven control, LTI system, MPC.

Introduction

Approximately in the middle of the last century, the mathematical theory of control was formed and developed, which was based on the well-known mathematical model of the process that was to be controlled. Systems with lumped parameters which were described by systems of ordinary differential or difference equations, and systems with distributed parameters, described using partial derivatives equations and boundary conditions were considered. The control theory was composed of both analysis problems of controlled processes and synthesis problems. The key to the creation of highly efficient control systems was the problem of stability. The needs of practice have initiated a variety of areas of research, some of which have emerged as independent areas of theory. Thus, the theory of optimal control, game approach to control problems, including differential games, and a number of others, were fruitfully developed. Synthesis of control based on a given mathematical model was reduced to finding a control law that provides the specified dynamic properties of the controlled plant. This a priori synthesized control law was then implemented in various control systems. With the help of adaptive procedures and parameter tuning, the synthesized laws were corrected, thus reacting to possible changes in the environment.

All the created theoretical base worked well in practice if the mathematical model of the plant was known. Most often, it was built on the basis of the known laws of mechanics, physics, and others. The need to control design for objects whose mathematical model was unknown led to the creation of a theory and methods for systems identification. Models in such cases were built on the given experimental data. However, the presence of errors in the available data very often, especially for complex systems, led to problems that are difficult to solve, including ill-conditioning. The main emphasis was focused on stochastic identification, which aimed at consistency of estimation. Despite the abundance of created methods for solving identification problems, one cannot speak of a complete solution to this problem. Especially when, for various reasons, it is difficult to choose the appropriate structure of the mathematical model for the system under study.

Another aspect is related to the recent rapid development of computer technology. Digital information technologies have begun to actively penetrate into all spheres of human activity, including has become widely used in control systems. The use of such tools only for the implementation of previously synthesized control laws depreciated the capabilities of computing tools in control systems. In many cases, it has become quite possible to solve control problems directly in the process of system operation. All this stimulated the research of other approaches and methods of control systems design. As a result, relatively recently, new areas of research have been formed that make it possible to set and solve control problems in a different way. Among them, two such directions aroused the greatest interest. These are model predictive control (MPC) and data-driven control, when instead of solving the identification problem using them, control that provides the given system dynamic may be evaluated directly from the data.

The great interest in the above new trends in control theory is evidenced by publications that have appeared over the past five years only in the journals «Automatica» (about 200 articles) and «International Journal of Control» (more then 40 articles).

Therefore, in this article an attempt is made to present some of the theoretical results of these two conceptions in control theory as applied to linear discrete time-invariant (LTI) systems. The main goal is to interest the reader in new approaches and methods of control and encourage further development of researches and especially their application in practice.

Trajectory description of LTI systems

Let us consider a discrete LTI controllable and observable system, the processes occurring in which can be described by the following system of difference equations

$$x_{t+1} = Ax_t + Bu_t, \quad y_t = Cx_t + Du_t, \tag{1}$$

where t is the current time, x_t is the state vector of the system at the moment t of dimension n, y_t is the measured output vector of dimension m, and u_t is the vector of input or controlled action of dimension r. Matrices *A*, *B*, *C*, *D* have dimensions corresponding to the specified variables. The controllability and observability properties of system (1) are determined by the matrices

$$\Omega_n = [B,\ AB, \ldots,\ A^{n-1}B],\ \ \Gamma_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}. \tag{2}$$

The system is fully controllable if the rank criterion is met

$$\operatorname{rank}\Omega_n = n$$

and fully observable when

$$\operatorname{rank}\Gamma_n = n.$$

It is generally accepted that description (1) is minimal if it corresponds to a fully controlled and fully observable system. We will also assume that the system is well controlled if the condition number of the matrix Ω_n has a low order. Good observability corresponds to good conditionality of the matrix Γ_n. With poor controllability, large control resources may be required to transfer the system from a given initial state to some terminal ones. With poor observability, the problem of estimating the full state vector from incomplete output observations can become ill-posed.

In systems with continuous time, using the Cauchy formula [1], one can go from a state-space description similar to (1) to input-output relations, which, unlike (1), gives a unique description to each specific system.

It is well known that for any system a set of descriptions (1) is admitted, interconnected by a non-singular transformation. They all give the same output response to any admissible input. The input-output ratio represents the mathematical model of the system through impulse transition matrices, which are actually matrices or Green's functions for controlled and observed systems with lumped parameters [2]. Such a description is often used in practice when solving various dynamic problems.

For discrete systems described by (1), it is easy to write an analogue of the Cauchy formula. It looks like

$$y_{t+k} = CA^{k-1}x_t + CBu_{t+k-2} + \ldots + CA^{k-2}Bu_t + Du_{t+k},\ \ k = 0, 1, 2, \ldots, \tag{3}$$

here the first term determines the final state of the free movement from the initial state x_t up to the moment $t+k$, and the subsequent terms determine the result of the forced movement at the same time instan. From vectors y_{t+k} for different values k from 0 to some value L^{-1}, we form a cascade vector of the following form:

$$y(t,\ L) = [y_t^{\mathrm{T}}\ \ y_{t+1}^{\mathrm{T}}\ \ldots\ y_{t+L-1}^{\mathrm{T}}]^{\mathrm{T}}, \tag{4}$$

whose dimension is $m \cdot L$. In (4) « T» is the transposition operation. By analogy with (4), we construct an expanded column vector composed of vectors u_t, i.e.

$$u(t,\ L) = [u_t^{\mathrm{T}}\ \ u_{t+1}^{\mathrm{T}}\ \ldots\ u_{t+L\text{-}1}^{\mathrm{T}}]^{\mathrm{T}}. \tag{5}$$

Cascade vectors (4), (5) actually represent a piece of the system (1) trajectory from time t to $t+L-1$, i.e. a sequence of vectors $\{y_t\ u_t\}_0^{L-1}$ from 0 to $L-1$, where the lower index is the beginning of the trajectory, and the upper index is its end. For this piece of the trajectory, based on (3), we can write a vector-matrix equation relating (4), (5), namely

$$y(t,L) = \Gamma_L \cdot x_t + \Phi_L \cdot u(t,L)\ , \tag{6}$$

where Γ_L is the observability matrix of dimension L, and Φ_L is a block triangular Toeplitz matrix of the form

$$\Phi_L = \begin{bmatrix} D & \dots & 0 & 0 \\ CB & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ CA^{L-2}B & \dots & CB & D \end{bmatrix}.$$

Let the system (1) be observed on the interval $t = l, l+1, \dots, l+T-1$ affected by the input $\{u_t\}$ corresponding to some its realization. Then the resulting process is modeled by the following matrix equation

$$Y = \Gamma_L X + \Phi_L U\ , \tag{7}$$

here $Y = [y(l, L), y(l+1, L), \dots, y(l+T-L, L)]$, $U = [u(l,L), u(l+1,L), \dots, u(l+T-L,L)]$ are block Hankel matrices of outputs and inputs, and the trajectory initial states of the system are assembled into a matrix $X = [x_l, x_{l+1}, \dots, x_{l+T-L}]$.

Matrix equation (7) represents the shift set of pieces of system trajectories on the observation interval. System (7) connects a set of trajectories with the parameters of systems represented as observability and impulse response matrices, and each of these trajectories is determined by its initial state. This system is the original 4SID (Subspace state-space system identification), a method for identifying multiply connected systems using trajectory data [3].

Now let's use (7) to construct a description that directly relates the trajectories of the LTI system generated by some informative action. In this case, their dependences on the parameters of the models described above and on the initial conditions are excluded. To do this, we multiply (7) on the right by a vector $g^{(i)}$ of dimension $T-L+1$.

We take the vector $g^{(i)}$ so that it satisfies the following system of equations

$$\begin{bmatrix} U \\ Y \end{bmatrix} \cdot g^{(i)} = \begin{bmatrix} u^i \\ y^i \end{bmatrix},\ i = 0, 1, \dots, (T-L+1)\ , \tag{8}$$

here $\{u^i, y^i\} = \{u(l+i, L)\ y(l+i, L)\}$ is one of the trajectories of set (7) represented by equation (6). Thus, with the help of (8), we pass from the set of trajectories specified by (7) to one corresponding to the i-th column of the matrices U, Y. This trajectory corresponds to the initial state x_{t+i}. Each i-th trajectory selected with the help of (8) will have its own value of the vector $g^{(i)}$. The system of equations (8) establishes a connection between the set of shear trajectories obtained from the trajectory over the entire observation interval $[l, l+T-1]$, with one shortened trajectory selected on this set. In fact, (8) specifies the inverse transformation of the transition from the matrix description of the LTI system (7) to equation (6) written for the time moment $t = l+i$. Therefore, (8) can be considered as a trajectory description of the LTI system at a given observation interval.

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Let us now introduce into consideration the concept of a persistently exciting input given on the observation interval $[l, l+T-1]$. The sequence $\{u_k\}_{k=l}^{k=l+T-1}$ on the observation interval $[l, l+T-1]$ will be persistently exciting of order L if the Hankel matrix U in (7), (8) has full row rank, i.e. $\mathrm{rank}\, U = Lr$ [4, 5].

In [6], a fundamental result was formulated on the solvability of system (8), which, according to the data on the observation interval, gives a data-based trajectory description of the LTI system. It is written in the form of a lemma on the connection between the trajectory on the observation interval and its individual fragments, which substantiates the trajectory description (8). Its essence is as follows. System (8) is resolvable if the input action on the observation interval is persistently exciting of order $L+n$.

Thus, the system of equations (8) can be considered as an alternative description of the LTI plant, which we will call the data-based model, since it links the data corresponding, taking into account the shift invariance, to various fragments of the complete trajectory on the interval $[l, l+T-1]$.

Model predictive control

This control method is based on the construction of a predictive control input based on a known model that determines the desired behavior of the system on a finite or semi-infinite interval (horizon), starting from a certain point in time. In the process of implementation, according to the current data measurements, it is estimated how much the real process coincides with the predicted one, and, if necessary, the control input is corrected on a new horizon. In this way, feedback is implemented, which ensures a stabilized movement along a given trajectory. Control synthesis, as a rule, is reduced to solving various optimization problems on a sliding horizon. A large number of MPC problems with different descriptions of the controlled process and optimality criteria are considered and solved. A large number of papers have been published on the formulation of problems and methods for their solution. A generalization of the obtained results can be found, for example, in relatively recently published monographs [6, 7]. Linear and nonlinear systems with discrete and continuous time, with control by state and by current measurements were considered. For LTI discrete systems, description (1) was taken as the initial one. In other cases, other similar equations describing that describe the local dynamics of the controlled and observed system were used.

Let us first consider one of the possible optimal control problems on a finite time interval $[t, t+N-1]$, where t is the current time, and N determines the finite control horizon. In the sliding interval mode with control synthesis at each step, taking into account the current data measurements, it is possible to implement MPC with feedback. If model (1) is known, then various formulations of optimal control problems are possible on its basis. We present here one of them considered in [8]. In addition to (1), it is also assumed that restrictions are imposed on the input and output

$$u_t \in U, \; y_{t+k} \in Y, \; k = 0, 1, \ldots, N-1, \tag{9}$$

where $U = \{u_t \in \mathbf{R}^r : u_{\min} \le u \le u_{\max}\}$ (as a rule $u_{\min} = -u_{\max}$) and $Y = \{y_t \in \mathbf{R}^m :$ $G(t) \cdot y \le h(t), G(t) \in \mathbf{R}^{q \times m}, h(t) \in \mathbf{R}^q\}$. In a real process, due to the presence of disturbances at the input and measurement errors at the output, as well as due to the inaccuracy of the description, not the process that is predicted by model (1) is realized, but another, which we denote as y_t^p and u_t^p. With small perturbations and measurement noise, as well as a not very large control horizon N, the real and predicted process should not differ significantly. In [8], it was assumed that there are no disturbances at the input, and the output is measured with an additive bounded noise ξ_t, i.e.

$$y_t^p = y_t + \xi_t, \ \xi_t \in \Xi,$$

for each moment of system operation. Wherein $\Xi = \{\xi_t \in \mathbf{R}^m : \|\xi_t\|_\infty \le \varepsilon, \varepsilon > 0\}$. In addition, about the initial state of the system at the moment t, i.e. at the beginning of the interval, it is only known that it satisfies the condition $x_t \in X_0$, where $X_0 = \{x_t \in \mathbf{R}^n : x_{\min} \le x_t \le x_{\max}\}$ it determines all admissible initial states of the system. The optimal control problem is to minimize the quadratic criterion

$$\sum_{t}^{t+N-1} \|u_t\|^2 \tag{10}$$

under constraints (1), (9). It is also important that the desired control be robust with respect to all possible initial states and measurement errors.

When solving the problem posed in this way, the ideas put forward earlier in [9, 10] were used, as well as the separation principle for linear systems, which is described in [11]. As a result, in [8], an original method for implementing feedback was proposed and developed, which ensures the robustness of the control system.

Based on (1), other formulations of the optimal control problem for the MPC are admissible.

Now we consider the problem of MPC design based on description (3). Let us formulate it as a terminal control problem on the same sliding interval $[t, t+N-1]$ with horizon N. Let the total state vector $x\ (C = E)$ and $D = 0$ be measured. Then the state of the system at the moment $t+N$ according to (3) is determined by the relation

$$x_{t+N-1} = A^{N-1}x_t + Bu_{t+N-2} + BAu_{t+N-3} + \ldots + BA^{N-2}u_t. \tag{11}$$

In the terminal control problem, x_{t+N-1} is given, but x_t is measured and, therefore, $x_{t+N-1} - A^{N-1}x_t = \bar{x}(t, N)$ is calculated. Then (11) can be written as

$$\Omega_{N-1} \cdot u(t, N) = \bar{x}(t, N), \tag{12}$$

where $u(t, N) = [u_{t+N-2}^{\mathrm{T}} \ u_{t+N-3}^{\mathrm{T}} \ \ldots \ u_t^{\mathrm{T}}]^{\mathrm{T}}$. The stabilization problem corresponds to $x_{t+N-1} = 0$ and $\bar{x}(t, N) = -A^{N-1}x_t$, i.e., with exact model, data and calculations, solution (11) with $u_{t+N-1} = 0$ brings the system to the zero equilibrium state. In the general case, terminal control it is needed using non-zero u_{t+N-1}, u_{t+N}, etc. in order to keep the system in a state of x_{t+N-1}.

The solvability of the system of linear algebraic equations (SLAE) (12) depends on the properties of the matrix Ω_{N-1} and, first of all, on its conditionality. When $r < n$, then we assume $N-1 \ge n-r$, i.e. system (12) must be square or underdetermined. In addition, in these cases $\operatorname{rank}\Omega_{N-1}$ should be equal to n. No less important is the value of the condition number Ω_{N-1}. Under poor conditionality, problem (12) becomes ill-posed when the right-hand side is specified with an error or the elements of the matrix Ω_{N-1} are not yet accurately specified. This occurs if the measurements contain noise and the system model is approximate, for example, found from the solution of the identification problem. Therefore, the choice N should be tied to the condition number of the matrix Ω_{N-1}. To do this, it is enough to plot the condition number $\kappa(\Omega_{N-1})$ as a function of N. The most preferable will be the one N for which $\kappa(\Omega_{N-1})$ is closer to unity. Note that for values N for which $\operatorname{rank}\Omega_{N-1} < \overline{n}$, the condition number is taken equal ∞ (the system is degenerate). After choosing an appropriate N optimal control, it can be found from the formulation of the problem close to the one considered above on the basis of equations (1). The optimal solution will be the element $u(t, N)$ that delivers the minimum (10) under constraints (12) and $u(t, N) \in U$, where the last constraint is similar to the one specified in (9).

For small N and large $\|\overline{x}(t, N)\|$, such a problem may turn out to be incorrect due to the indicated restrictions on control. The control resource for its solution may not be enough. Therefore, it is proposed to use an original method for solving such a complicated problem, which allows finding a solution that satisfies the restrictions. Moreover, on its basis, one can iteratively form a control that brings the system closer to a given goal. The method presented below is sufficiently universal for solving arbitrary SLAEs.

SLAE solution based on SVD decomposition

Let a SLAE be given to be solved

$$\Phi z = \phi, \tag{13}$$

where Φ is a matrix of dimension $n \times m$, ϕ and z are vectors of dimensions n and m, respectively. For $n = m$, we have a square system, and for $n > m$ and $n < m$ (13) are an overdetermined and underdetermined SLAE. We use the SVD decomposition of the matrix Φ, which will allow us to find solutions in all these cases. The matrix Φ is represented using the SVD decomposition as

$$\Phi = Q\Sigma V, \tag{14}$$

where Q and V are orthogonal matrices of dimensions $n \times n$ and $m \times m$, respectively. The matrix Σ is rectangular in dimension $n \times m$, on the diagonal of which singular values are located in a non-increasing order [12].

In the case of an overdetermined system $n > m$, we have $\Sigma = \begin{bmatrix} \Sigma_m \\ 0 \end{bmatrix}$, where Σ_m is a square matrix of dimension m and for $\sigma_m \neq 0$ (σ_m is the m-th singular number, i.e., Σ_m is nondegenerate). Then the solution z is written as

$$z = V \cdot \Sigma_m^{-1} \cdot Q_m^{\mathrm{T}} \phi, \tag{15}$$

where Q_m is the matrix formed from the first m columns of the matrix Q. Solution (15) coincides with the one obtained using the standard LSM for overdetermined SLAEs.

When $n < m$, we have an underdetermined system and the matrix Σ in the SVD decomposition takes the block form $\Sigma = [\Sigma_n \;\; 0]$. Solution (15) in this case is written as

$$z = V^n \cdot \Sigma_n^{-1} \cdot Q^{\mathrm{T}} \phi, \tag{16}$$

where V^n is the matrix formed by the first n columns of the matrix V. The written solution coincides with the normal solution (15), i.e., among the set of solutions of the underdetermined system, the one that delivers the minimum to the quadratic form $z^{\mathrm{T}} \cdot z = \|z\|^2$ is taken.

For a square system $n = m$, the solution for a nonsingular matrix Φ has the form

$$z = V \cdot \Sigma^{-1} \cdot Q^{\mathrm{T}} \phi , \tag{17}$$

which coincides with the standard solution (13). When the matrix Φ is degenerate, then, as in the case of an underdetermined system, a normal solution is found. In this case, the matrix Σ is represented as $\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$, where 0 are the corresponding vectors or blocks. By analogy, blocks of matrices V and Q are written in (16).

Problem (13) becomes ill-posed for an ill-conditioned matrix Φ [13]. Then, to find an approximate regularized solution, one should use a stabilizer. When solving the SLAE based on the SVD decomposition, the stabilizer can be formed from the values of the singular values of the matrix Σ. As such, it is proposed to take a stabilizer

$$\alpha \Sigma_s^q , \tag{18}$$

where $\alpha\ (\alpha \geq 0)$ is the regularization parameter, $q\ (q > 0)$ is the tuning parameter, and the diagonal matrix Σ_s has the form

$$\begin{bmatrix} \sigma_n/\sigma_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \sigma_n/\sigma_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_n/\sigma_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} .$$

By adding this stabilizer to the matrix Σ, i.e. in (17), we obtain a regularizing operator that allows us to find an approximate regularized solution to problem (13). It looks like

$$z^{\alpha} = V (\Sigma + \alpha \Sigma_s^q)^{-1} Q^{\mathrm{T}} \phi . \tag{19}$$

The regularization parameter can be found in various ways, for example, from the residual principle [13] or by taking its quasi-optimal value, as suggested in [13, 14]. Problems (15), (16) can also become ill-posed. They can be regularized in the same way, i.e. replacing Σ with an expression $\Sigma + \alpha \Sigma_s^q$.

System output control by a predictive model

In practice, most often it is necessary to control not the state of the system, determined by the vector x_t, but the variables that are measured. Especially when the model (1) is found from the solution of the identification problem, since the vector x in this case plays a connecting role between the input and output variables. In other words, they are internal generalized variables, the physical meaning of which cannot always be interpreted, taking into account the non-uniqueness of their representation. For MPC-based control, it is important to control the behavior of those variables that have an explicable physical meaning, namely, these are the output measured variables. Therefore, in this section, we will consider the problem of local-terminal control on a sliding interval by the measured characteristics of the system, i.e. vector y_t. Description (3) is best

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suited for the synthesis of such an MPC. Let N be the control horizon of the sliding interval $[t, t+N-1]$. The value of the output variable at the end of the interval is determined by equation (3). Its predictive value is determined from the following equation:

$$\bar{y}(t, N) = \Lambda_{N-1} \cdot u(t, N), \tag{20}$$

where $\Lambda_{N-1} = [CA^{N-2}B \, \ldots \, CBD]$ — output controllability matrix, $\bar{y}(t,N) = y_{t+N\text{-}1} - CA^{N\text{-}1}x_t$, $u(t, N) = [u_t^{\mathrm{T}} \; u_{+1t}^{\mathrm{T}} \; \ldots \; u_{t+N-1}^{\mathrm{T}}]^{\mathrm{T}}$, calculated value with known x_t.

For the solvability of (20), it is necessary that $rN \geq m$. The quality of control essentially depends on the properties of the matrix Λ_{N-1}. It is important to $\operatorname{rank} \Lambda_{N-1} = m$. Other properties will be specified after the SVD decomposition of the matrix Λ_{N-1}, which will also be used to solve (20). We perform the SVD decomposition of the matrix Λ_{N-1} and obtain

$$\Lambda_{N\text{-}1} = Q\Sigma V^{\mathrm{T}}. \tag{21}$$

If the matrix Λ_{N-1} in (20) is square, then, taking into account the previously indicated properties of the matrices included in expansion (21), the solution of SLAE (20) is written as

$$u(t, N) = \mathrm{V}\Sigma^{\text{-}1}Q^{\mathrm{T}}\,\bar{y}(t, N). \tag{22}$$

When $rN > m$, then matrices V and Σ are represented in block form

$$V = [V^m \;\; V^{rN-m}],\; \Sigma = [\Sigma_m \;\; 0],$$

where V^m contains the first m columns and V^{rN-m} are the $rN-m$ remaining columns of the matrix V. The matrix Σ_m is square, and 0 is a zero $m \times (rN-m)$ dimension matrix, and solution (20) is written as

$$u(t, N) = V^m \cdot \Sigma_m^{-1} \cdot Q^{\mathrm{T}} \cdot \bar{y}(t, N). \tag{23}$$

The main properties of solutions (22) and (23) of SLAE (20) are determined by the properties of the matrix Σ and Σ_m, i.e., values of singular numbers. The condition number $\kappa(\Lambda_{N-1}) = \dfrac{\sigma_1}{\sigma_m}$ of the controllability matrix characterizes the sensitivity of the solutions obtained with respect to the errors of the initial state X_t, system parameters, i.e. matrix elements A, B, C, D, as well as calculation errors. With good conditionality, $\kappa(\Lambda_{N-1})$ it is close to unity.

In addition, the larger the value of σ_i, the more control resource the system has in the presence of restrictions. All this should be taken into account when choosing a control horizon. Obviously, the horizon should not be large if forecasting is carried out according to an approximate model and the errors at the input and output are quite significant, since this will lead to a large forecast error at large horizons. As a result, it will be necessary to solve more complex problems with frequent correction of predictive control.

Control strategies

In this section, LTI control strategies based on MPC will be discussed. In classical control theory, it is very common to design a control system that implements a strategy which includes program control providing the movement of the system along a given trajectory and a stabilization system that ensures stable motion along this trajectory. In simpler cases, the problem of stabilizing a given equilibrium state is solved. In the presence of restrictions on control, not every trajectory of motion can be realized. In such cases, the trajectory of motion is searched for, which is the closest to the given one or calculated taking into account the constraints.

The problem of terminal control is often considered, when it is not so important along which specific trajectory we approach a given terminal set within the available resources for the control. Many other control strategies are also considered in control theory.

Within the framework of the MPC approach, since the synthesis of control is carried out directly in the process of the system functioning, there is no need to use the strategies described above. The problems of program control and stabilization can be combined into one more complex problem that implements the principle of feedback on current measurements made or estimates obtained on their basis. Many of them are described in the extensive literature on the implementation of the MPC. Based on the results already obtained, it is enough for the designer of the control system to decide which of them is more suitable for his particular case.

In this section, we describe one of the strategies for the movement of the system along a stabilized trajectory to a given final state. By its very nature, it is close to the task of pursuit, or rather, approaching some object. This strategy is applicable to both state and measured variables control problems. Let us describe the implementation of such a strategy. With regard to (12) or (20), first, the value of the control vector without restrictions is found, which has the form

$$u(t,N) = V^n \cdot \Sigma_n^{-1} \cdot Q^{\mathrm{T}} \cdot \overline{x}_{t+N-1} \tag{24}$$

or (23). We have an exact normal solution with minimal $\|u(t,N)\|_2$. If this solution satisfies the control constraints, then the original problem is solved. When it goes beyond the boundary, we look for a solution (12) or (20) with the right side $\beta\overline{x}_{t+N-1}$ ($0<\beta<1$), in which β is the maximum admissible, under which the constraints are satisfied. This β always exists, since $\beta\overline{x}_{t+N-1}$ for $\beta\in[0,1]$ is a segment connecting the point with the zero value of the right side of (12), (20) and the point $\overline{x}_{t+N-1}$. Moreover, this maximum admissible β can be evaluated. As a result we have

$$\beta = \begin{cases} \left|\dfrac{u_{j\min}}{u_j(t,N)}\right|, \text{ if } \; u_j(t,N) < u_{j\min}, \\ 1, \text{ if } \; u_{j\min} \le u_j(t,N) \le u_{j\max}, \\ \left|\dfrac{u_{j\max}}{u_j(t,N)}\right|, \text{ if } \; u_j(t,N) > u_{j\min}, \end{cases}$$

where $u_{j\min} \le u_j \le u_{j\max}$, $j \in \overline{1, rN}$.

A similar result is recorded for other constraints on u and output variable. When $\beta<1$, which means that the resource is not enough to fulfill the control goal in the first

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chosen interval, the following control strategy can be implemented. For the selected control that satisfies the constraints, calculate the predicted value of the controlled variable $\bar{x}_{t+N-1}$ and compare it with the measured one, thereby evaluating the effectiveness of the selected controlling action. Close values indicate good quality of predictive modeling and/or favorable realizations of errors.

When $\beta < 1$ it is proposed to achieve the control goal iteratively. To do this, at the next step, using the measured or estimated value x_{t+N-1}, calculate the right side of equations (12) or (20) as the control goal for the next interval. We perform the same actions that were performed at the previous interval. If in the result β is equal to one, then the control goal is achieved. At $\beta < 1$ the iteration continues until the goal is reached. In the future, only stabilization of the achieved state should be provided. In so doing control is refined not at the end of the horizon, but at each step, which will make it possible to provide better feedback. It is assumed that at this stage of control its resources are sufficient to suppress effectively disturbances affecting the system.

On the basis of the described approach to the implementation of MPC with feedback, it is possible to form other control strategies that are more suitable for the specific application task under consideration. In particular, if the terminal point changes according to a known law or this change can be estimated from the results of observations, then advance pursuit can be implemented.

Thus, the control scheme described above makes it possible to implement control strategies with variable feedback, starting with a one-step correction moving to longer intervals, for example, with a step equal to the length of the horizon.

The proposed control method based on MPC with feedback is suitable for both stable and unstable processes. It is only important that the control resources allow it. Particular in relation to the one considered is the problem of stabilization, in which the zero state is terminal. The goal is achievable if the sequence β_i after a certain number of steps becomes equal to one, and the control resource is not enough to achieve it when it converges to a value β that is less than one. This applies equally to the problem of stabilization with unstable eigenvalues.

State estimation on a sliding interval backwards

The methods considered in the previous MPC sections are implemented if the state x_t is known at the beginning of each control interval. As a rule, only variables y_t measured at each moment of time are known, whose dimension is less than n, i.e. $m < n$. According to these data, in order to implement MPC, it is necessary to estimate x_t. To do this, we take a sliding interval backwards $[t-M+1, t]$ from the point t and, using the values y_{t-j} ($j = 0, 1, ..., M-1$) measured on it, we will restore the state vector we need. Let us use equation (6), which, under the data on the specified interval, we write in the form

$$y(t-M+1) = \Gamma_M x_{t-M+1} + \Phi_M u(t-M+1), \tag{25}$$

where $y(t-M+1) = [y_{t-M+1}^{\mathrm{T}}\ y_{t-M+2}^{\mathrm{T}} \dots y_t^{\mathrm{T}}]^{\mathrm{T}}$, $u(t-M+1) = [u_{t-M+1}^{\mathrm{T}}\ u_{t-M+2}^{\mathrm{T}} \dots u_t^{\mathrm{T}}]^{\mathrm{T}}$.

Since we know the input and output values on the interval back from the point t, we introduce into consideration a vector $f(t, M)$ of dimension mM

$$f(t, M) = y(t-M+1) - \Phi_M u(t-M+1),$$

which can be calculated from known data. Then to find x_{t-M+1} we have an overdetermined SLAE

$$\Gamma_M x_{t-M+1} = f(t, M). \tag{26}$$

Dimension (26) depends on the properties of the observability matrix Γ_M and, first of all, on its condition number. Therefore, we will perform its SVD decomposition for a sufficiently large M. The dependence on M of the condition number is determined from the relation $\kappa(\Gamma_j) = \frac{\sigma_1}{\sigma_j}$, where *j* — varies. From this relation with variable *j*, we choose the appropriate value of *M*, at which the condition number gives solution (26) the least sensitive to data errors. After choosing *M*, solution (26) can be written in the form (15).

If the condition number is bad for any *M*, then regularization should be used, which leads to the solution (19).

In addition to the described approach to estimating the current state vector of the system, it is possible to find a solution to the estimation problem in other similar ways. Some of them are described in [15, 16].

Data-driven predictive control

Great interest is currently shown to MPC problems, which use the trajectory description of systems (8). Most of them are reduced to the synthesis of optimal control on a finite horizon. Let's consider some of them. Presume a control horizon is given with data in the following form $[t-n,\ t-n+1,\ \ldots,\ t-1,\ t,\ t+1,\ \ldots,\ t+N-1]$. Here, the total interval with data is composed of two. The first of them $[t-n,\ t-1]$ is the prehistory interval, on which we have a constantly exciting input action, i.e. from the sequence of outputs, one can form a Hankel matrix with row rank rn. In this case, any initial state (1) specifies a single trajectory, which is provided by data on an interval of length at least n. In this case, we know a priori the dimension of model (1), although we do not know it itself. In the general case, when description (1) is absent, the length of the prehistory interval is taken to be equal $[t-n-L,\ t-1]$ and L is chosen so as to guarantee that $L+n$ was certainly no less than n, and hence the uniqueness of the trajectory is guaranteed. The second interval $[t,\ t+N-1]$ of the length N is the horizon on which the predictive, in most cases, optimal control is synthesized. Then the Hankel matrices in (8) can be written in the block form

$$U = \begin{bmatrix} U_p \\ U_f \end{bmatrix},\ Y = \begin{bmatrix} Y_p \\ Y_f \end{bmatrix}, \tag{27}$$

where U_p and Y_p correspond to the implemented informative process that guarantees the uniqueness of the trajectory, U_f and Y_f correspond to the future predictive process on which the given control goal is realized. The control horizon is chosen in such a way that the control goal, taking into account the existing restrictions on input and output, is feasible.

Blocks U_p and Y_p are formed from the first $n+L$ blocks of matrices U and Y, while the blocks U_f and Y_f are composed of the remaining blocks of these matrices that define the control horizon. The very idea of control according to the prediction model (8), taking into account (2), is formulated in the form of the following lemma [17].

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Lemma. Let the matrix U_p consist of permanently exciting input actions of the order $L+n$ of a completely observable system. Then for the trajectory of the corresponding prehistory and the given input action on the forecast interval the statements are valid

a) there is at least one vector $g^{(0)}$ satisfying

$$\begin{bmatrix} U_p \\ Y_p \\ U_f \end{bmatrix} \cdot g^{(0)} = \begin{bmatrix} u^{(p)} \\ y^{(p)} \\ u^{(f)} \end{bmatrix}, \tag{28}$$

b) the prediction $y^{(f)}$ is unique and is determined by the relation

$$y^{(f)} = Y_f \cdot g^{(0)} \tag{29}$$

for any $g^{(0)}$ satisfying (28).

It is easy to see that the union of (28) and (29) gives (8) for the value i corresponding to the beginning of the prehistory interval.

In fact, this lemma is the basis for solving the analysis problem, namely, it establishes what the system output will be for any given input on the prediction interval contained in the matrix U_f. In this case, all the requirements that make the trajectory description realizable must be met. Here we note that all of them are not rigid enough and admit a set of suitable descriptions (8), (28), (29). In [18], some conditions are given that should be satisfied when choosing the parameters of the trajectory model for MPC, namely $N \geq r(L+2n)+n-1$, and when $r=1$ it is required that $L \leq N+1-3n$.

With the correct choice of suitable parameters for the trajectory description, the lemma formulated above makes it possible to solve the direct problem quite simply, namely, to calculate from (29) the output variable on the prediction interval $[t,\ t+N-1]$ for any given $u^{(f)}$. To do this, first from (28) $g^{(0)}$ is found, and the value of the output variable on the prediction interval is found from (29). More difficult is the problem of predictive control synthesis. As a rule, they are reduced to solving optimization problems. In a rather general and at the same time simple case [17], the MPC problem is formulated as follows. The control providing on the interval $[t,\ t+N-1]$ the movement along the trajectory $\{y_k\}_0^{N-1}$ the closest to the given one $\{y_k^*\}_0^{N-1}$ can be found from the solution of the minimization problem

$$\min_{u,y,g^{(0)}} \sum_{k=0}^{N-1} \left(\left\langle y_k - y_k^*, G_y (y_k - y_k^*) \right\rangle + \left\langle u_k, G_u u_k \right\rangle \right) \tag{30}$$

under restrictions (28), (29).

Since the system model is based on trajectory data, which may contain an error, the problem (30) is complicated and reduced to finding a robust control. If we assume that the error contains only the measured input variable, moreover, corresponding to the interval of prehistory, then instead of the exact $y^{(p)}$ we have an approximate one, determined by the expression

$$\bar{y}^{(p)} = y^{(p)} + \xi . \tag{31}$$

The noise ξ in (32) can satisfy various types of constraints, which we will not dwell on here. Then problem (30) is transformed into the following one:

$$\min_{u^{(f)},y^{(f)},g^{(0)}} \max_{\xi} \sum_{k=0}^{N-1} \left(\left\langle y_k - y_k^*, G_y (y_k - y_k^*) \right\rangle + \left\langle u_k, G_u u_k \right\rangle \right) \tag{32}$$

under restrictions

$$\begin{bmatrix} U_p \\ Y_p + \Xi \\ U_f \\ Y_f \end{bmatrix} \cdot g^{(0)} = \begin{bmatrix} u^{(p)} \\ y^{(p)} + \xi \\ u^{(f)} \\ y^{(f)} \end{bmatrix}, \tag{32′}$$

where ξ and Ξ are the trajectory realization of noise consistent with the constraints.

Such a min-max problem is quite complex and difficult to solve. In certain cases, it can be reformulated in such a way that the search for its solution is simplified. So instead of (32) we can solve a problem close to it

$$\min_{u,\gamma,g^{(0)},y} \gamma \tag{33}$$

under restrictions (32′) on ξ and

$$\sum_{k=0}^{N-1} \left(\left\langle y_k - y_k^*, G_y (y_k - y_k^*) \right\rangle + \left\langle u_k, G_u u_k \right\rangle \right) \le \gamma .$$

Some approaches to solving such problem were proposed in [17].

The original formulation of the problem of finding the optimal robust MPC was considered in [8]. It is assumed that the output in the already realized process of prehistory is known approximately according to (31), in which ξ satisfies the condition

$$\|\xi\|_\infty \le \varepsilon . \tag{34}$$

Then it is proposed the optimal control by trajectory model of prediction to find from the minimization problem. To find:

$$\min_{g^{(0)},u^{(f)},y^{(f)},\bar{y}^{(p)}} \left\| u^{(f)} \right\|^2 \tag{35}$$

under constrains (28), (29), $\left\| \bar{y}^{(p)} - y^{(p)} \right\|_\infty \le \xi$, $u^{(f)} \in U$, $y^{(f)} \in Y$, where U is the domain of admissible controls, and Y is of admissible values of the output on the prediction intervals. A method for solving such a problem can be found, for example, in [9, 19]. However, if there are errors in the past data with a single trajectory solution ($g^{(0)}$ is unique) does not guarantee the robustness of the solution found in this way. Therefore, in [8], a modified formulation of the problem is proposed, in which two problems are separated: the problem of estimation and optimal control, as proposed in [11]. The work [8] describes in detail the procedure for constructing a robust optimal control by solving first the problem of estimating the membership set of all values of the parameter $\delta g^{(0)}$ consistent with permissible errors, determined by the relation $g^{(0)} = \bar{g}^{(0)} + \delta g^{(0)}$, where $\bar{g}^{(0)}$ is its nominal value, which is found from the available

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data, and $\delta g^{(0)}$ is all its permissible variations. After that, the problem of finding the optimal control on the prediction interval is solved, taking into account the found guaranteed estimates. All this is described in more detail in [8].

When implementing feedback based on MPC, it is advisable to have not very large prediction interval. The presence of constrains on control can significantly affect its choice. Moreover, within the framework of the problem statements considered above, there are no guarantees of the feasibility of the control goal when using small prediction horizons.

Therefore, in the section «Control strategies», an iterative scheme for implementing MPC was considered when achieving the final goal through solving terminal control problems on a sequence of small horizons. This increases the efficiency of the feedback, and as a consequence of this, the robustness as well.

Let us consider one of the possible approaches for implementing such strategies using the data-based description of LTI systems. To do this, we choose an appropriate control horizon that satisfies the condition $rN \geq n$. Let us write for it the following terminal control problem

To find $$\min_{u,\, g^{(0)}} \sum_{k=0}^{N} \|u_k\|^2 \qquad (36)$$

under constrains (28) and (29), $y_{t+N-1} = y^*$.

In order to solve the problem (36) we begin first with case, when no constrains on control are imposed. As a result, we get a normal solution $u(t, N)$. If the controls are subject to restrictions similar to those specified in the «Control strategies» section, then we check whether the found solution satisfies them or not. When the constraints are fulfilled, the problem is solved completely. When they go beyond the allowable area, we use the procedure described after formula (24) for finding the parameter β , with the help of which we ensure the feasibility of the restrictions. After the control chosen in this way, on the basis of (28) $g^{(0)}$ is found and then from (29) the predicted output is calculated.

The considered statement of the terminal control problem and the approach to its solution are equivalent to (20) and solution (24) using the parameter β to satisfy the given constraints. It is quite difficult to establish the feasibility of the control goal in the class of normal solutions on a sequence of terminal problems with a given horizon. Solvability is guaranteed only in the zero-terminal stabilization problem for a stable LTI system. In a number of cases, for given initial and final states of the system, it is possible, on the basis of a computational experiment, to check the solvability of the problem. With known matrices A, B, C , using numerical-analytical procedures, it is possible to estimate reachability domains.

In recent years, a large number of papers have been published with different formulations of control problems and approaches to their solution using informative trajectory data, i.e. based on (8). It is almost impossible to list and analyze them. Moreover, this is a fairly intensively developed direction of research and applied development, and new results should be expected in the near future.

Conclusion

The main states of the control theory based on the data of the trajectory description were considered for discrete linear stationary LTI systems. These include a fairly large number of real systems encountered in practice. Nevertheless, there remains a fairly large class of systems that do not fall into this class. This is especially concerning nonlinear systems, which in practice are more common than linear ones. A natural question arises about the possibility of extending the considered approaches and methods to the class of nonlinear systems. When considering controlled and predictable processes in systems with nonlinear models, various methods of linearization of the original nonlinear equations are widely used with the further prospect of using the mathematical apparatus developed for linear systems. As a rule, its own linearized model is constructed on different sections of the trajectory. This is quite consistent with MPC when the prediction horizon is finite and not very large. When using the trajectory description (8), there is no need to find an approximate description through the linearization of the original nonlinear equations. The trajectory data over not very large interval will just match the linearized model if it admits an acceptable approximation, i.e. the scatter of trajectory data over the considered prediction interval is small. At the next prediction horizon, the new data may correspond to a different linearized model. As a result, an appropriate choice of horizon can match data measurement errors with linearization errors.

If we are dealing with a system continuous in time, then the use of discrete data is equivalent to approximating a continuous system to a discrete one, and here it is also advisable to match the errors.

Based on the foregoing, we can conclude that the control method according to the MPC scheme using trajectory data is universal.

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НОВІ НАПРЯМИ ТЕОРІЇ КЕРУВАННЯ

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У статті викладено концептуальні засади нових трендів у теорії керування, які інтенсивно розвиваються останнім часом. На відміну від класичної теорії керування, яка сформувалася у минулому столітті і базується на відомих математичних моделях керованих процесів у вигляді локальних рівнянь, у нових підходах стосовно лінійних стаціонарних систем використовуються співвідношення вхід–вихід, що випливають безпосередньо з формули Коші як для неперервних, так і дискретних систем. На основі цього ж опису можна обґрунтувати і отримати так звані траєкторні моделі, які безпосередньо прив’язані до даних, що формують на інтервалах спостереження траєкторії уже реалізованих попередніх та майбутніх процесів, для яких слід синтезувати керування. Такий підхід орієнтований насамперед на знаходження керування за моделлю передбачення. При цьому поточні вимірювання, здійснювані на об’єкті, дають змогу реалізувати зворотний зв’язок і у разі розбіжностей прогнозу від реального процесу провести корекцію прогнозного керування, тобто стабілізувати його. Керування за траєкторною моделлю передбачення дає можливість виключити ідентифікацію моделі за траєкторними даними, а керування здійснювати безпосередньо за ними. Оскільки дані містять похибки, найважливішим у аналізованому підході є питання робастності обраного керування. Йому присвячується велика кількість опублікованих робіт, де гарантований підхід, орієнтований на несприятливу реалізацію похибок даних, є найбільш затребуваним. Найчастіше синтез керування зводиться до розв’язання різних оптимізаційних задач переважно на скінченному горизонті передбачення. Значна увага у статті приділена методам розв’язування задач синтезу на основі SVD-розкладання. Щоб зменшити складність вирішуваних задач пропонується зводити її до термінального керування на горизонті невеликої тривалості. Тоді реалізується ітеративна стратегія керування, яка за рахунок зворотного зв'язку забезпечує здійсненність глобальної цілі керування.

Ключові слова: теорія керування, траєкторна модель, керування за даними, лінійна стаціонарна система, керування за моделлю передбачення

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